Analytic continuation of the three-particle amplitudes

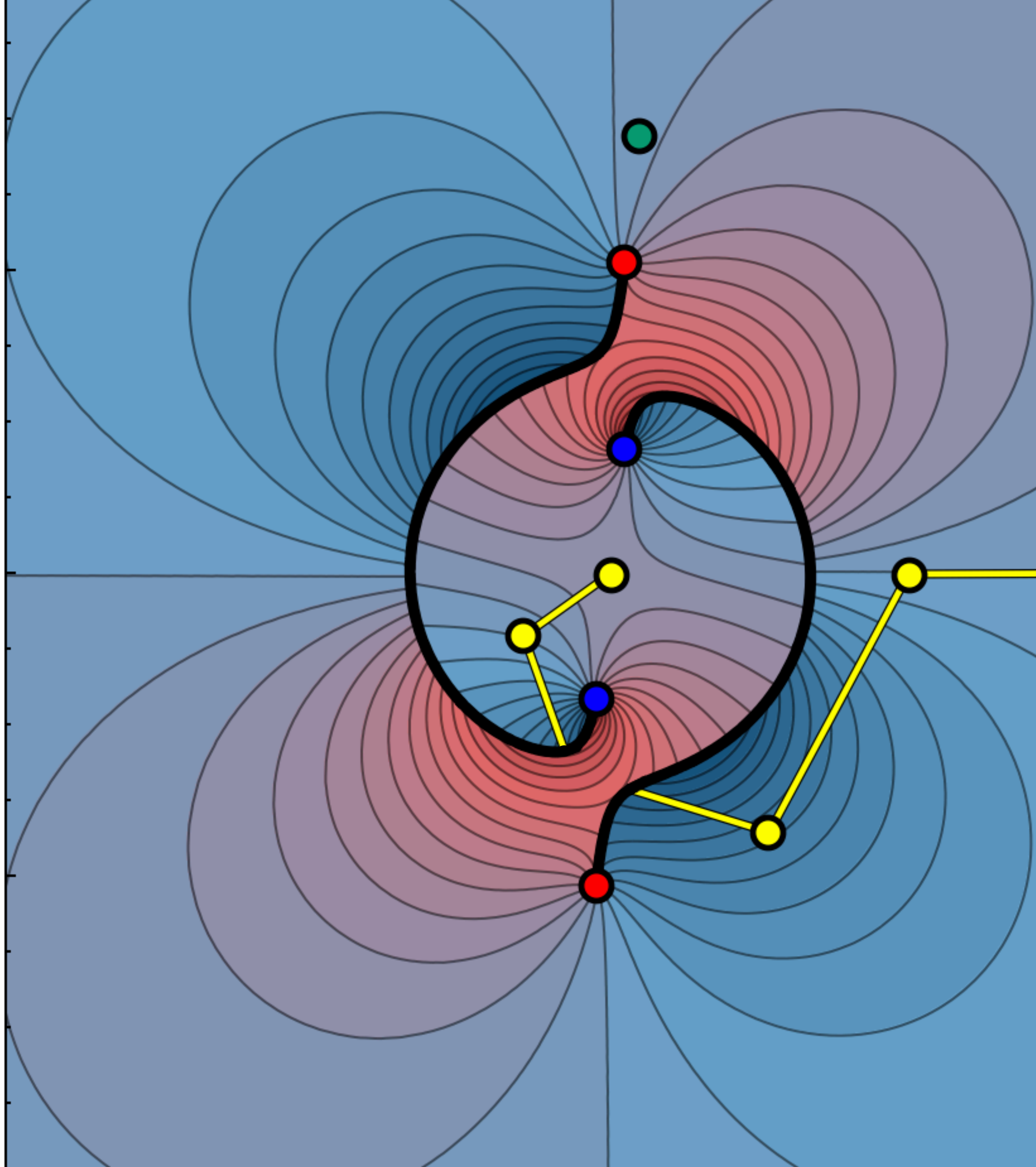
Sebastian M. Dawid

with

Md Habib E. Islam (ODU)

Raúl A. Briceño & Andrew Jackura (UC Berkeley)

UNIVERSITY of WASHINGTON



Outline

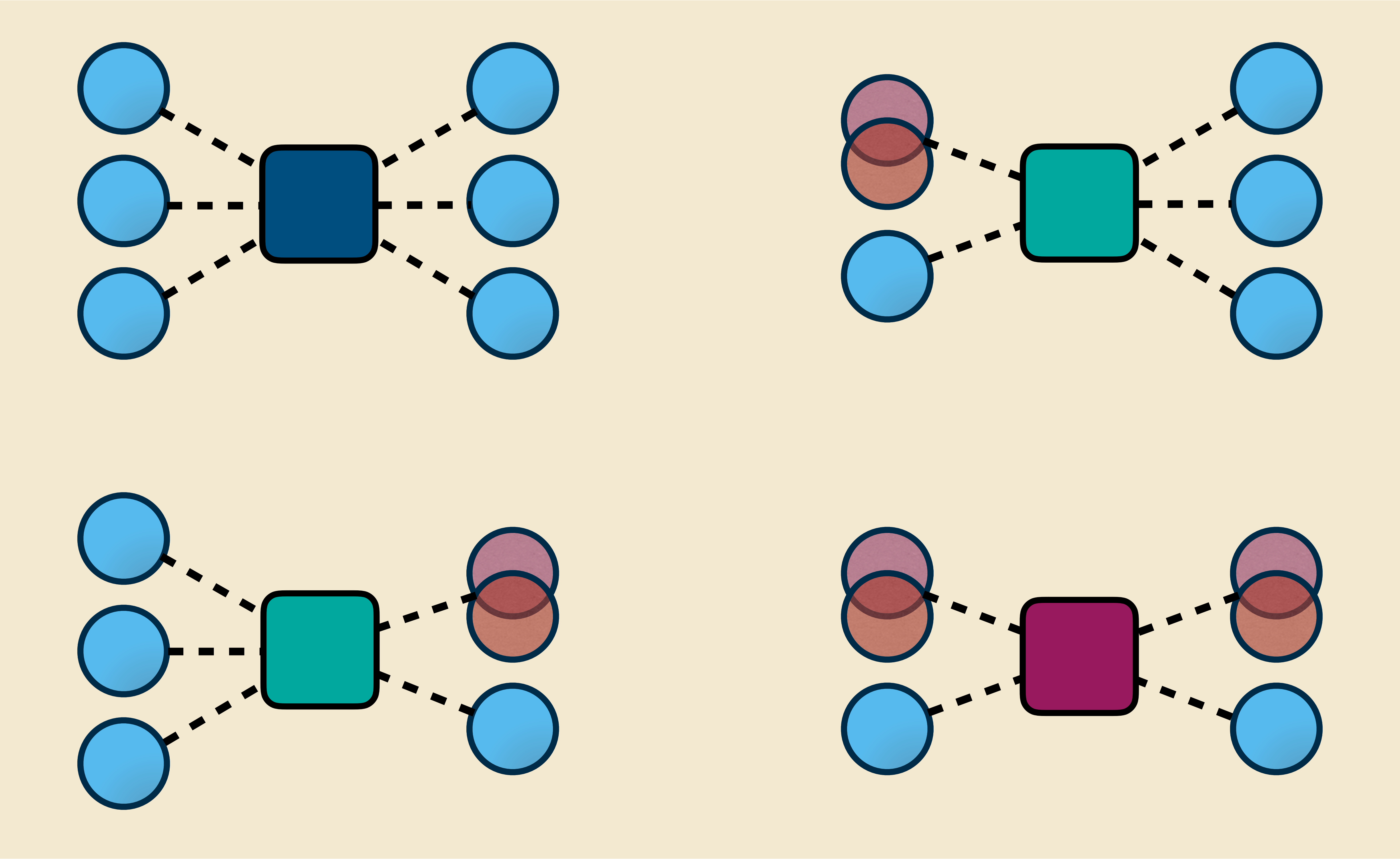
- Need for amplitude analysis
- Three-body integral equations
- **Analytic continuation**
- Scattering observables in a toy model

Solving relativistic three-body integral equations in the presence of bound states
Jackura, Briceño, Dawid, Islam, McCarty, Phys. Rev. D104 (2021) 1, 014507

Analytic continuation of the relativistic three-body amplitudes
Dawid, Islam, Briceño, arXiv:2303.04394

Evolution of Efimov states in relativistic scattering theory
Dawid, Islam, Briceño, Jackura, in preparation

Coupled 3ϕ and ϕb system

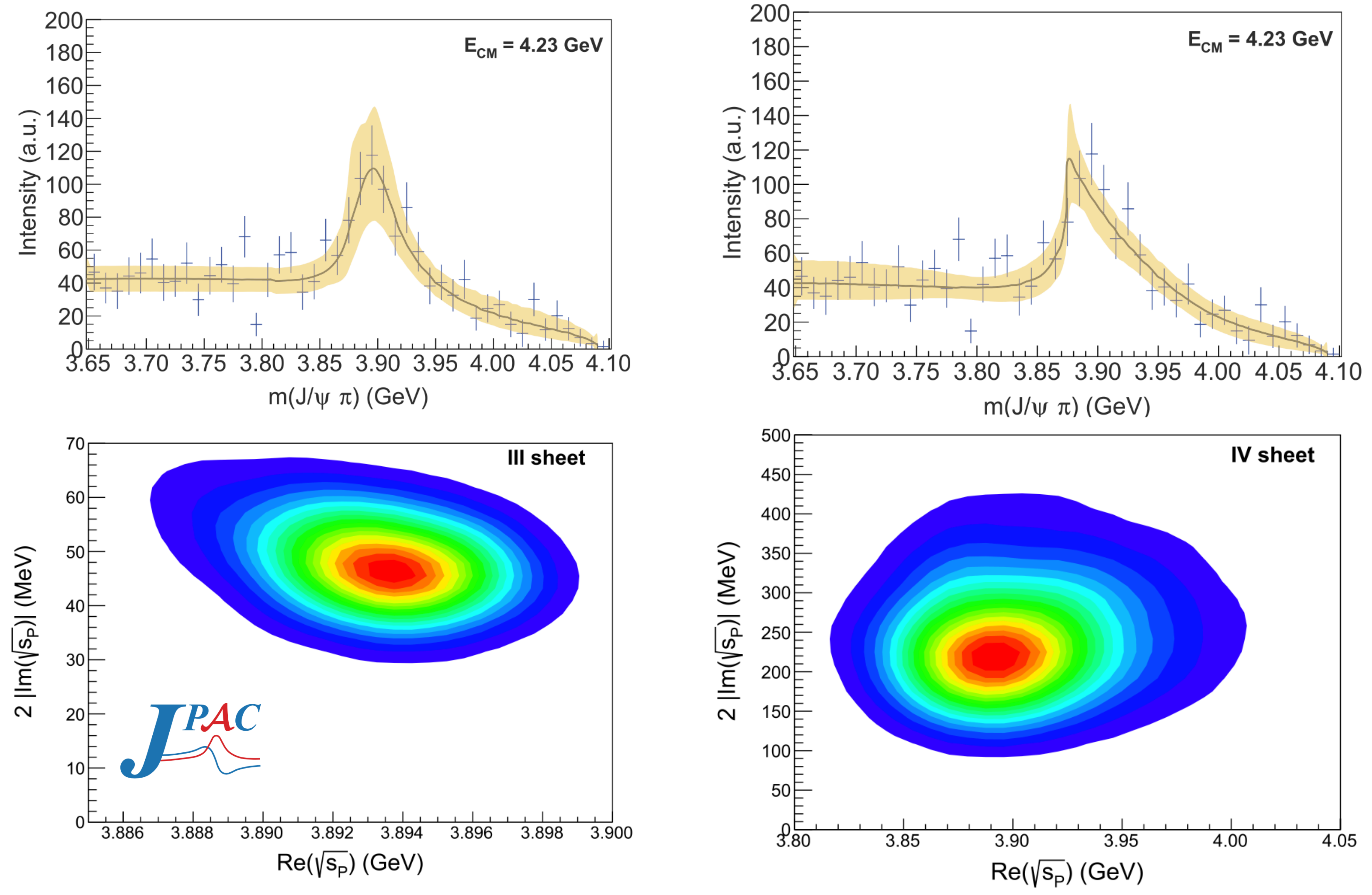


Need for amplitude analysis

Plenary talk by by A. Hanlon

Experiment — $Z_c(3900)$ pole

Amplitude analysis and the nature of the $Z_c(3900)$
Pilloni et al. (JPAC), Phys. Lett. B 772, (2017) 200-209

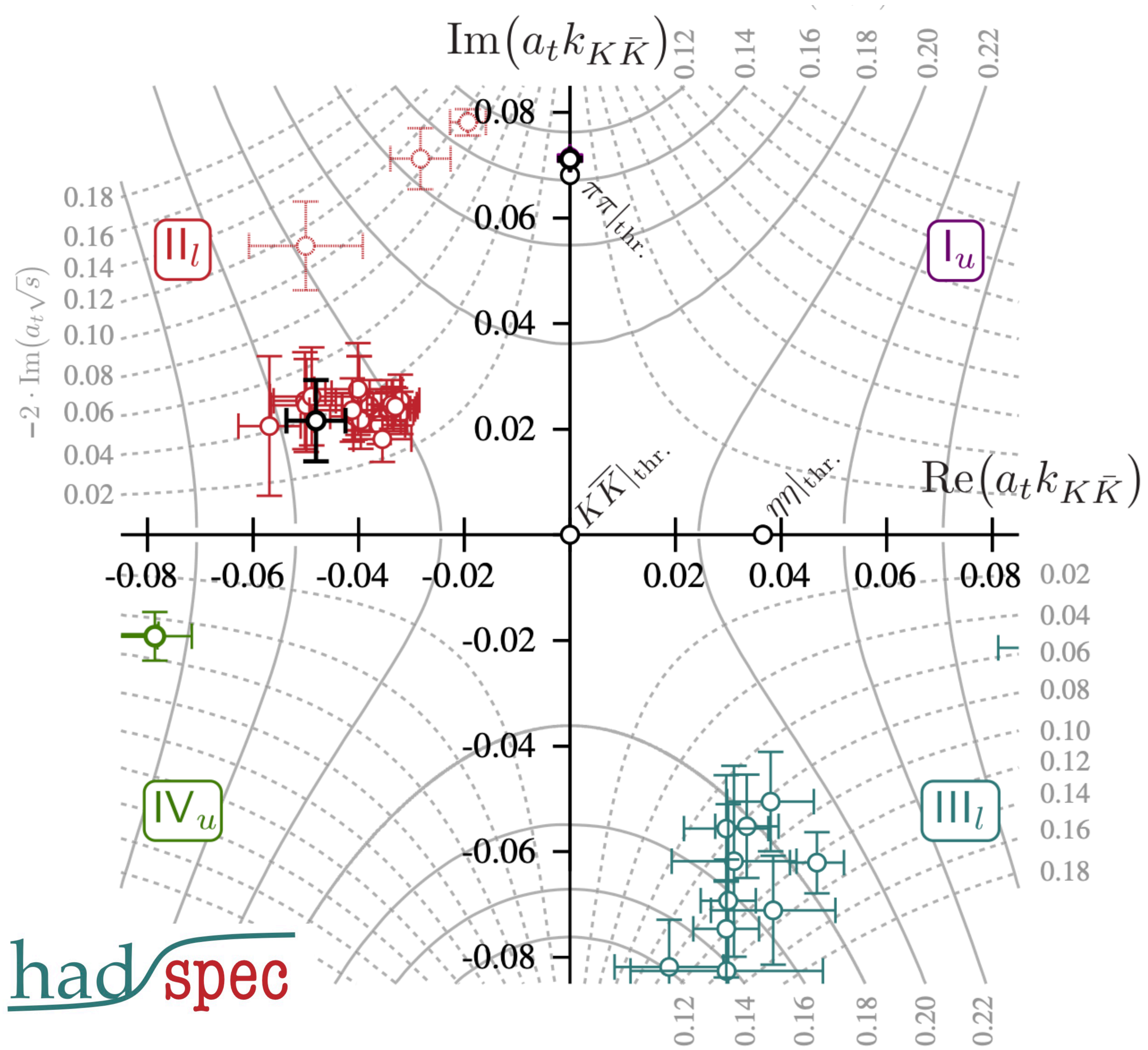


Pole on 3rd sheet

Pole on 4th sheet + triangle

Lattice — f_0 isoscalar pole

Isoscalar ππ, KK, ηη scattering and the σ, f_0, f_2 mesons from QCD
Briceño et al. (HadSpec), Phys. Rev. D 97, (2018) 054513



Three-body program

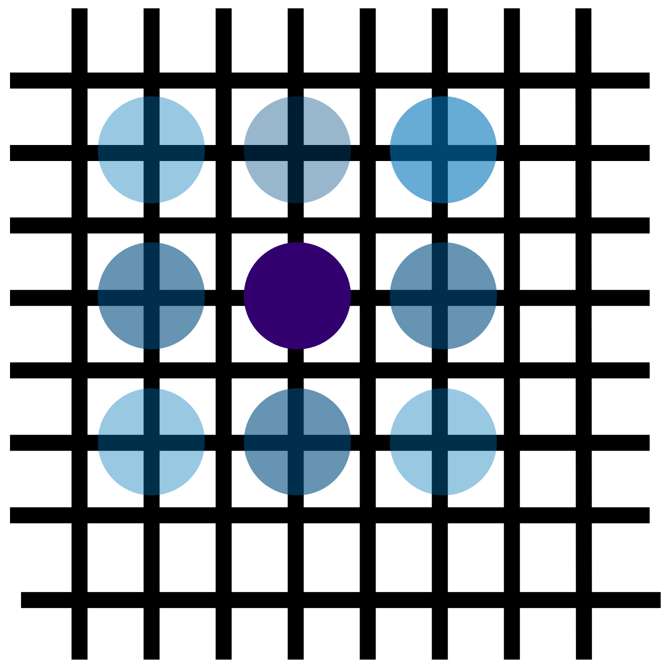
Talk by S. Sharpe | Talk by Z. Draper | Talk by M. Sjö

Relativistic, model-independent, three-particle quantization condition
Hansen, Sharpe, Phys. Rev. D 90 (2014) 11, 116003

