QCD topology with background electromagnetic fields on the lattice

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Outline

- Topology in QCD with EM fields
- The topological susceptibility
- The axion-photon coupling
- Conclusions and further work
Topology in QCD with EM fields
Definition of $Q_{\text{top}}$:

$$Q_{\text{top}} = \int d^4 x \, q_{\text{top}}(x), \quad q_{\text{top}} = \frac{1}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \Tr G_{\mu\nu} G_{\rho\sigma}.$$

Adding electric or magnetic fields *separately*: no changes in topology.

$$\langle Q_{\text{top}} \rangle = 0.$$

If $F_{\mu\nu} \neq 0$ such that $\vec{E} \cdot \vec{B} \neq 0$ it can be interpreted as an effective $\theta$-therm D'Elia et al., 2012.

Hence, non-orthogonal EM fields $\iff$ non-trivial topology.

$$\langle Q_{\text{top}} \rangle \neq 0.$$
The topological susceptibility
Is the second moment of $Q_{\text{top}}$: $\chi_{\text{top}} = \frac{\langle Q_{\text{top}}^2 \rangle}{V_4}$

It is also the mass of the axion:

$$f_a^2 \frac{\delta^2}{\delta a^2} \log Z(a) \bigg|_{a=0} = \frac{\partial^2}{\partial \theta^2} \log Z(\theta) \bigg|_{\theta=0} \quad \leftrightarrow \quad m_a^2 f_a^2 = \chi_{\text{top}}.$$ 

Hence, an analysis of $\chi_{\text{top}}$ gives information on $m_a$.

Current estimate from ChPT at zero $T$: $\chi_{\text{top}}(LO) = [75.5(5)\text{MeV}]^4$ Cortona et al 2016.

- Lattice calculations give almost the same central value but with a bigger error, $\chi_{\text{top}} = [75.6(1.8)(0.9)\text{MeV}]^4$ Borsanyi et al 2016.
- ChPT also predicts a mild enhancement with B at low $T$ Adhikari 2022.
Index theorem says $\mathcal{D}$ has zero modes when $Q_{\text{top}} \neq 0$.

Staggered operator lacks these zero modes $\rightarrow$ huge lattice artifacts, specially for high temperatures (we are talking of several orders of magnitude!).

One possible solution: substitute the smallest eigenvalues of $D_{\text{stagg}}$ with their continuum values Borsanyi et al 2016.

How? Reweighting each configuration by:

$$
\prod_f \prod_{i=1}^{2|Q_{\text{top}}|} \prod_{\sigma = \pm} \left( \frac{2m_f}{i\sigma \lambda_i + 2m_f} \right)^{n_f/4}.
$$
Simulation setup for $\chi_{top}$

- Improved staggered quarks with 2+1 flavours and physical quark masses.
- $N_s \times N_t = 24^3 \times 6, 24^3 \times 8, 28^3 \times 10, 36^3 \times 12$.
- $T = 110$-300 MeV, $eB = 0, 0.5, 0.8 \text{ GeV}^2$.
- Gradient Flow used to reduce the UV fluctuations and control the topology
  Lüscher 2010.
$$\chi_{\text{top}}$$: preliminary results

Topology controlled after applying the gradient flow.

$$28^3 \times 10, T = 211 \text{ MeV}, \langle Q \rangle = -0.1 \pm 0.06$$
Gradient flow evolution of $\chi_{\text{top}}$. Note the plateaus.
Separation between chiral and non-chiral eigenvalues. $Q = -3$. 
χ_{top}: preliminary results

Effect of the reweighting.

T \approx 150 \text{ MeV}, eB = 0
Effect of the reweighting.

$T \approx 150 \text{ MeV, } eB = 0$

$\chi_{\text{top}} \, \text{fm}^{-4}$

PRELIMINARY

No RW
$eB = 0$
1606.07494, Borsanyi et al.
Continuum limit for a single temperature.

\( T \approx 150 \text{ MeV}, \ eB = 0 \)
The axion-photon coupling
The photon-axion coupling $g_{a\gamma\gamma}^{QCD}$

- The axion couples directly and indirectly to photons.

- ChPT calculations show that the coupling decomposes into two terms, one model dependent and one model independent.