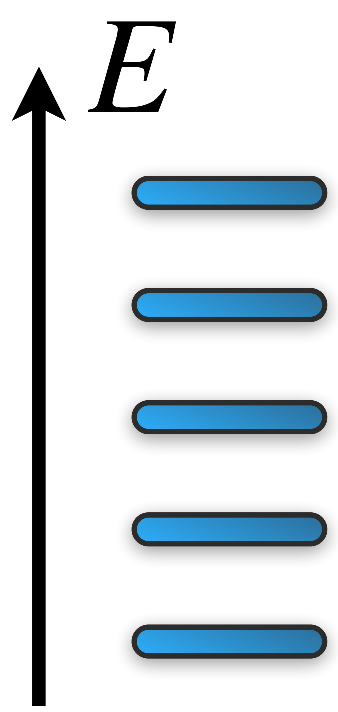
Three-body unitarity in finite volume
Mai, Döring, Eur. Phys. J. A 53 (2017) 12, 240

Relativistic-invariant formulation of the NREFT three-particle quantization condition
Müller, Pang, Rusetsky, Wu, JHEP 02 (2022), 158

Finite Volume



Lattice QCD

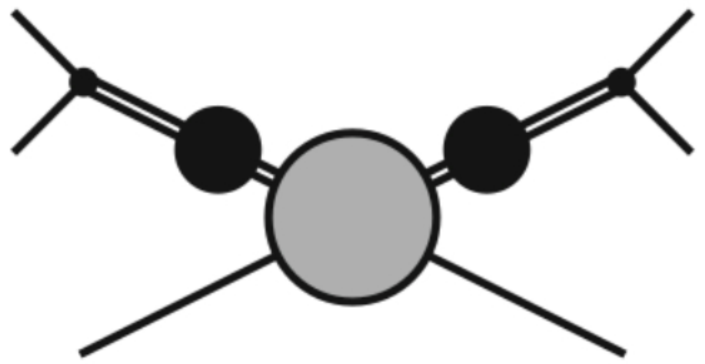


Spectrum

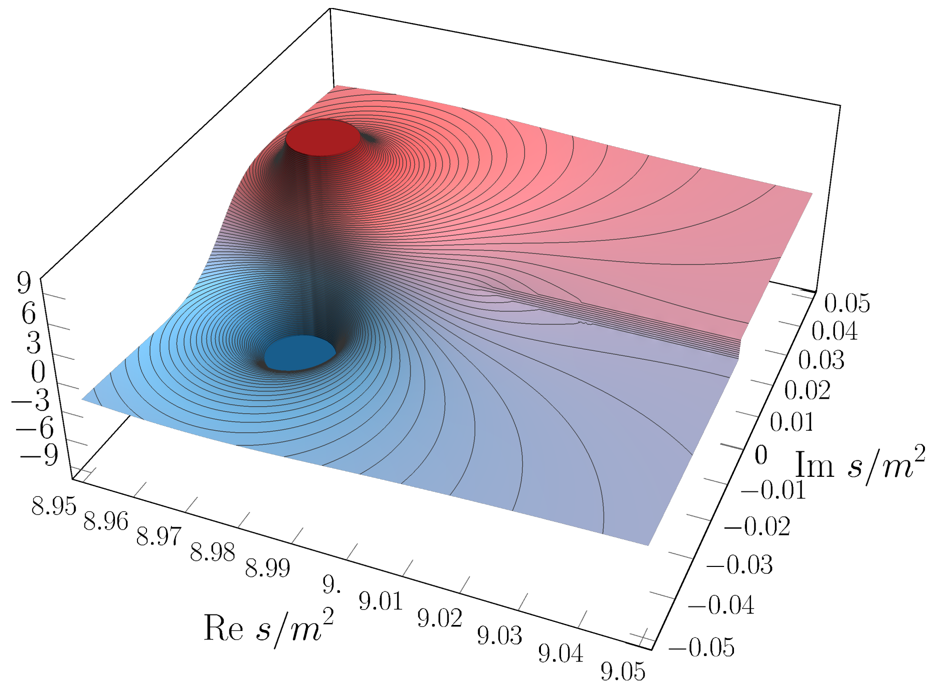
$\mathcal{K}_3$

Quantization Condition

Infinite Volume



Amplitude

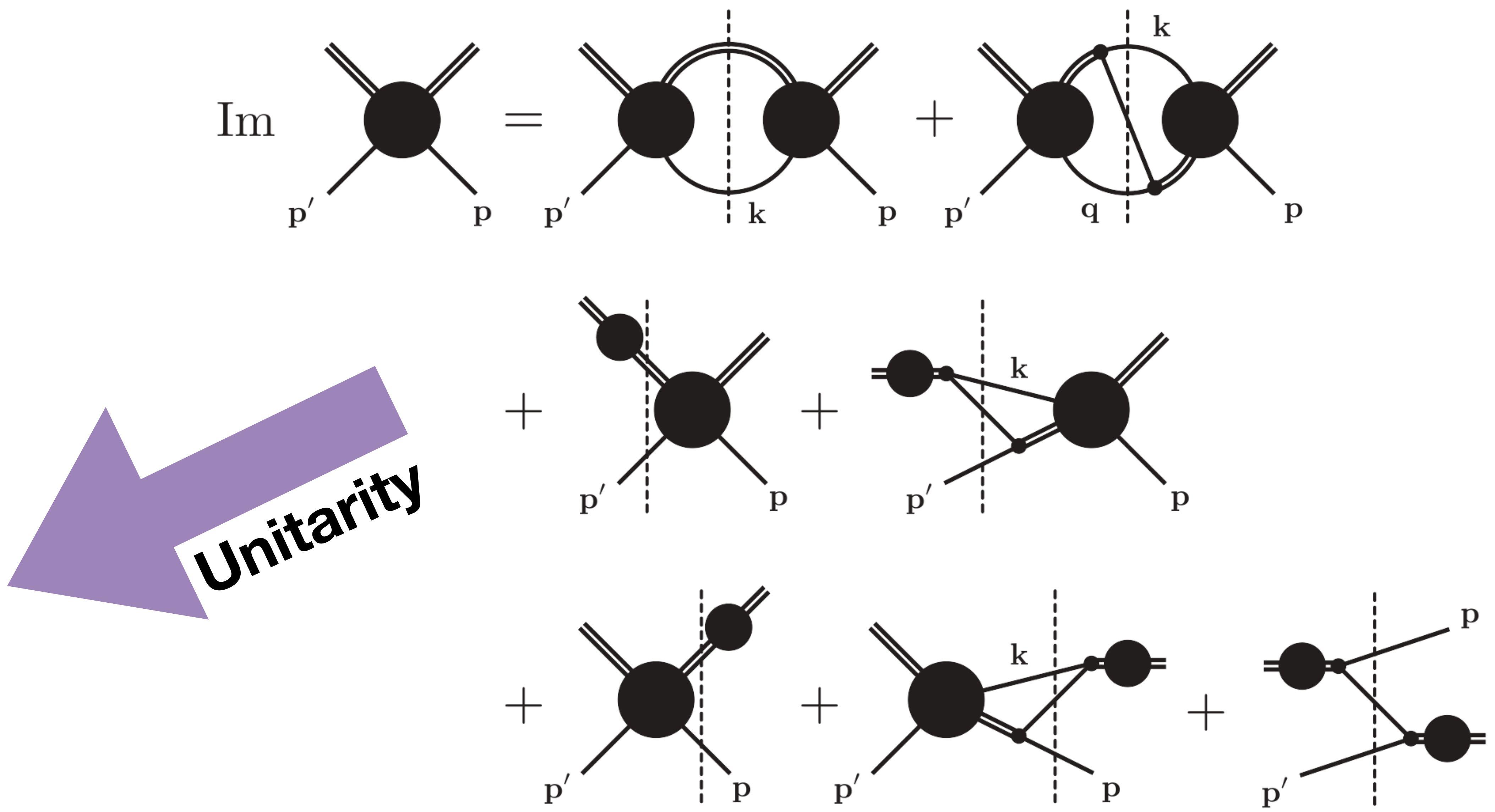


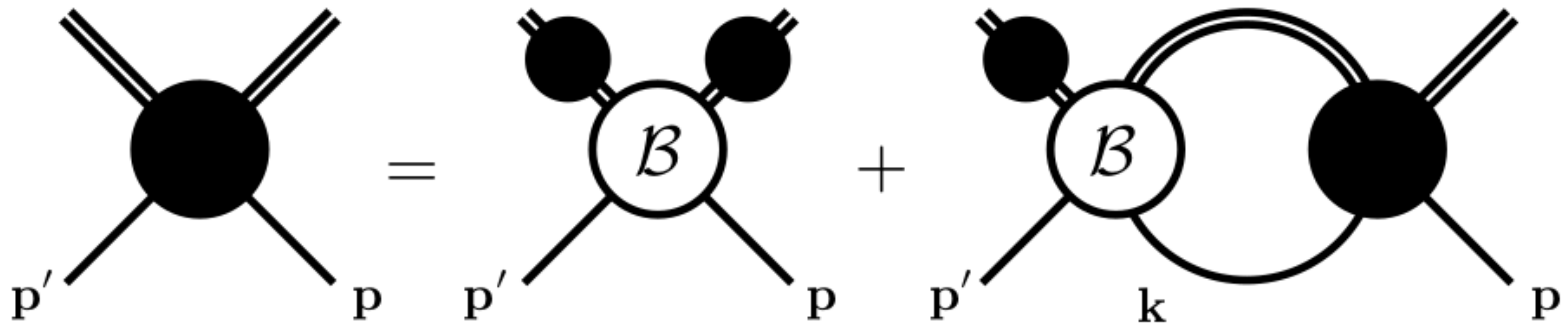
Resonances

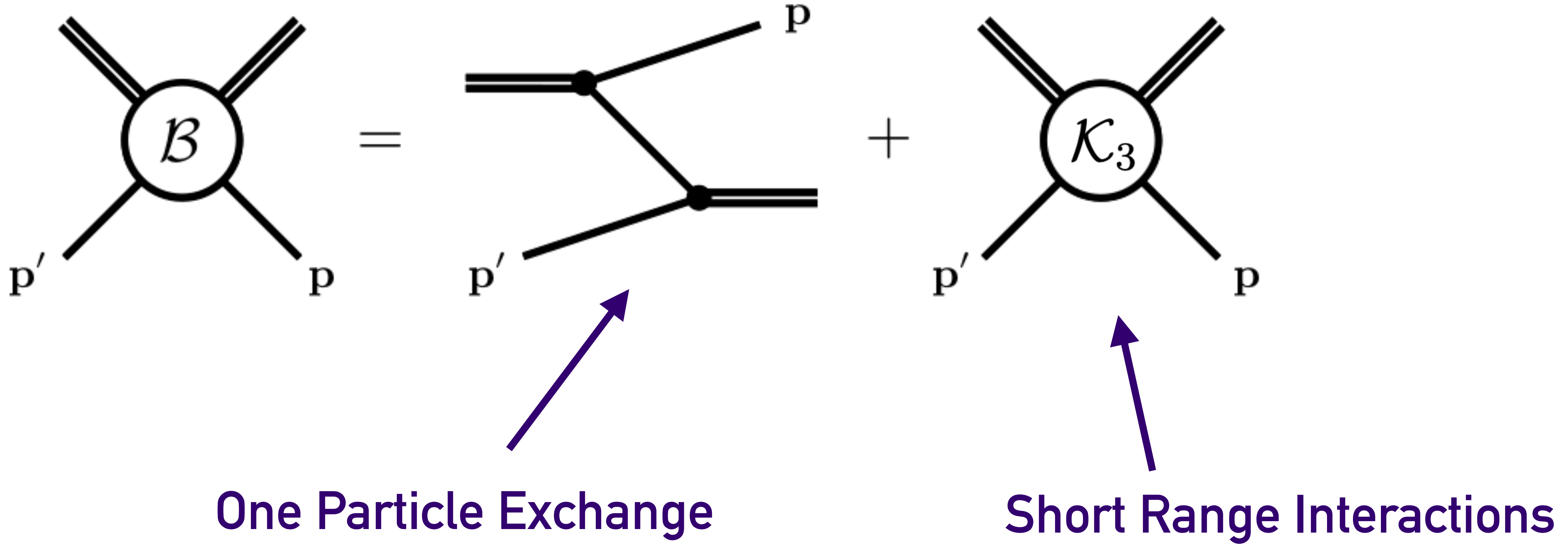
$$\det\left[\mathcal{K}_{\mathrm{df},3}(s) + F_3(s, \boldsymbol{P}, L)^{-1}\right] = 0$$

S-matrix parametrization

All diagrams by Andrew Jackura







Three-body amplitude

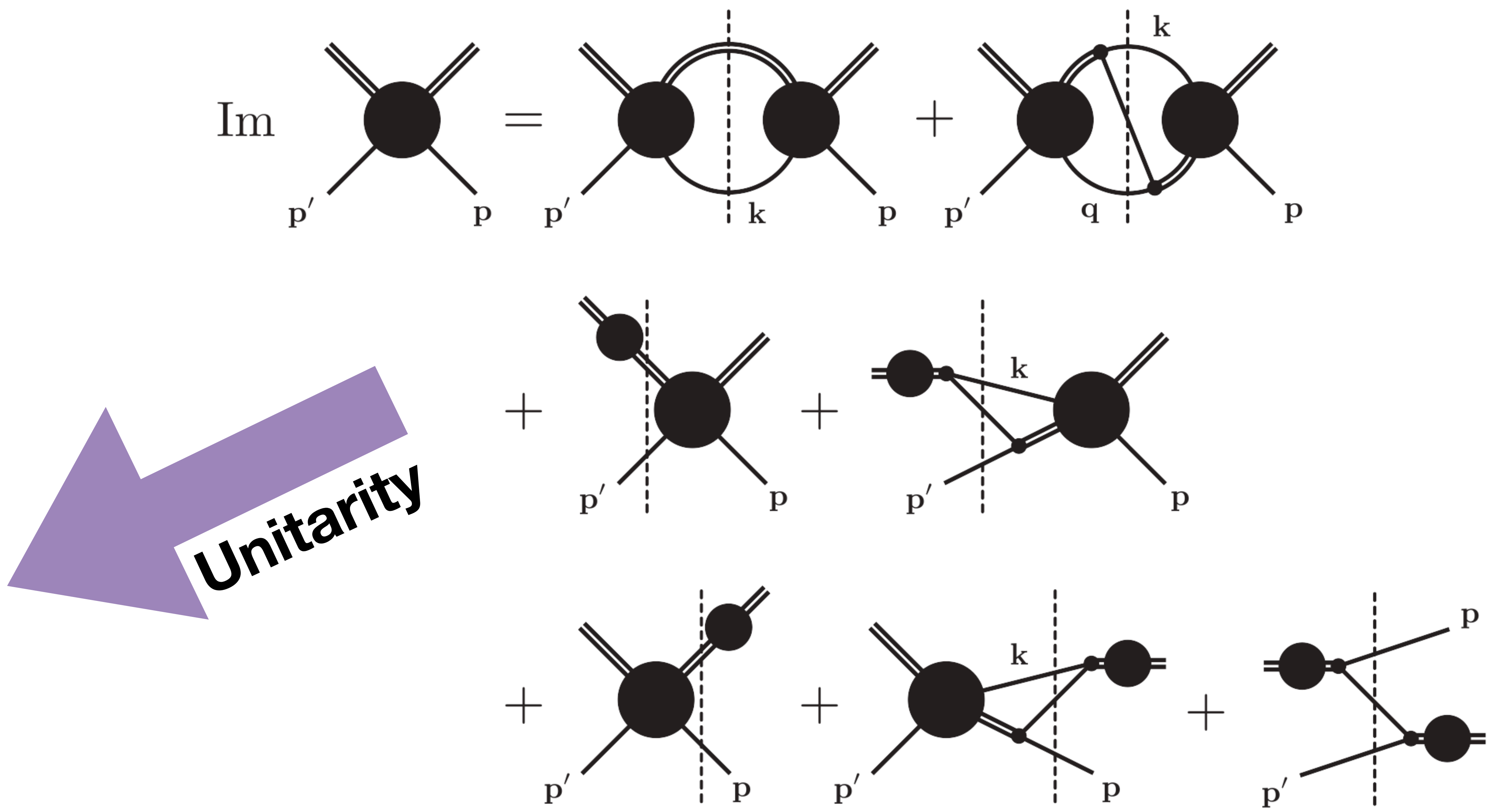
$$[\mathcal{M}_3]^J_{\ell' m'_\ell;\ell m_\ell}(p', s, p)$$

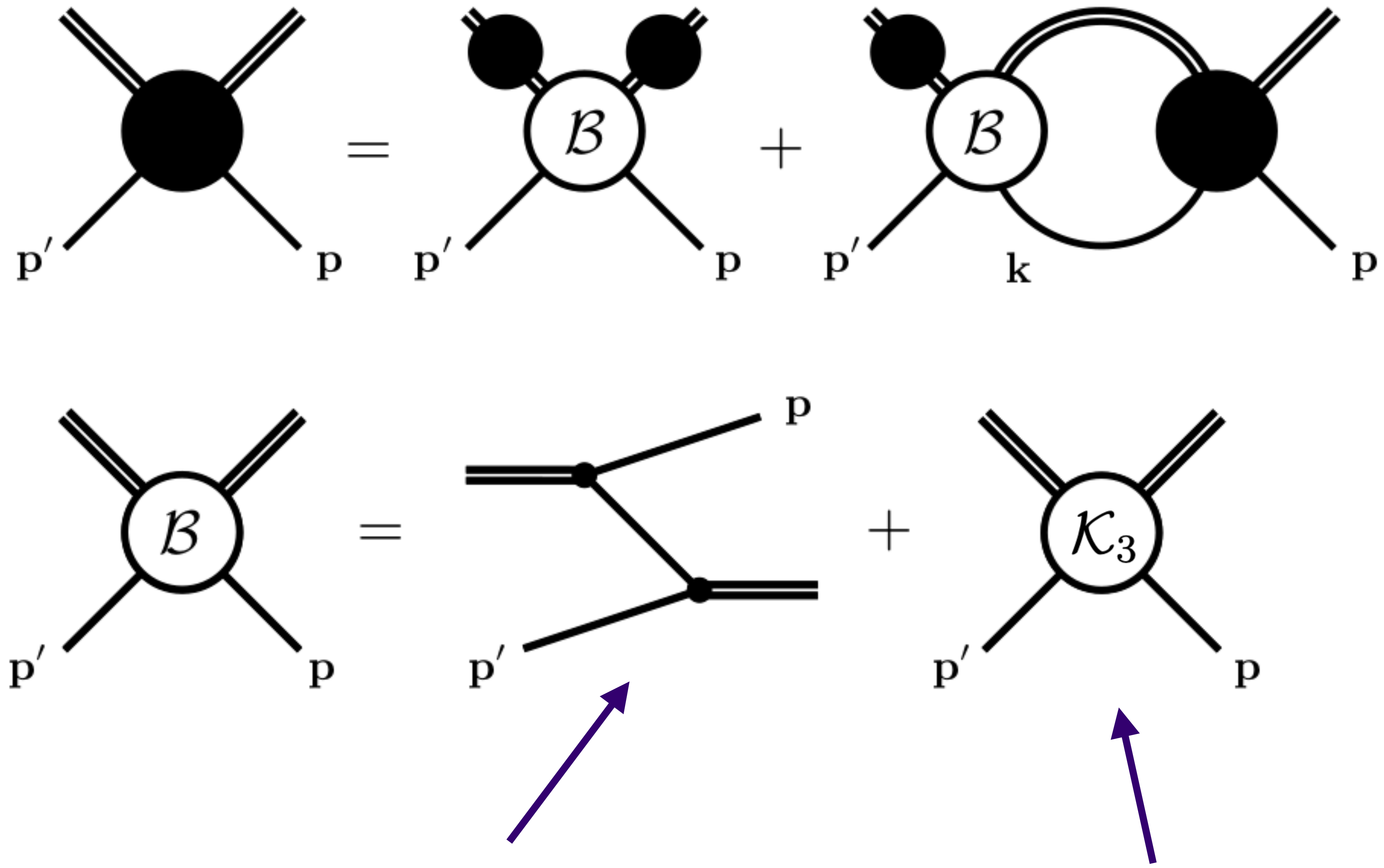
- pair-spectator
- partial waves
- symmetrization

$$\mathcal{M}_3 = \mathcal{M}_2 \mathcal{B} \mathcal{M}_2 + \mathcal{M}_2 \int \mathcal{B}\, \rho_3\, \mathcal{M}_3$$

S-matrix parametrization

All diagrams by Andrew Jackura





One Particle Exchange

Short Range Interactions

Three-body amplitude

$$[\mathcal{M}_3]^J_{\ell' m'_\ell;\ell m_\ell}(p', s, p)$$

- pair-spectator
- partial waves
- symmetrization

$$\widetilde{\mathcal{M}}_3 = \mathcal{B} + \int \mathcal{B}\,\mathcal{M}_2\,\rho_3\,\widetilde{\mathcal{M}}_3$$

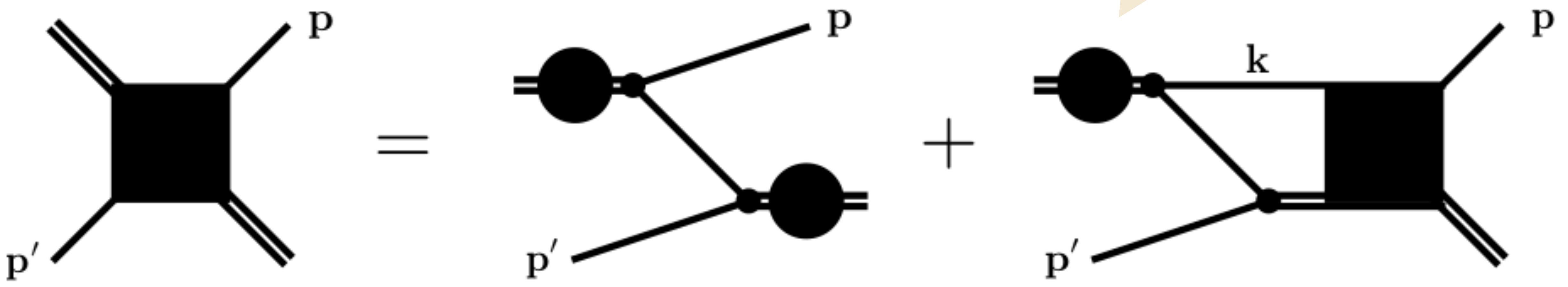
$\mathcal{B}\,\rho_3$

$$\mathcal{M}_2 = \mathcal{K}_2 + \mathcal{K}_2 i\rho_2\mathcal{M}_2$$

Ladder equation

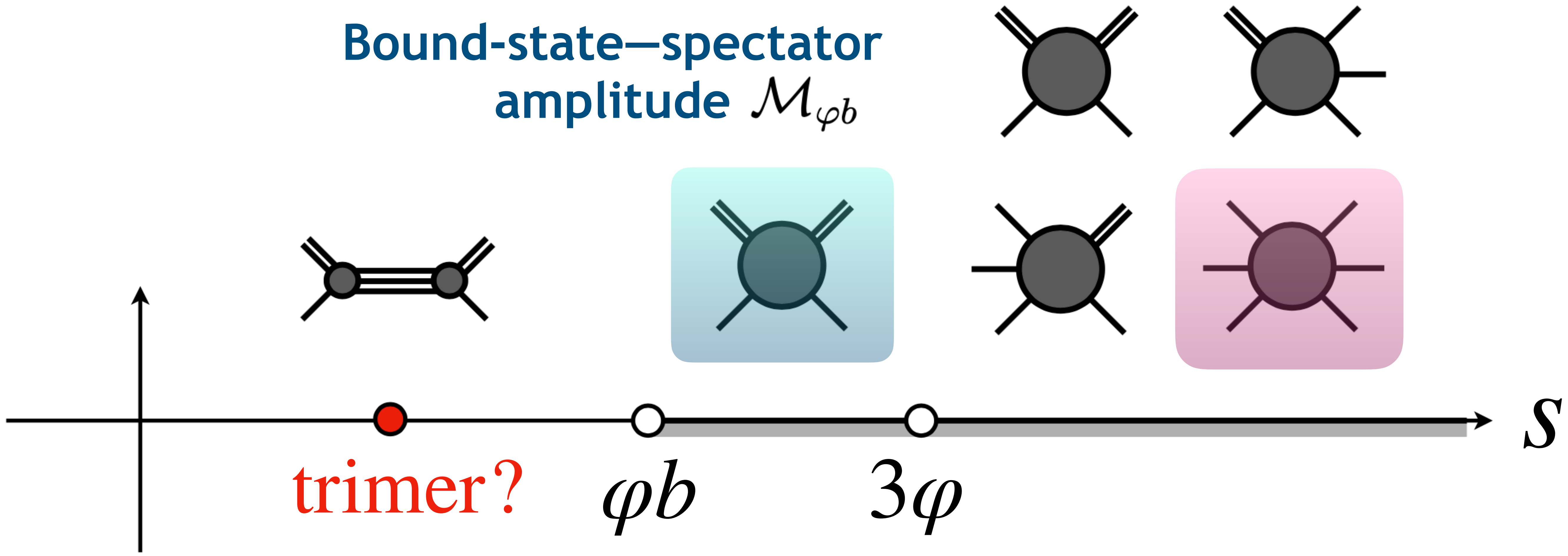
$$\mathcal{M}_2^{-1} \sim -\frac{1}{a} - i\rho_2$$

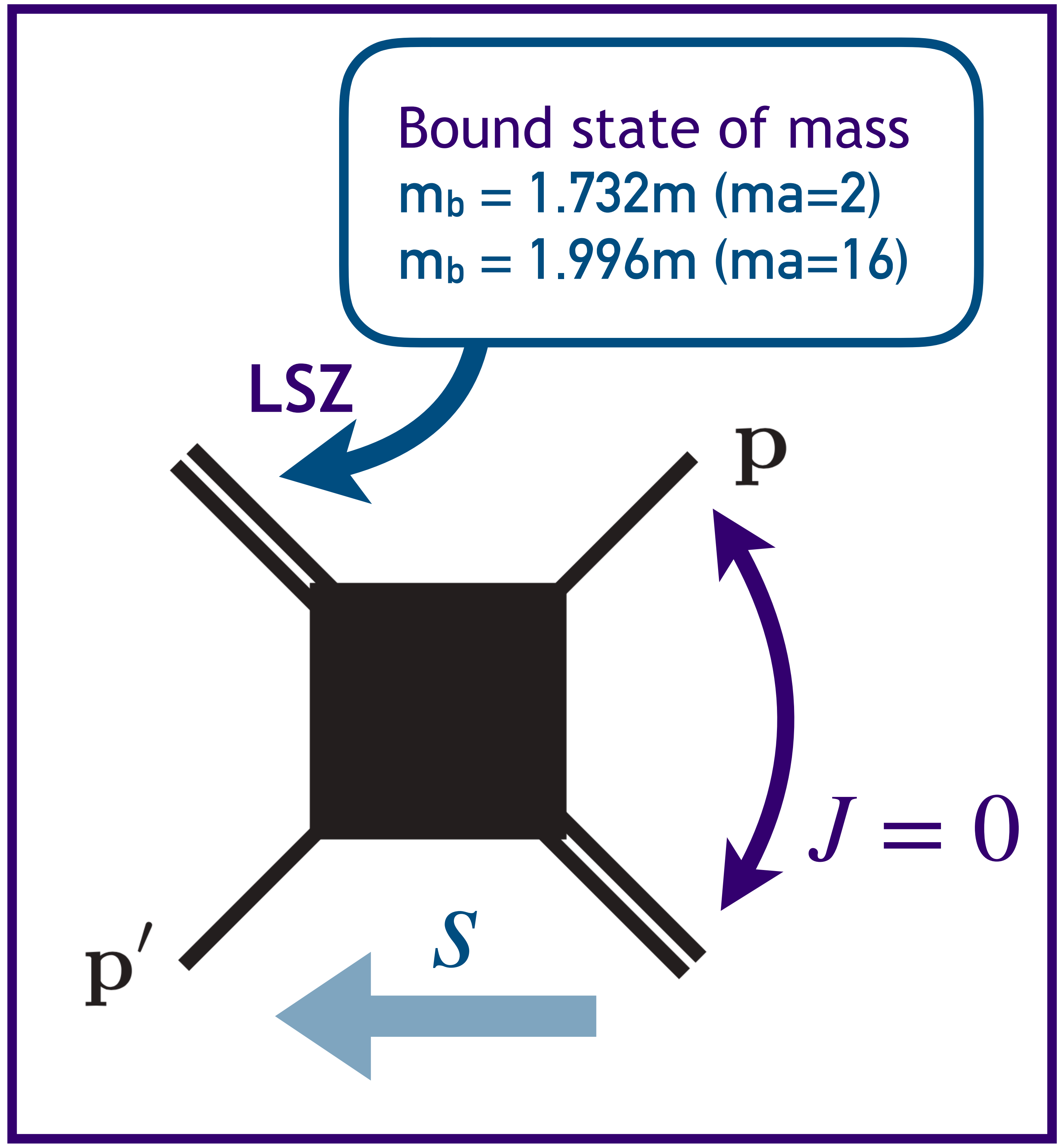
Ladder amplitude $\widetilde{\mathcal{M}}_3 = d(p', s, p)$



Three-body amplitude

Bound-state—spectator amplitude $\mathcal{M}_{\varphi b}$

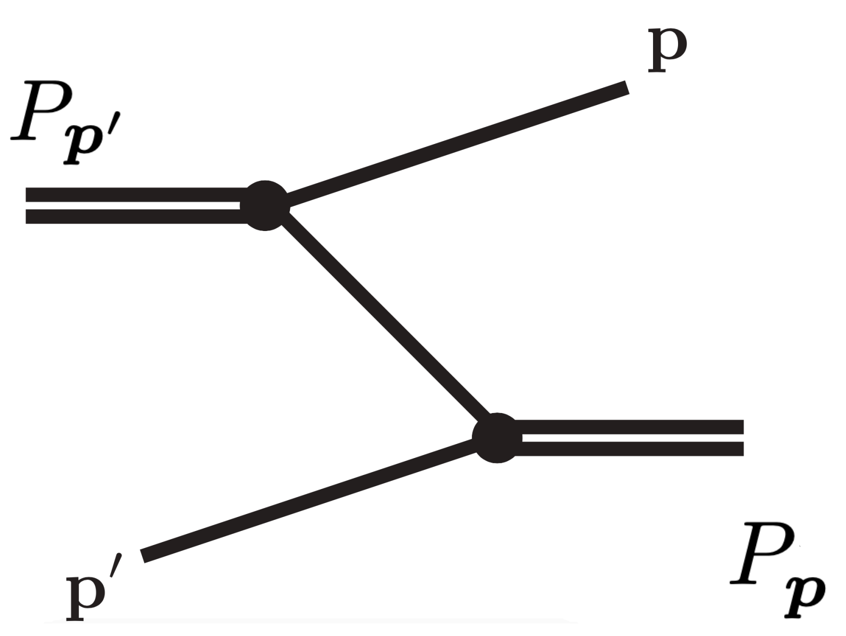




The Born term

Analytic continuation of the relativistic three-body amplitudes
Dawid, Islam, Briceño, arXiv:2303.04394