- Current estimate from ChPT: $g_{a\gamma\gamma} = g_{a\gamma\gamma}^0 + g_{a\gamma\gamma}^{QCD} = \frac{\alpha_{em}}{2\pi f_a} \left( \frac{E}{N} - 1.92(4) \right)$
  

- We want to compute the QCD dependent part of the coupling $\longrightarrow$ no need to include axions on the lattice!
If we include both electric and magnetic background fields, the only CP odd operators in our theory are:

\[ \text{Tr} \tilde{G}^{\mu \nu} G_{\mu \nu} \quad \& \quad E \cdot B. \]

So by symmetry arguments, \( Q_{\text{top}} \) can only be (for weak fields):

\[ Q_{\text{top}} \propto E \cdot B + \mathcal{O} \left( [E \cdot B]^3 \right). \]

By looking at \( Z \):

\[ \frac{\delta \log Z(a)}{\delta a} \bigg|_{a=0} = \frac{\langle Q_{\text{top}} \rangle_{E,B}}{f_a} \rightarrow g_{a \gamma \gamma} f_a = \frac{T}{V} \frac{\partial}{\partial (E \cdot B)} \langle Q_{\text{top}} \rangle_{E,B} \bigg|_{E,B=0}. \]

So for homogeneous, static and weak EM fields

\[ \frac{T}{V} \langle Q_{\text{top}} \rangle_{E,B} \approx \frac{g_{a \gamma \gamma}^{QCD} \cdot f_a}{e^2} e^2 E \cdot B \quad \text{and} \quad g_{a \gamma \gamma}^{QCD} < 0. \]
Improved staggered quarks with 2+1 flavours and physical quark masses.

\[ N_s \times N_t = 24^3 \times 32, \ 32^3 \times 48, \ 40^3 \times 48. \]

\[ T = 0. \]

We keep \( \mathbf{E} \cdot \mathbf{B} \) in the linear response region.

Imaginary electric fields (sign problem).

Gradient Flow used to reduce the UV fluctuations and control the topology.

Lüscher 2010.
Shift of $Q_{\text{top}}$ at non-zero $\mathbf{E} \cdot \mathbf{B}$. Effect also shown in D’Elia et al 2016.
Linear response for weak fields.

\[ T \langle Q \rangle_{E,B} \text{ GeV}^4 \]

\[ e^2 \vec{E} \cdot \vec{B} \text{ GeV}^4 \]

\[ \times 10^{-3} \]

\[ 24^3 \times 32, T \approx 0 \text{ MeV} \]

\[ \times 10^{-5} \]

Slope method

Lattice points
Topology controlled after applying gradient flow. (32³ × 48).
Approaching the continuum limit.
Conclusions and further work
Conclusions and further work

- We have shown:
  - that the topology is under control.
  - how the would-be zero modes introduce huge lattice artifacts for $\chi_{\text{top}}$.
  - a linear response of $\langle Q_{\text{top}} \rangle$ with $\mathbf{E} \cdot \mathbf{B}$ for weak fields.
  - we are getting closer to obtaining continuum limit extrapolations for $\chi_{\text{top}}$ and $g_{\alpha\gamma\gamma}^{QCD}$.

- Further work:
  - further understand the would-be zero modes at low temperatures.
  - generate more statistics and perform the continuum limit for both observables.
  - implement the reweighting technique for $g_{\alpha\gamma\gamma}^{QCD}$. 
Thank you for your attention!
Backup slides
EM fields can induce topologies in the gluon sector. But how? \[\rightarrow\] Index theorem.

The index theorem says (for QCD) Atiyah, Singer ’71:

\[
\text{Index}(\not{D}) \equiv n_- - n_+ = Q_{\text{top}}
\]

Since in QCD \(\langle Q_{\text{top}} \rangle = 0\), we don’t see imbalances in chirality.

But after including electromagnetic fields the situation is different:

\[
\text{Index}(\not{D}) \equiv n_- - n_+ = Q_{\text{top}} + Q_{U(1)}.
\]

We have two different topological contributions to the zero modes.

Path integral favours as little zero modes as possible: \(\det M \uparrow\uparrow\).

Hence, it selects gluon field configurations such that:

\[
Q_{U(1)} \uparrow \iff Q_{\text{top}} \downarrow.
\]