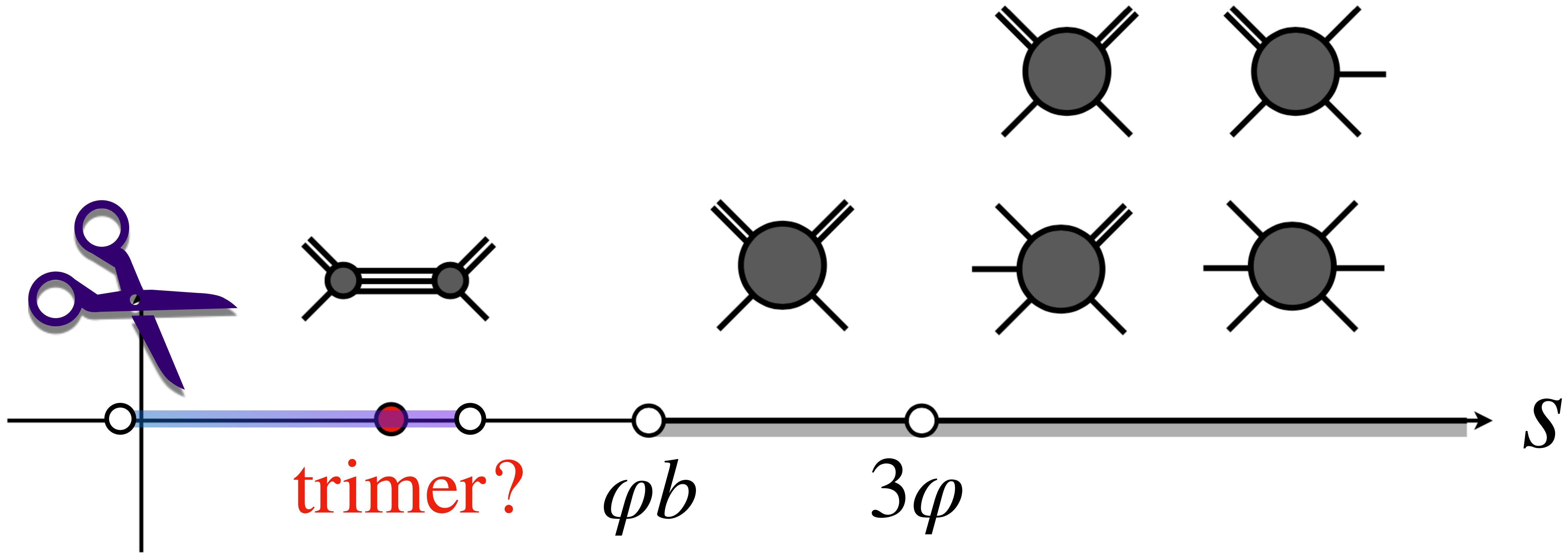
One-particle exchange propagator

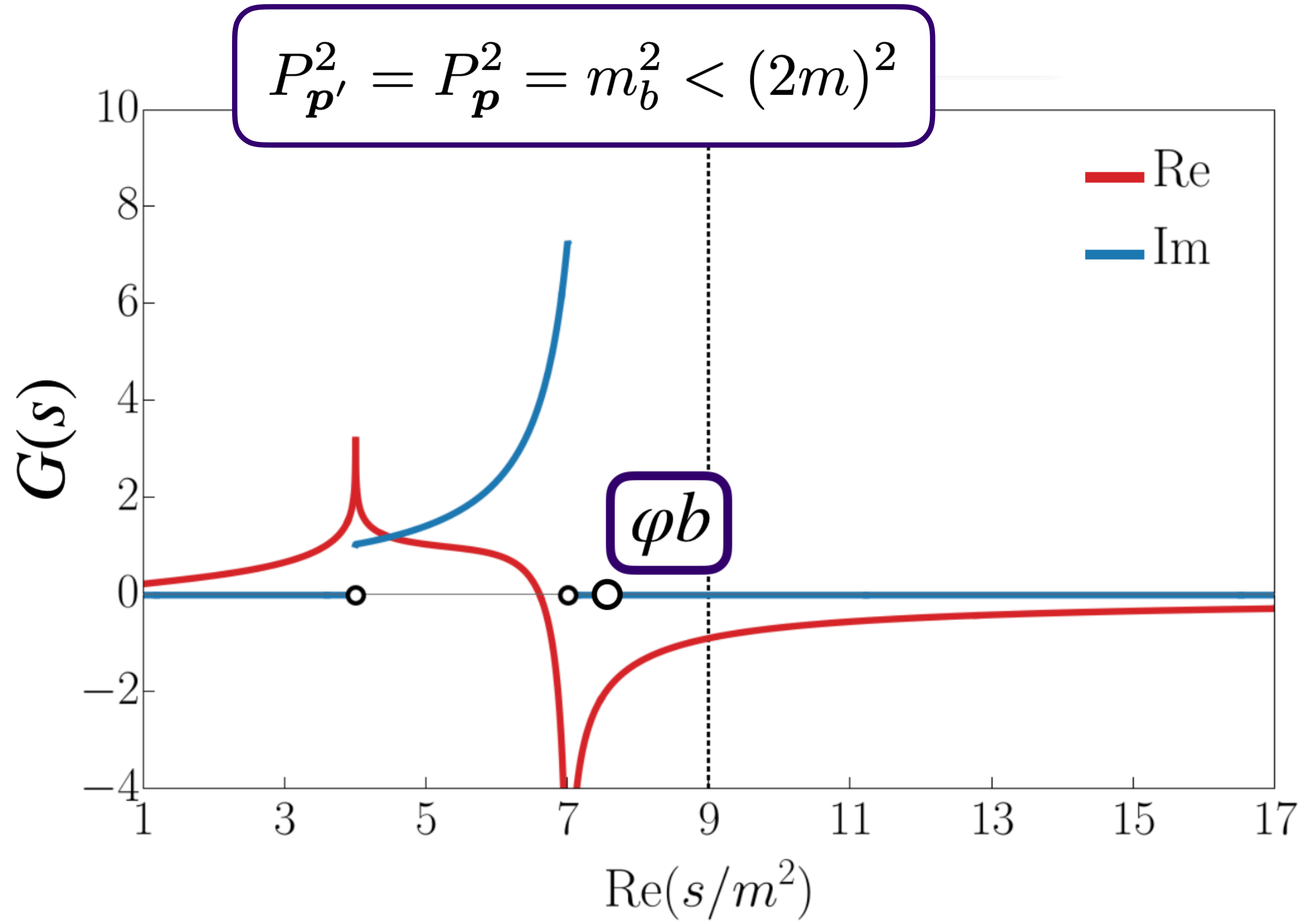


$$\propto \frac{1}{(P_{\mathbf{p}'} - p)^2 - m^2 + i\epsilon}$$

Partial-wave projection

$$G(p', s, p) \propto \log\left(\frac{1 + z(p', s, p)}{1 - z(p', s, p)}\right)$$





Analytic continuation of the integral equation

Singularities of the ladder equation

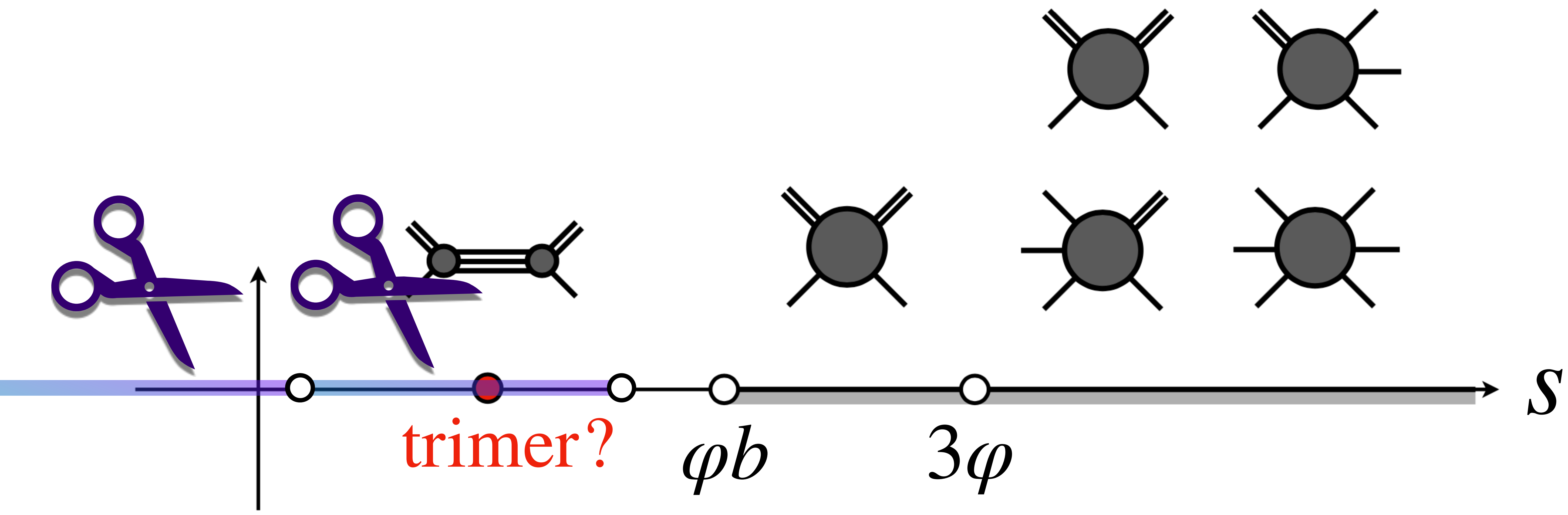
Inhomogeneous term

Homogeneous term

$$d(p', s, p) = -G(p', s, p) - \int_0^{q_{\max}} \frac{dq\, q^2}{(2\pi)^2 \omega_q} G(p', s, q) \mathcal{M}_2(q, s)\, d(q, s, p)$$

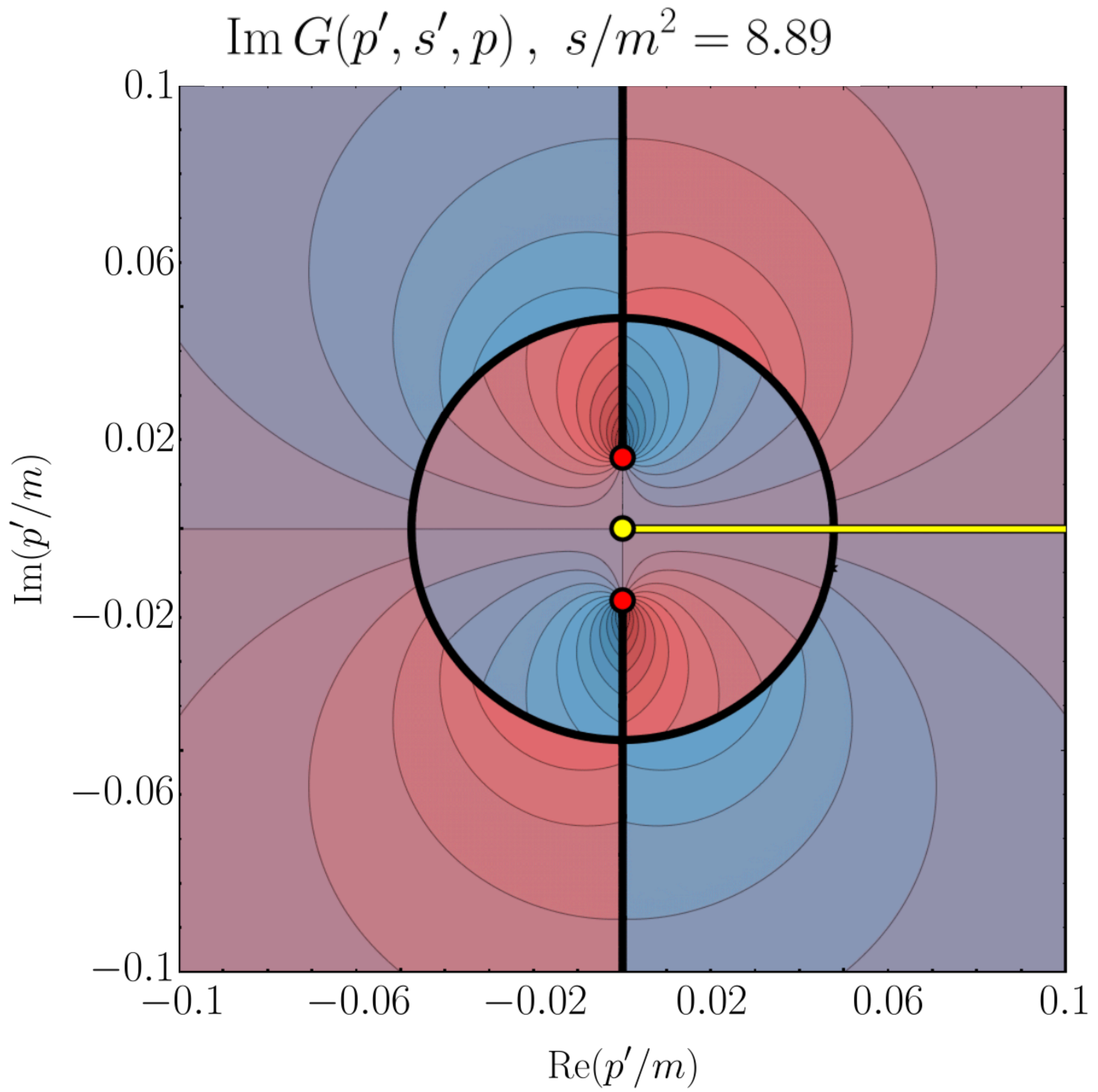
Integration kernel

Solution



In a nutshell

- Avoid crossing the singularities in the integration
- Deform the contour, add discontinuity, deform, …



Analytic continuation of the integral equation

In a nutshell

- Avoid crossing the singularities in the integration
- Deform the co[...] discontinuity,

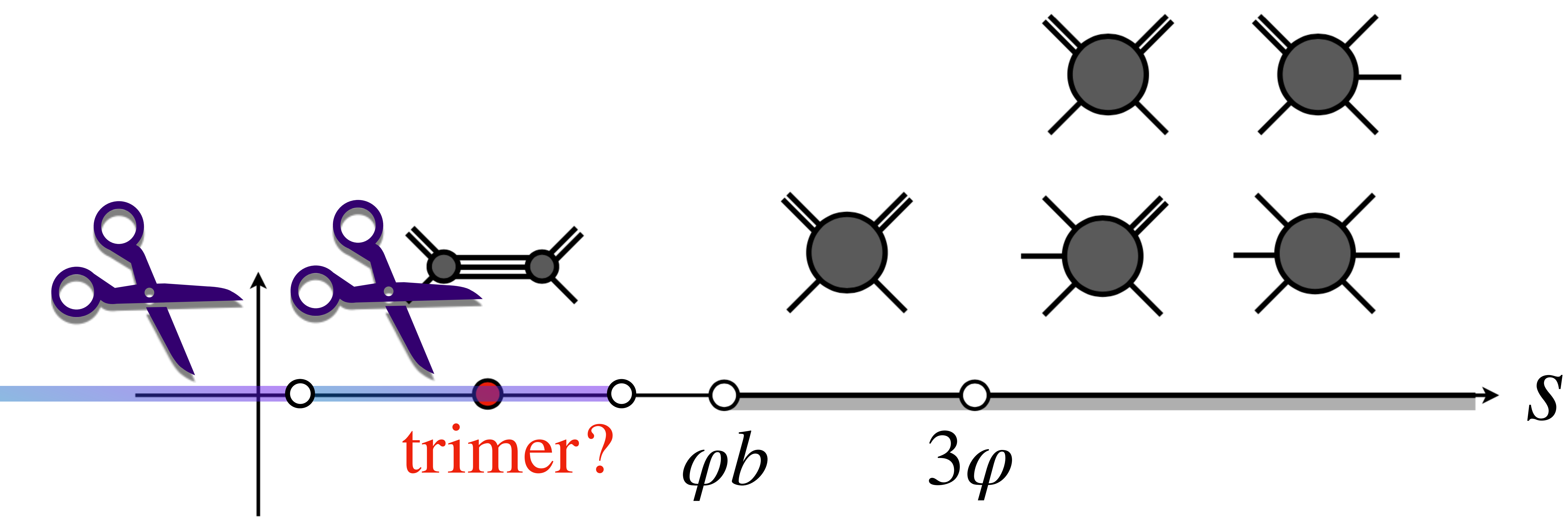
Negative Im(s)

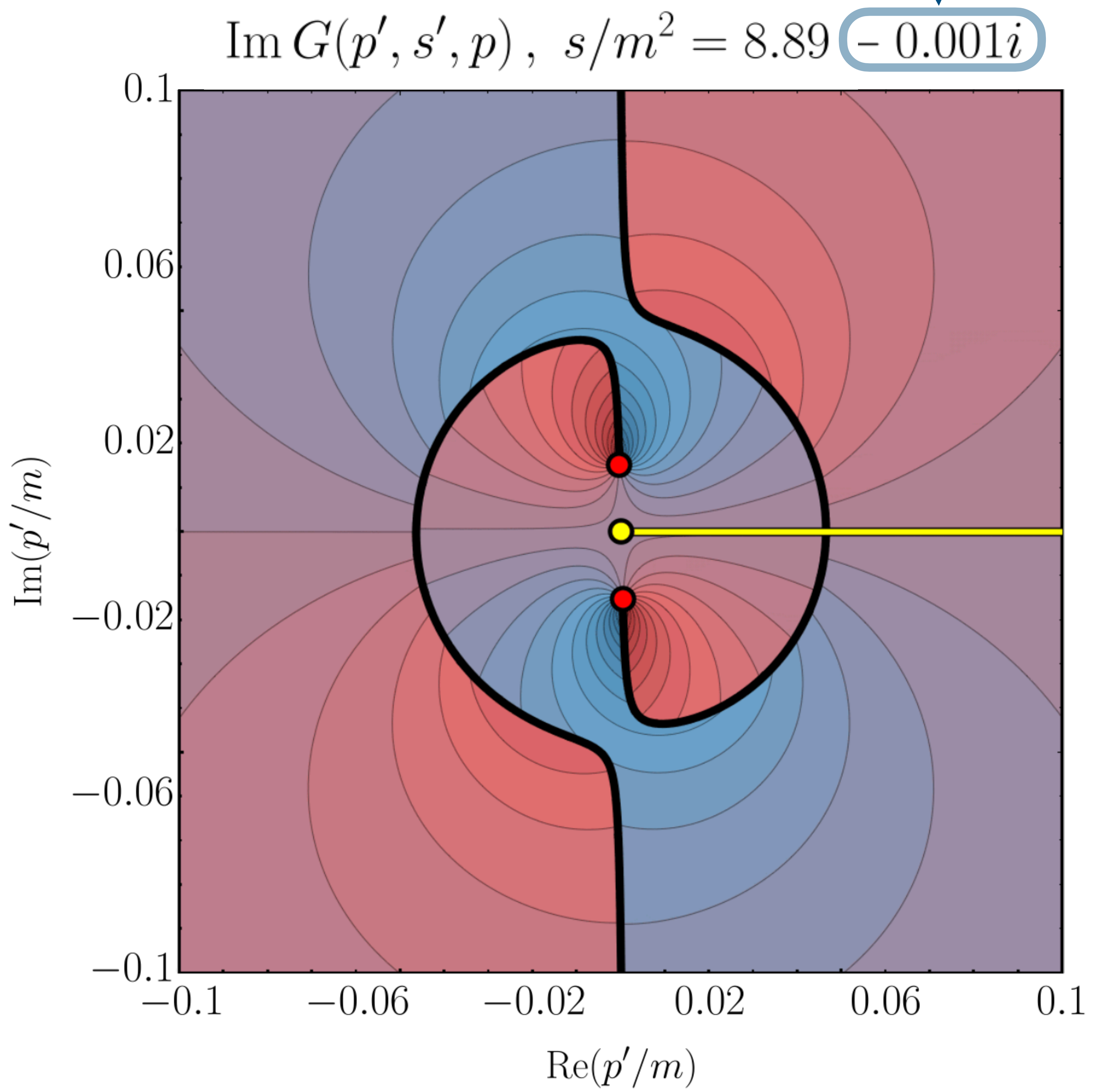
Singularities of the ladder equation

Inhomogeneous term — Homogeneous term

$$d(p',s,p) = -G(p',s,p) - \int_0^{q_{\max}} \frac{dq\, q^2}{(2\pi)^2\omega_q} G(p',s,q)\mathcal{M}_2(q,s)\, d(q,s,p)$$

Integration kernel — Solution





Self-consistency of the deformed contour

Singularities of the ladder equation

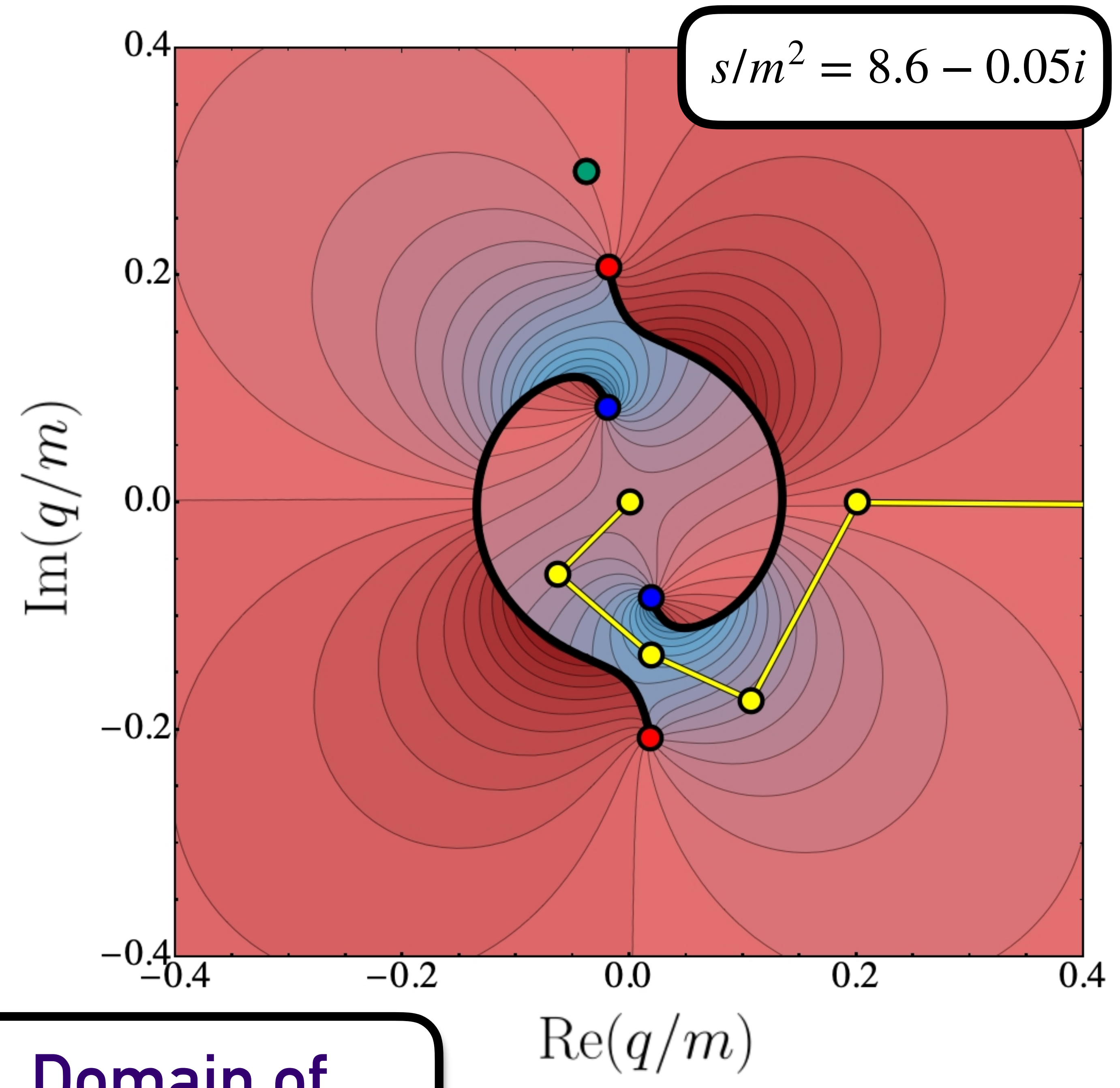
Deformed contour

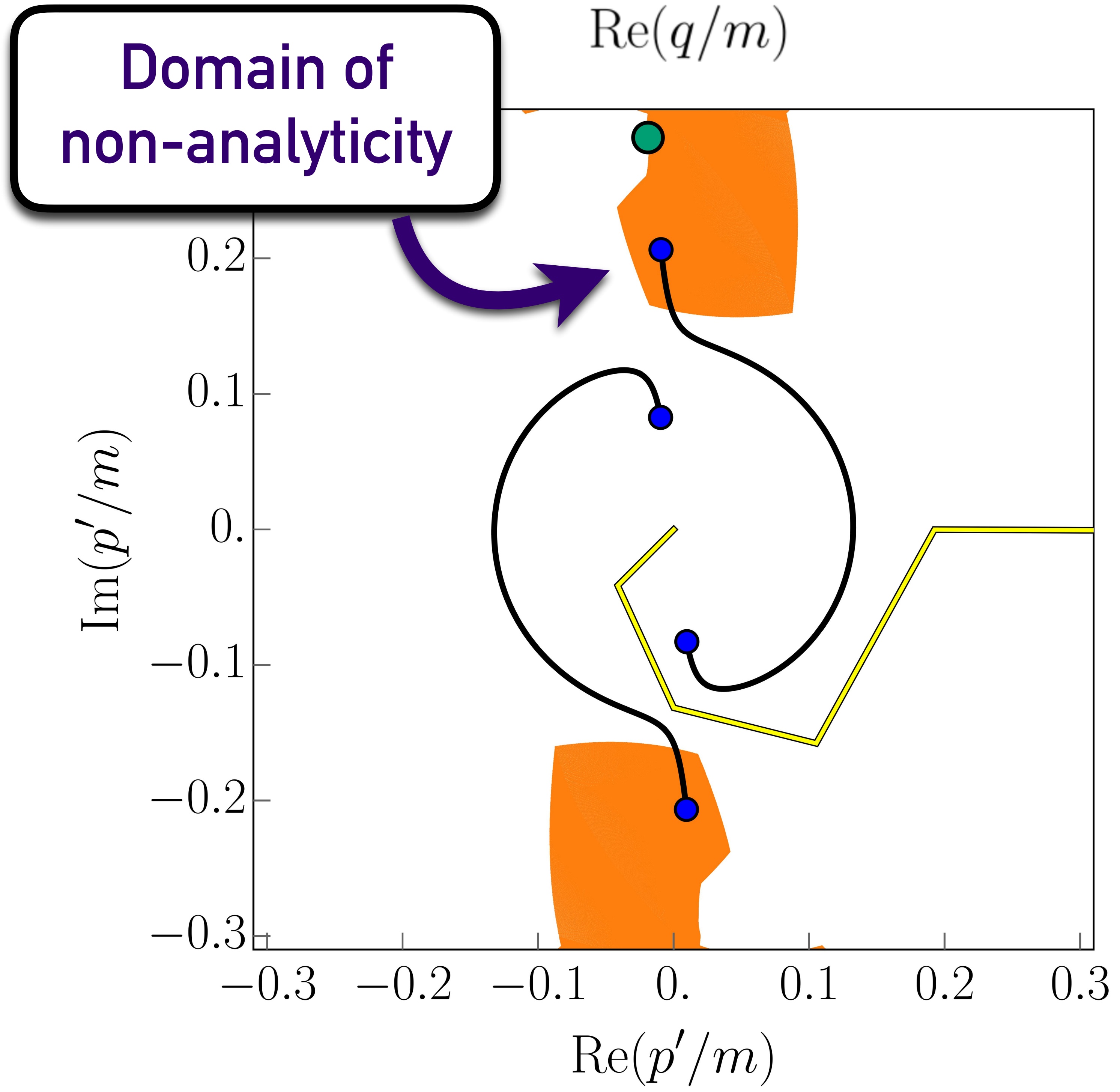
$$d(p', s, p) = -G(p', s, p) - \int_0^{q_{\max}} \frac{dq\, q^2}{(2\pi)^2 \omega_q} G(p', s, q) \mathcal{M}_2(q, s)\, d(q, s, p)$$

Integration kernel

Extrapolation to the desired momentum $p' \longrightarrow q_{\varphi b}$

Addition of discontinuity to the integration kernel $\Delta(p', s, p) \propto 2\pi i$

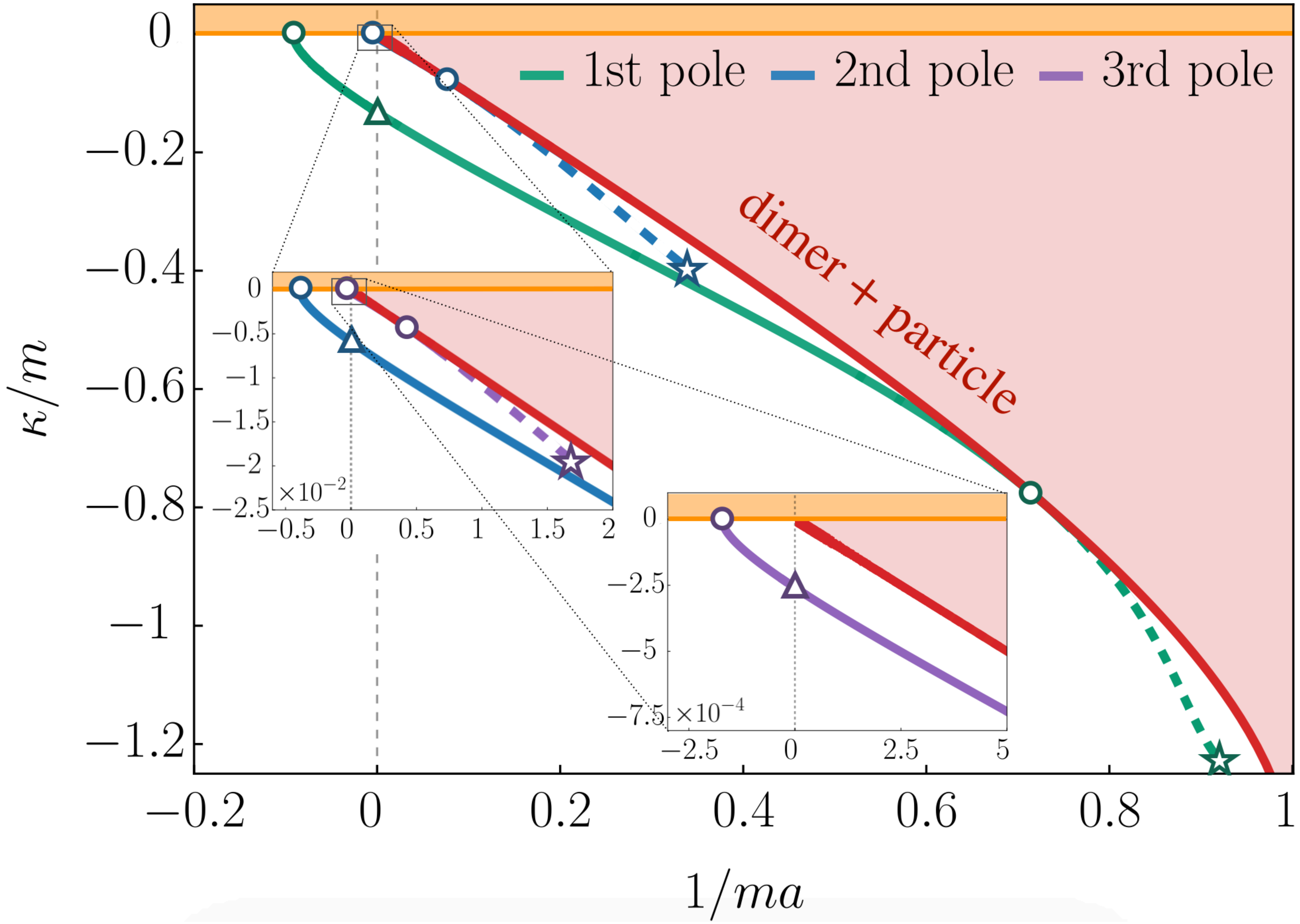




Efimov phenomenon in relativistic scattering

Evolution of Efimov states in relativistic scattering theory
Dawid, Islam, Briceño, Jackura, in preparation

Trajectories of the first three trimers: binding momentum $\kappa \propto \sqrt{E}$ vs $1/ma$



At unitarity, ratios of binding energies approach Efimov's constant λ^2=515

- Emergence of virtual state
- Two- and three-body threshold
- Value at unitarity

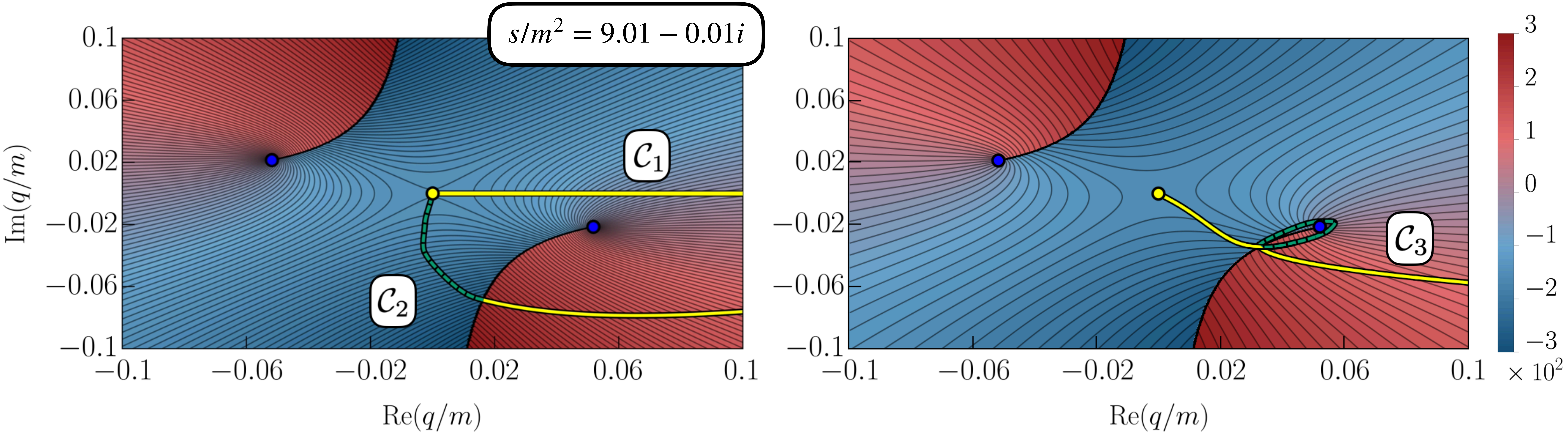
Beyond the three-body cut

Evolution of Efimov states in relativistic scattering theory
Dawid, Islam, Briceño, Jackura, in preparation

Collision of the integration contour with:

- pole of the pair amplitude ⟶ dimer-particle cut
- unitarity branch cut of the pair ⟶ three-body cut

$$- \int_0^{q_{\max}} \frac{dq\, q^2}{(2\pi)^2 \omega_q} G(p', s, q) \mathcal{M}_2(q, s)\, d(q, s, p)$$



Accessing the 2nd Riemann sheet

Accessing the 3rd Riemann sheet

Three-body resonances (2nd sheet)

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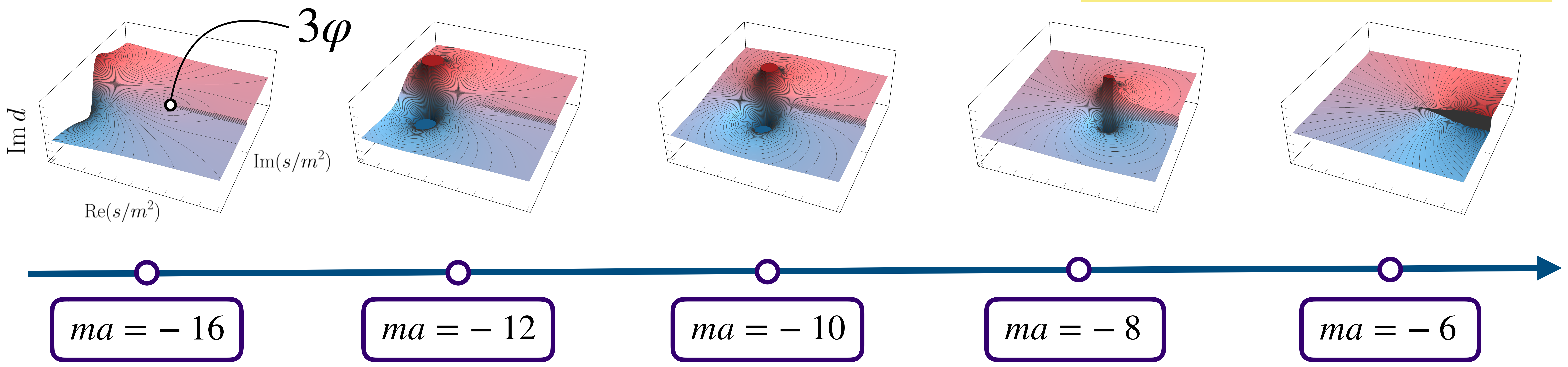

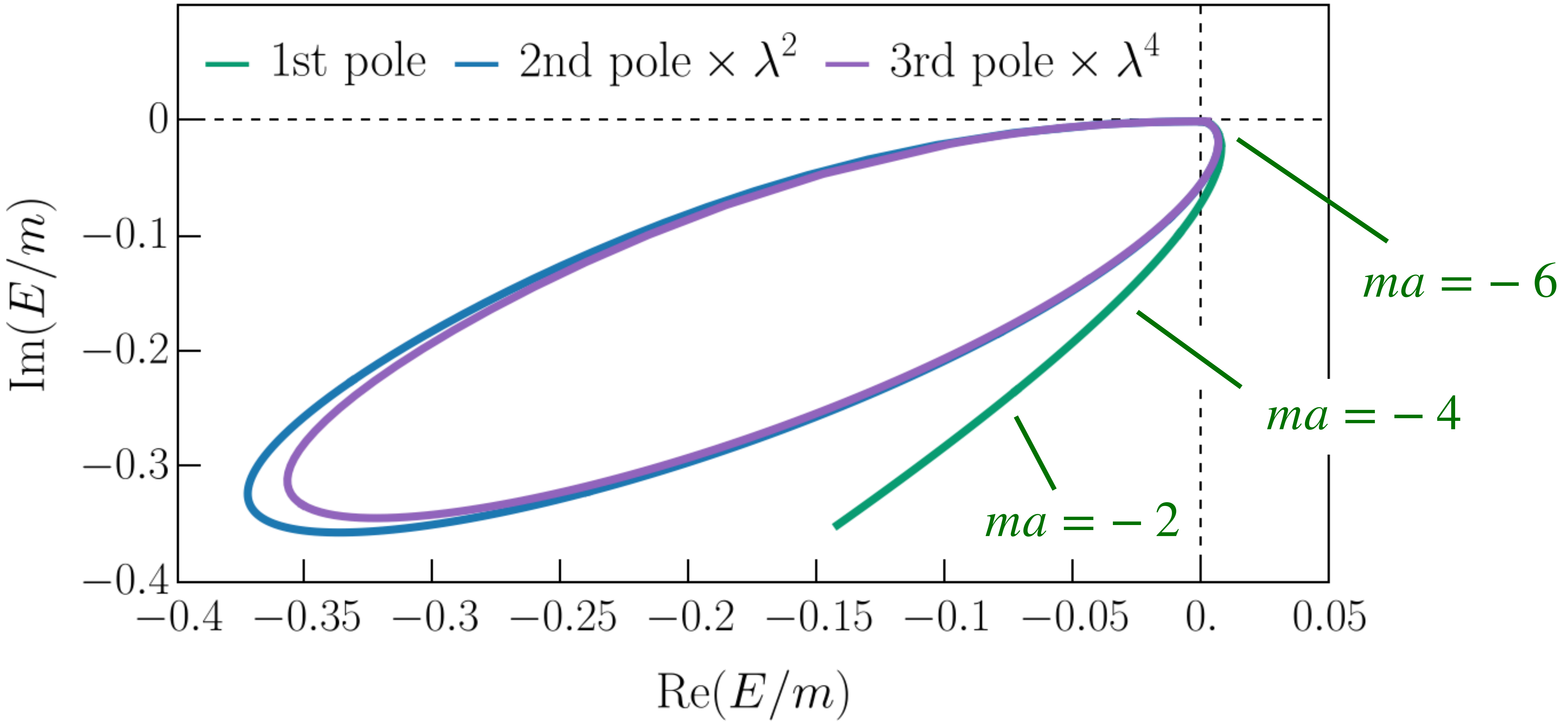

Poles travel on loop-like trajectories and "disappear" (Missing poles problem)

Three-body resonances (2nd sheet)

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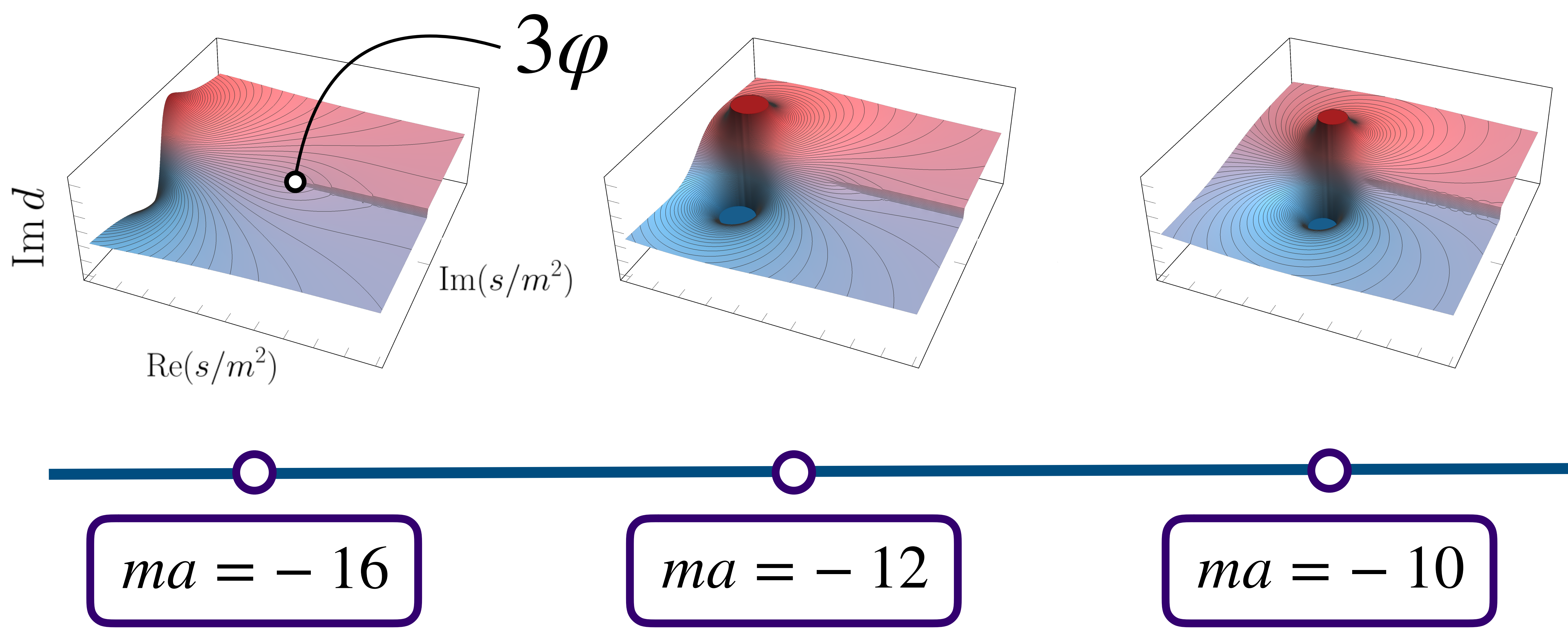
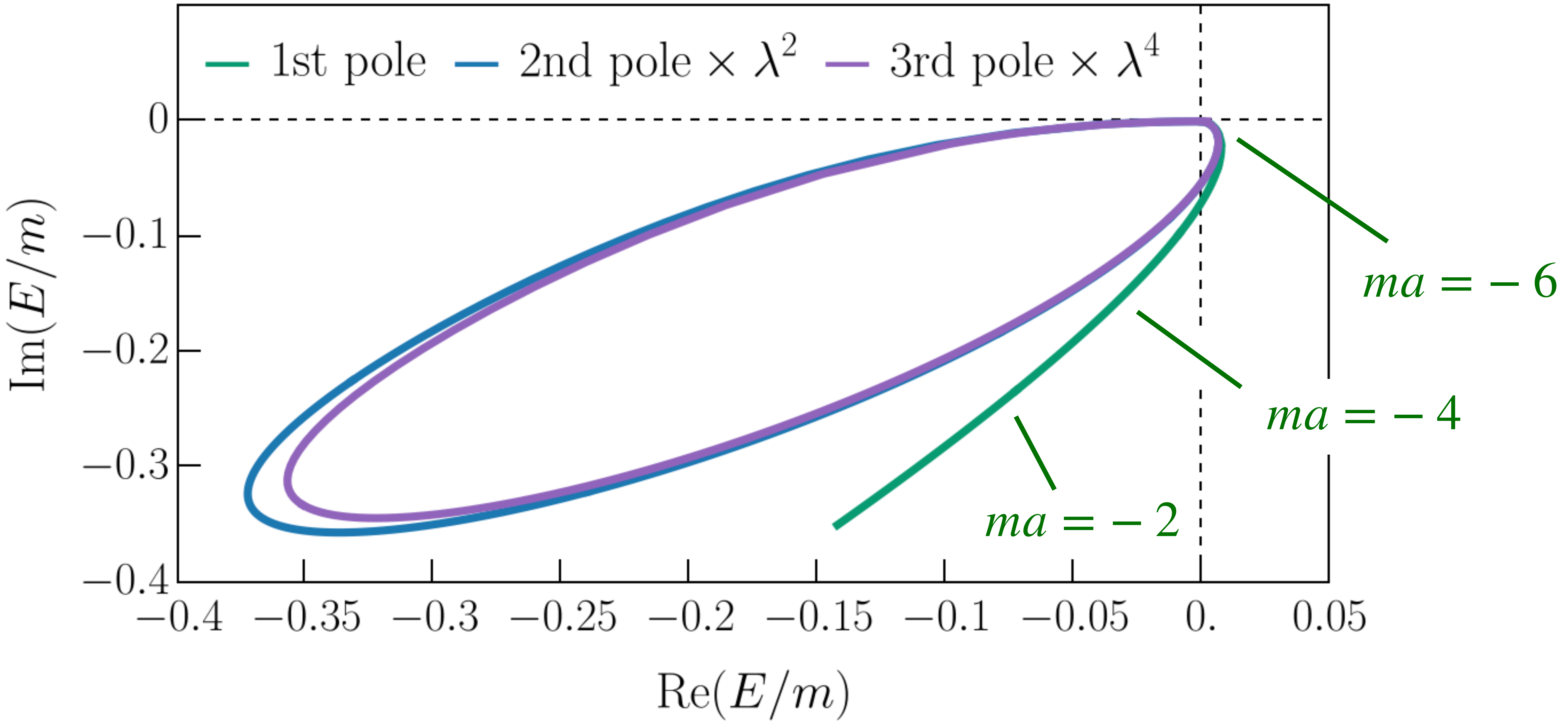
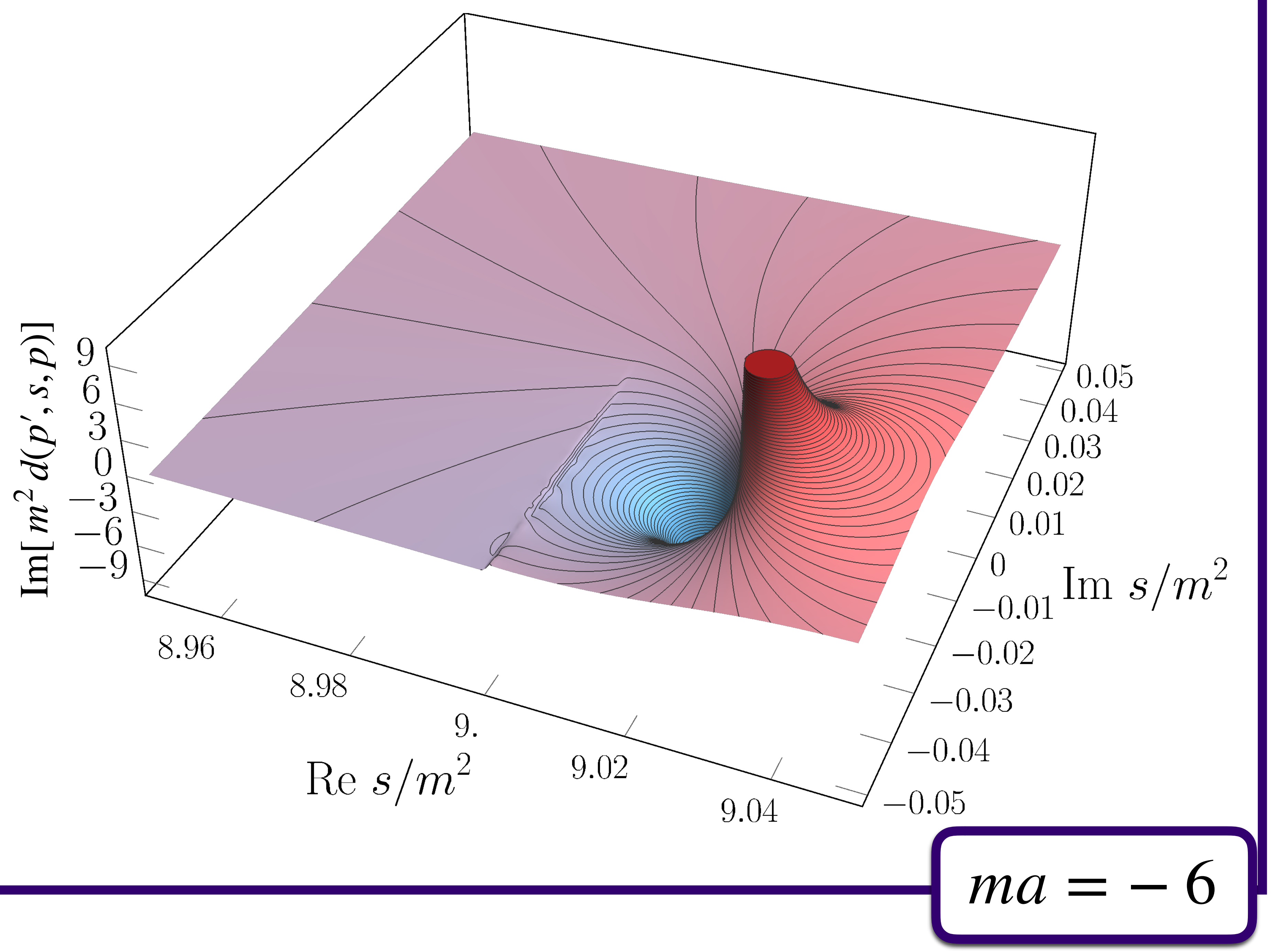




Continuation through the three-body cut



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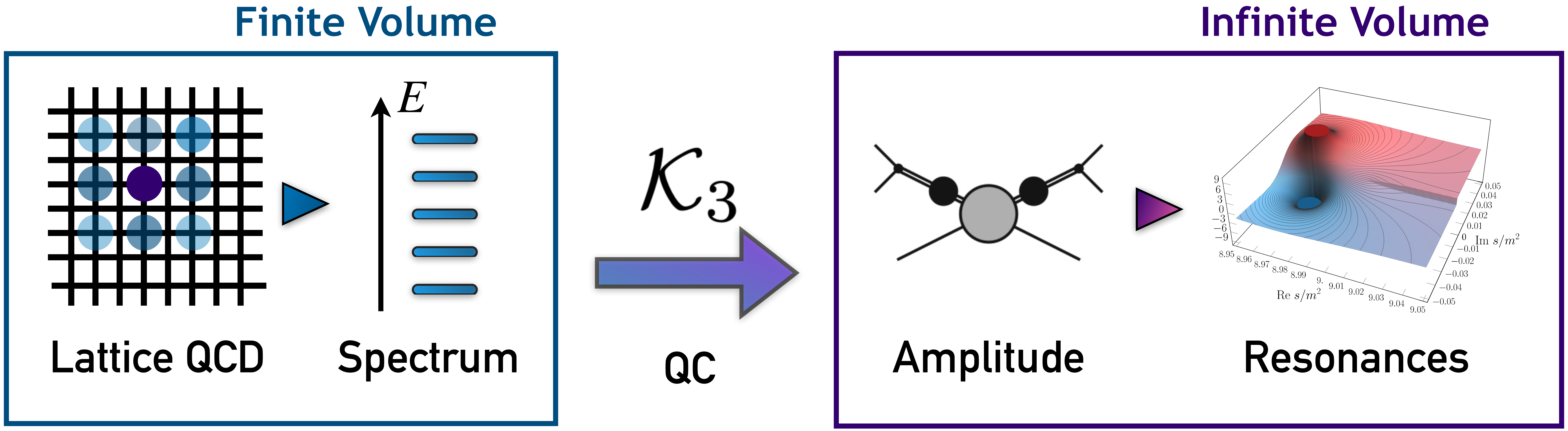
Summary



- Development of the formalism
  - ☑ Consistency check of the three-body framework
  - ☑ Breakdown of the Lüscher QC
  - ☑ Amplitude in the complex plane
  - ☑ Trajectories of Efimov trimers

- Future work
  - ☐ Higher partial waves
  - ☐ Application to realistic states
  - ☐ Multi-body, multi-channel formalism

Talk by M. Islam

Talk by S. Sharpe

THANK YOU

Personal choice of relevant literature

Three-body scattering and quantization conditions from S matrix unitarity
A. Jackura, arXiv:2208.10587 (2022)

Off- and On-Shell Analyticity of Three-Particle Scattering amplitudes
D. Brayshaw, Phys. Rev. 176, 5 (1968)

S-matrix pole trajectory in the three-neutron model
W. Gloeckle, Phys. Rev. C 18, 1 (1978)

S-matrix pole trajectory of the three-body system
A. Matsuyama, K. Yazaki, Nucl. Phys. A 534, 620 (1991)

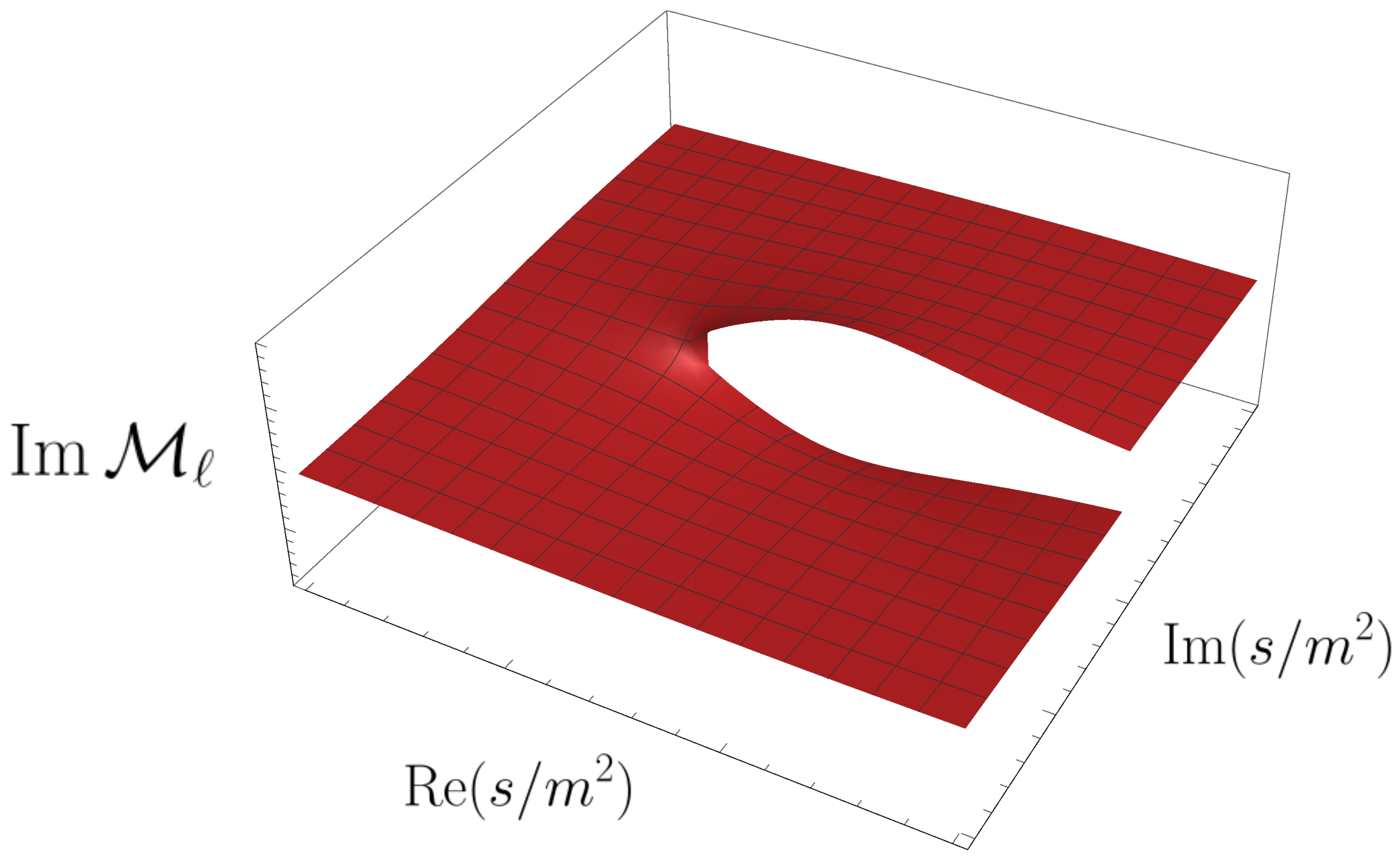
Fate of the Tetraquark Candidate $Z_c(3900)$ from Lattice QCD
Ikeda et al. (HAL QCD), Phys. Rev. Lett. 117 (2016) 24, 242001

The Three-Boson System with Short-Range Interactions
Bedaque, Hammer, van Kolck, Nucl. Phys. A 646 (1999) 444

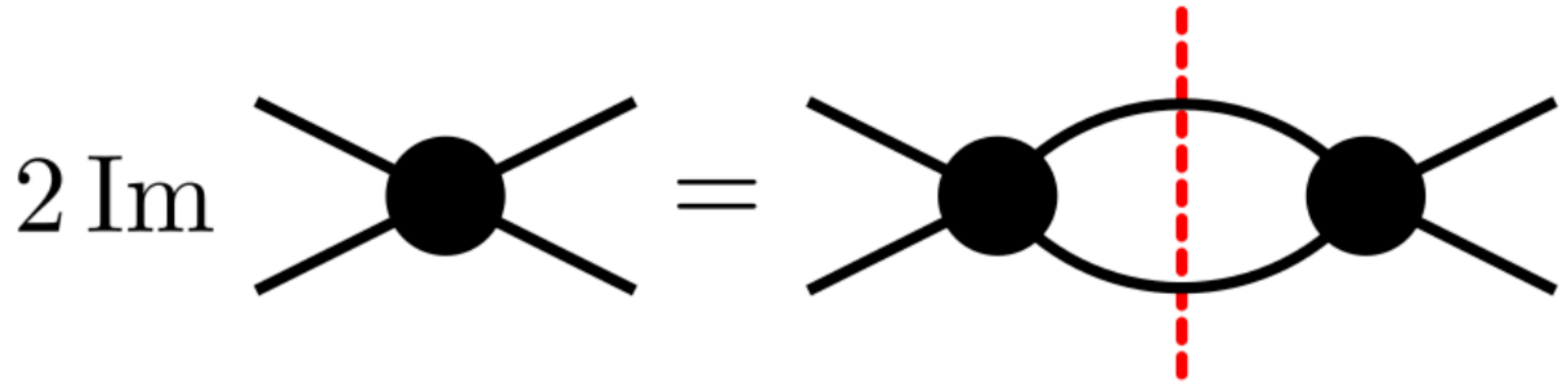
Energy-Dependent $\pi\pi\pi$ Scattering Amplitude from QCD
Hansen et al. (HadSpec), Phys. Rev. Lett. 126 (2021), 012001

Resonance, Ariel Davis for Quanta Magazine

Basics of scattering theory



- Analyticity on the first Riemann sheet
- Bound-states & resonances correspond to poles
- Branch cuts correspond to open channels



Properties of the S matrix

- Analyticity (causality)
- Unitarity (probability conservation)
- Poincaré symmetry (frame independence)
- Crossing symmetry (particles—antiparticles)
- Internal symmetries (charge, isospin, G-parity)

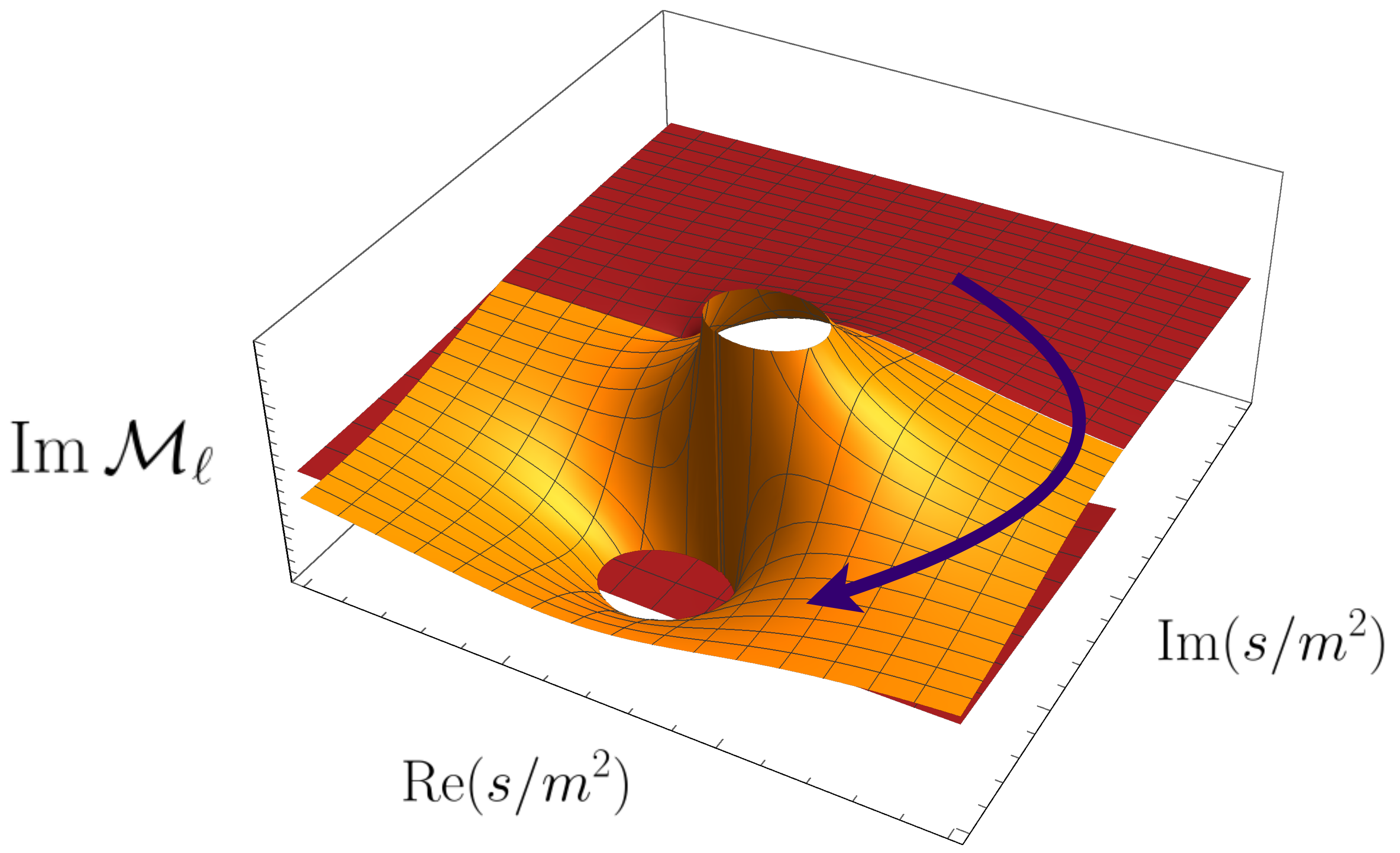
K-matrix parametrization

$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell^{-1}(s) - i\rho(s)}$$

Phase shift

$$\mathcal{K}_\ell^{-1}(s) = \frac{q^\star}{8\pi\sqrt{s}}\cot(\delta_\ell(s))$$

Basics of scattering theory



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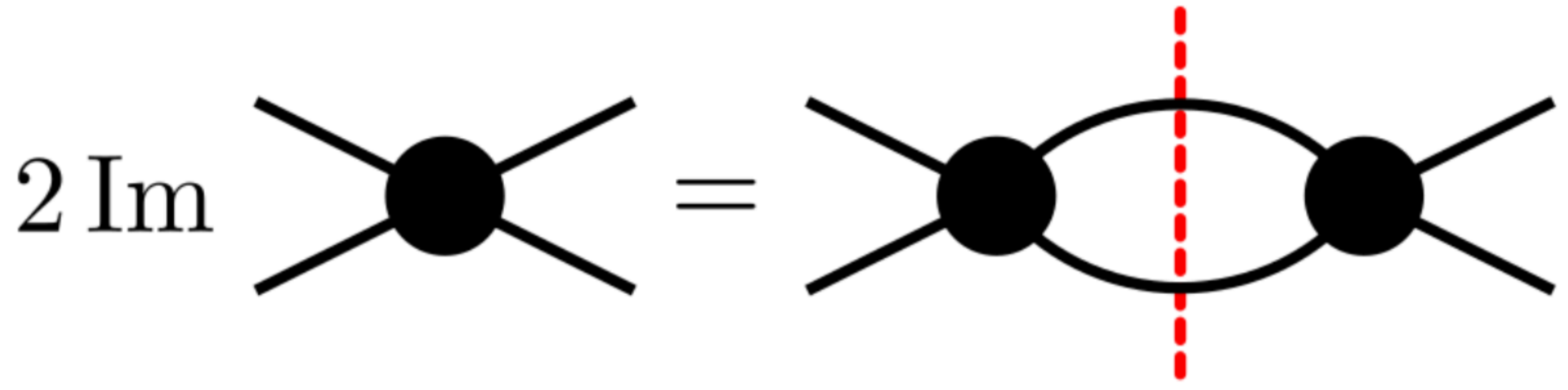

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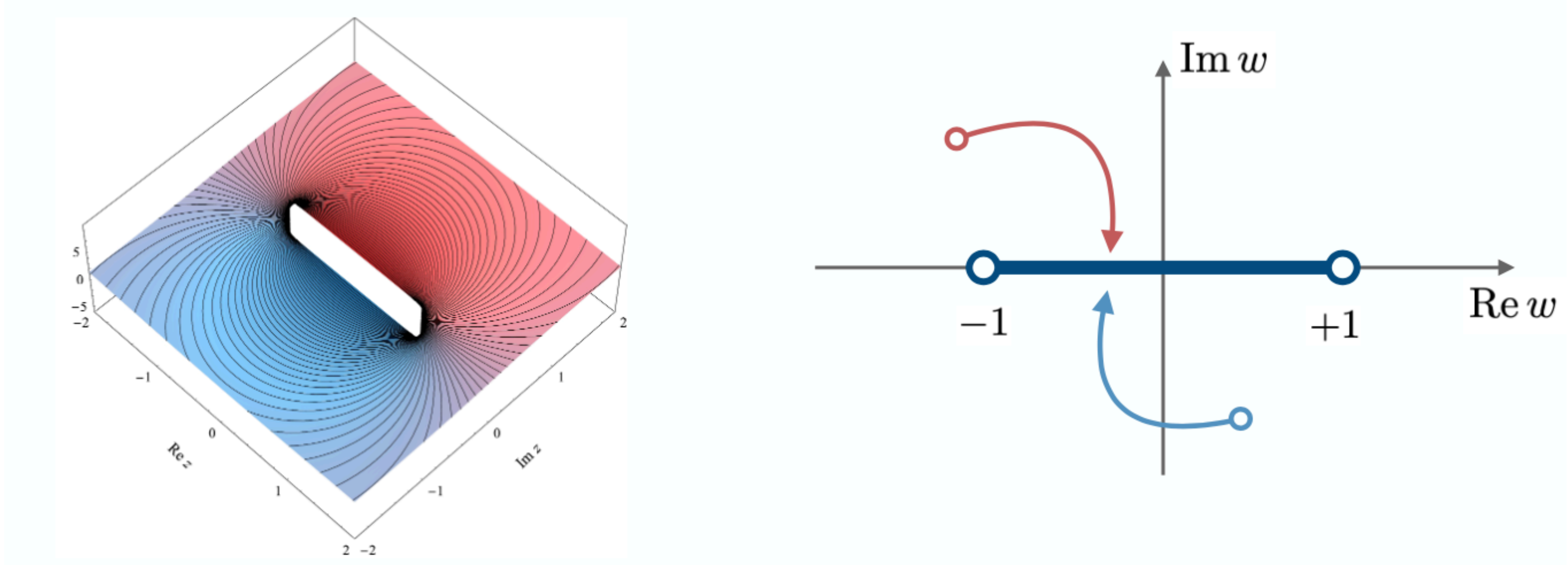
Analytic continuation of the relativistic three-body amplitudes
Dawid, Islam, Briceño, arXiv:2303.04394

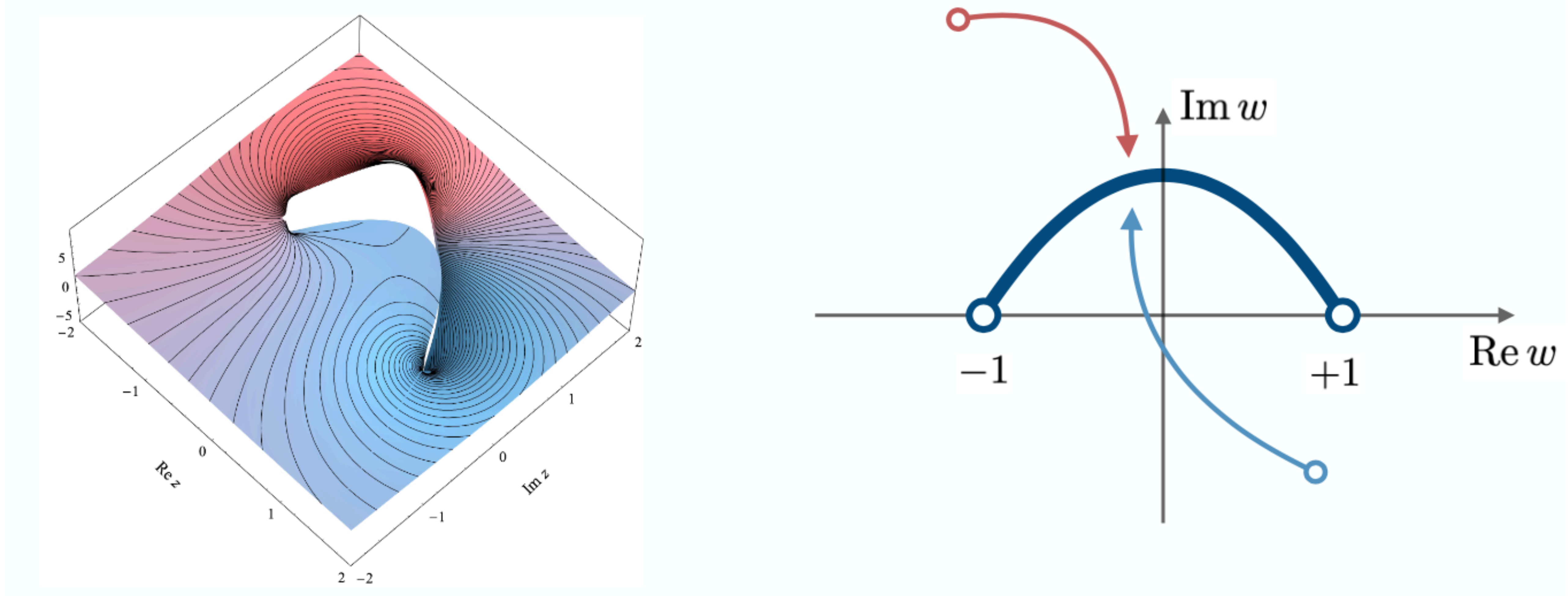
Brief introduction to analytic continuation

$$I(z) = \int_{\mathcal{C}(w_1, w_2)} f(w,z)\, dw$$



$$I(x) = \int_{-1}^{1} \frac{dw}{w-x} = \log\left(\frac{x-1}{x+1}\right)$$

$$x \in (-\infty, -1) \cup (1, \infty)$$



Unphysical left-hand cut

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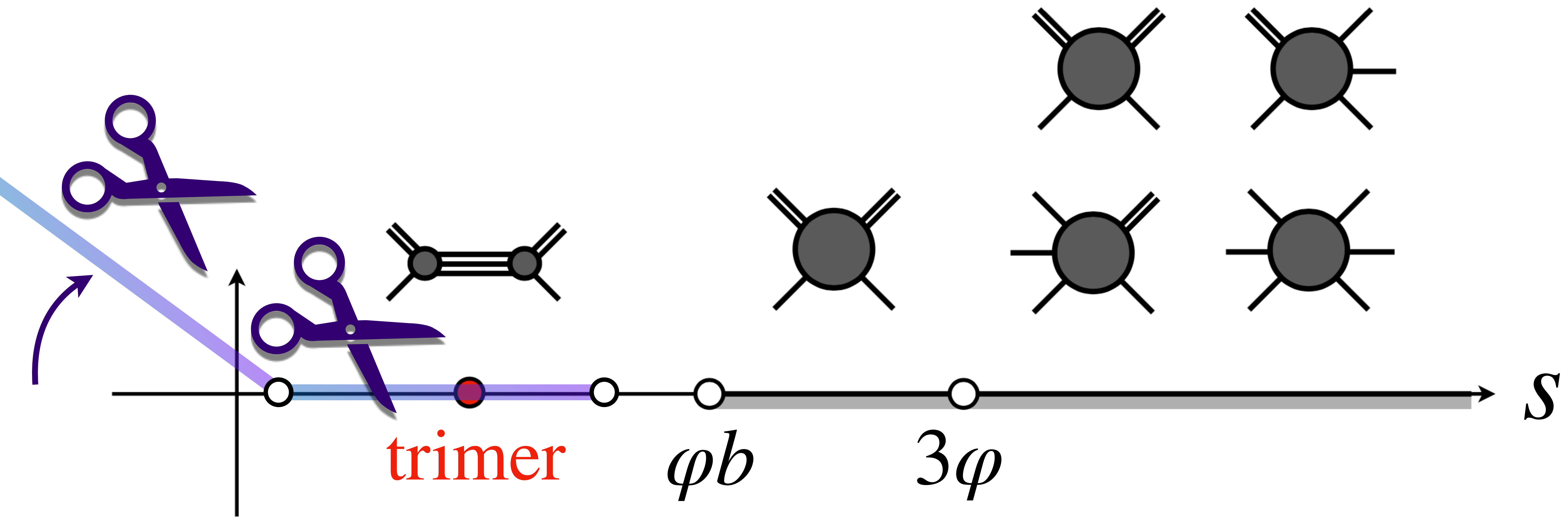
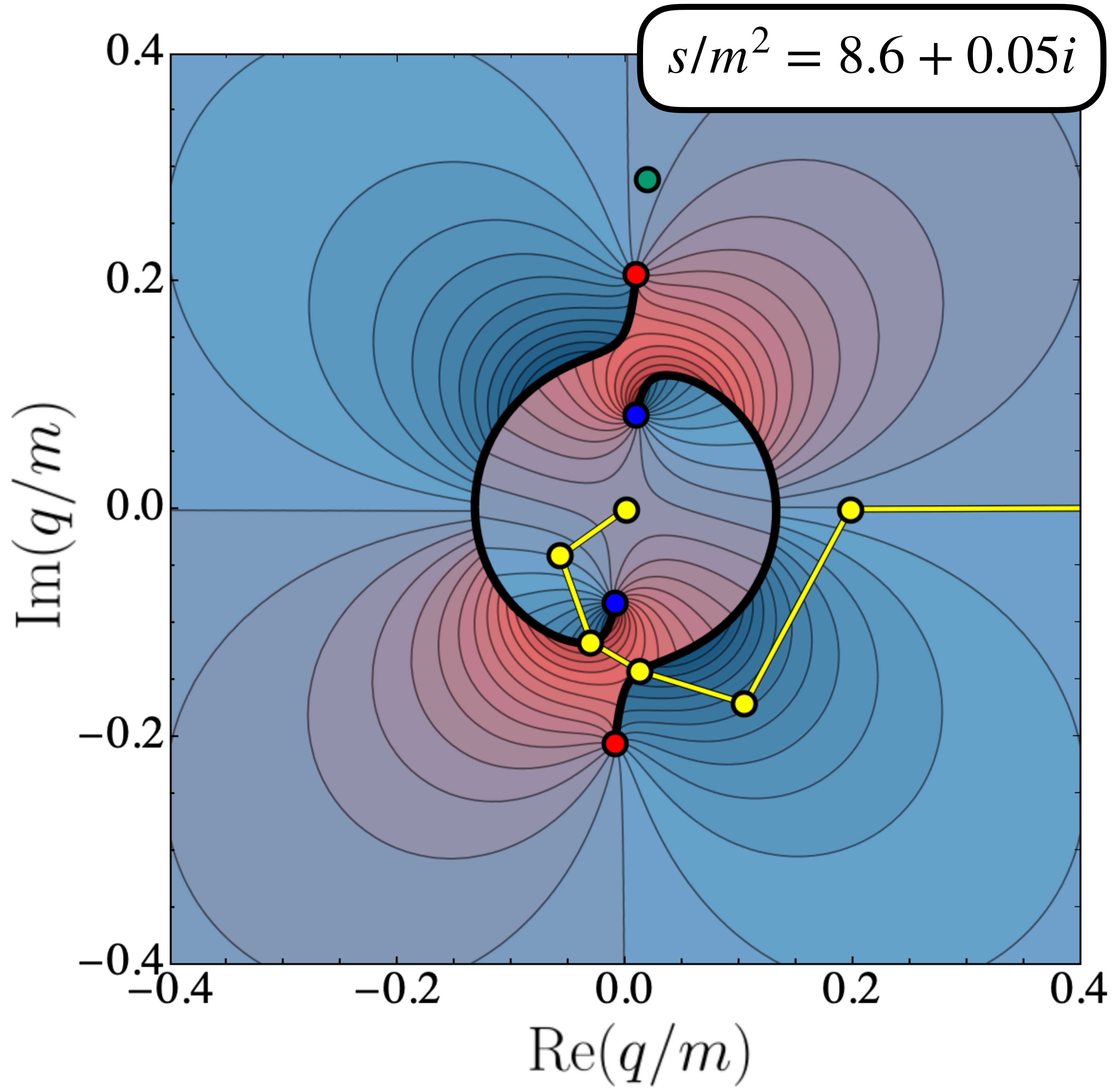
Dawid, Islam, Briceño, arXiv:2303.04394

- Complex conjugation of **s** reflects the cuts in **p'**
- We add discontinuity to the inhomogeneous part
- Check self-consistency of the contour





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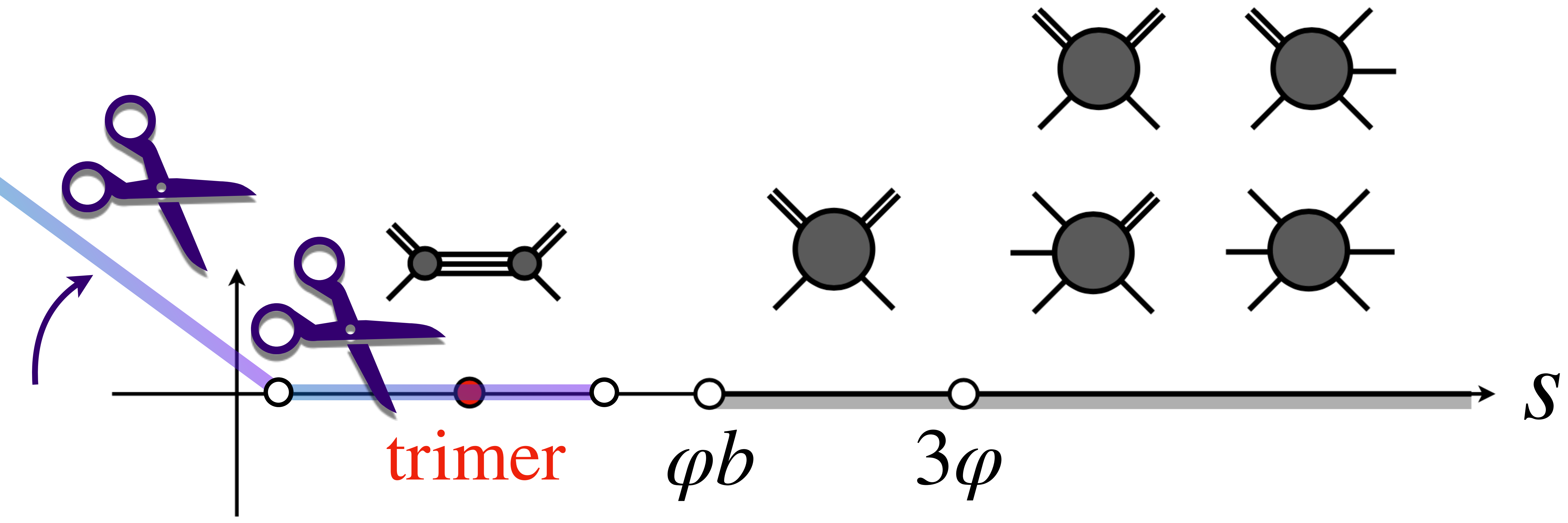
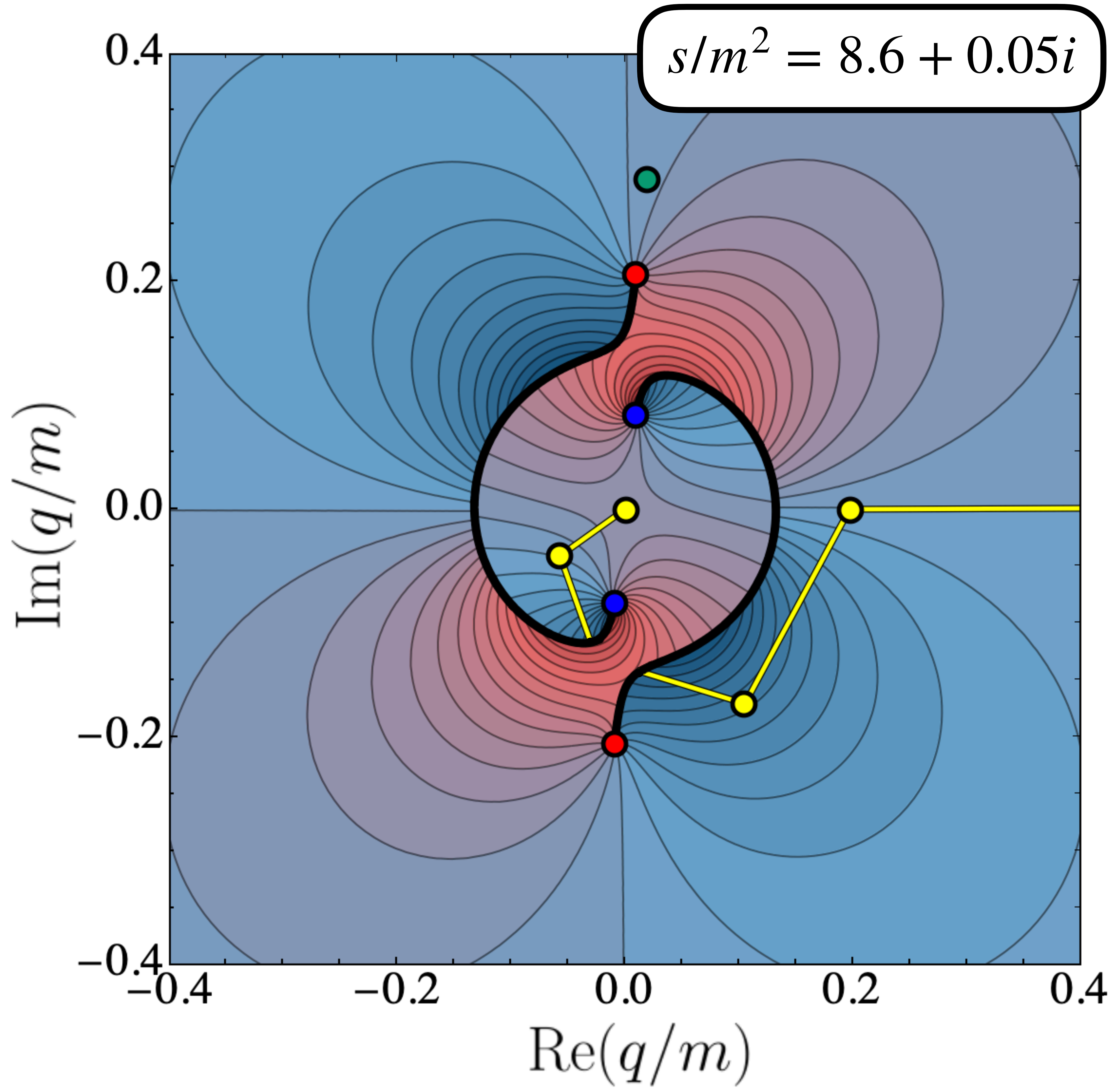
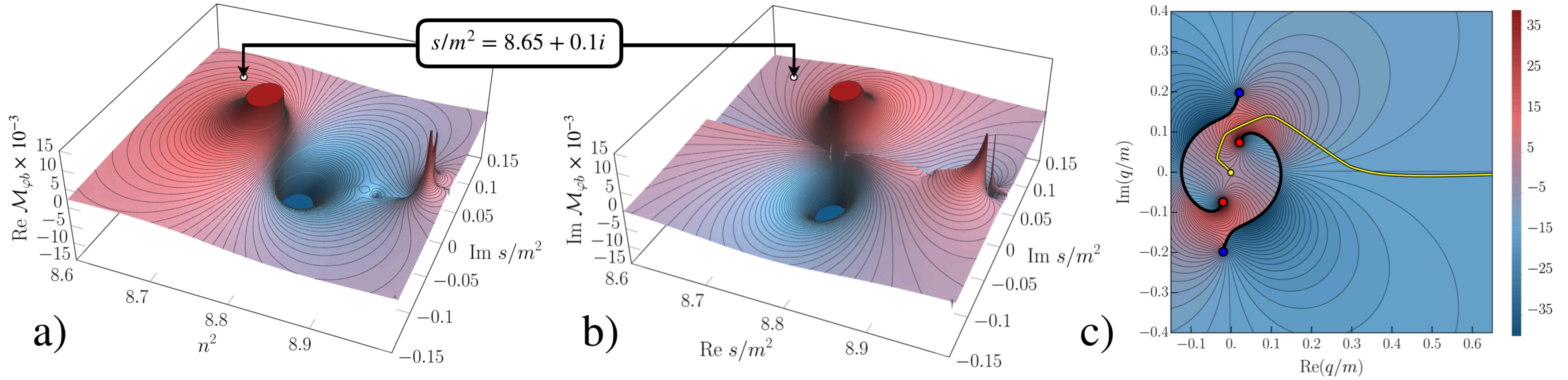
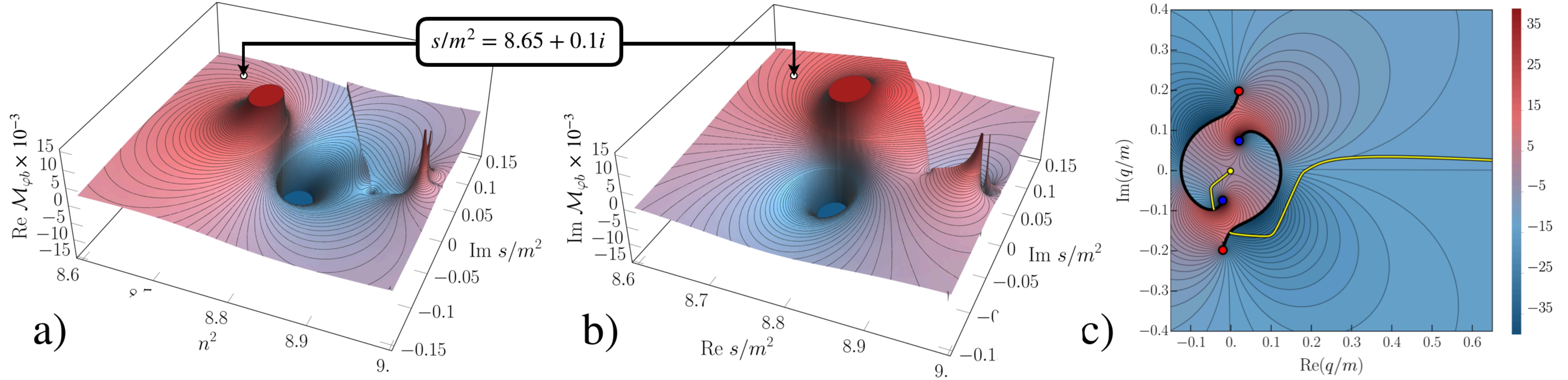


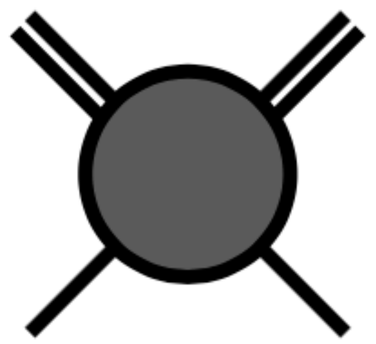


Rotating away the unphysical cut (ma=16)



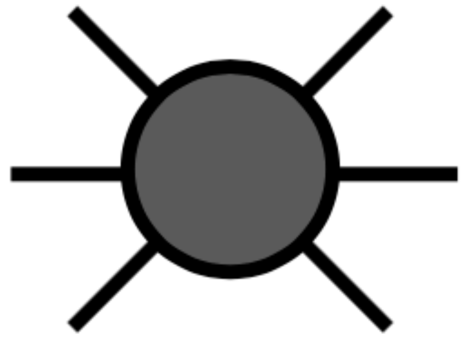


Complex-plane amplitude (ma=16)

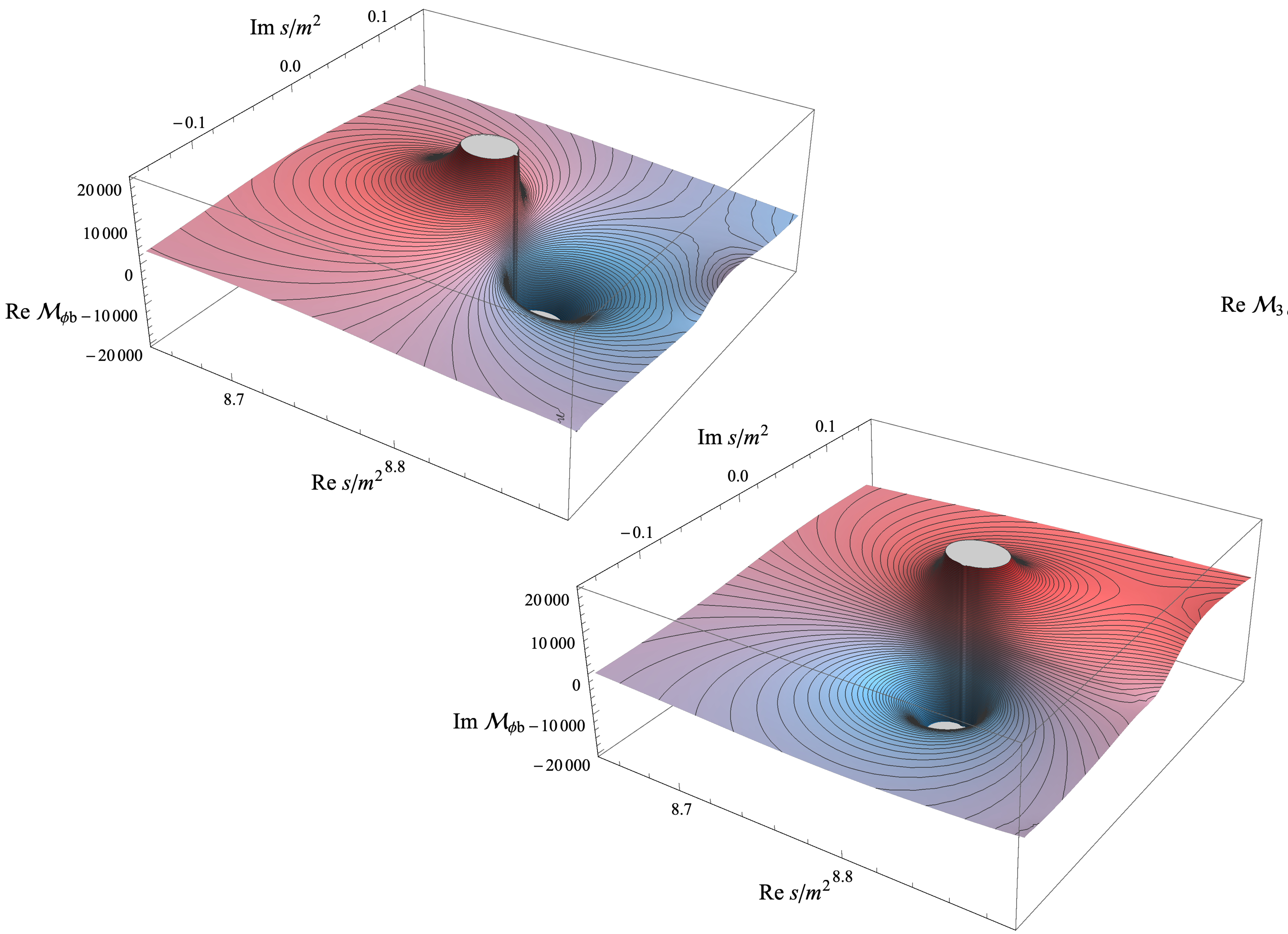


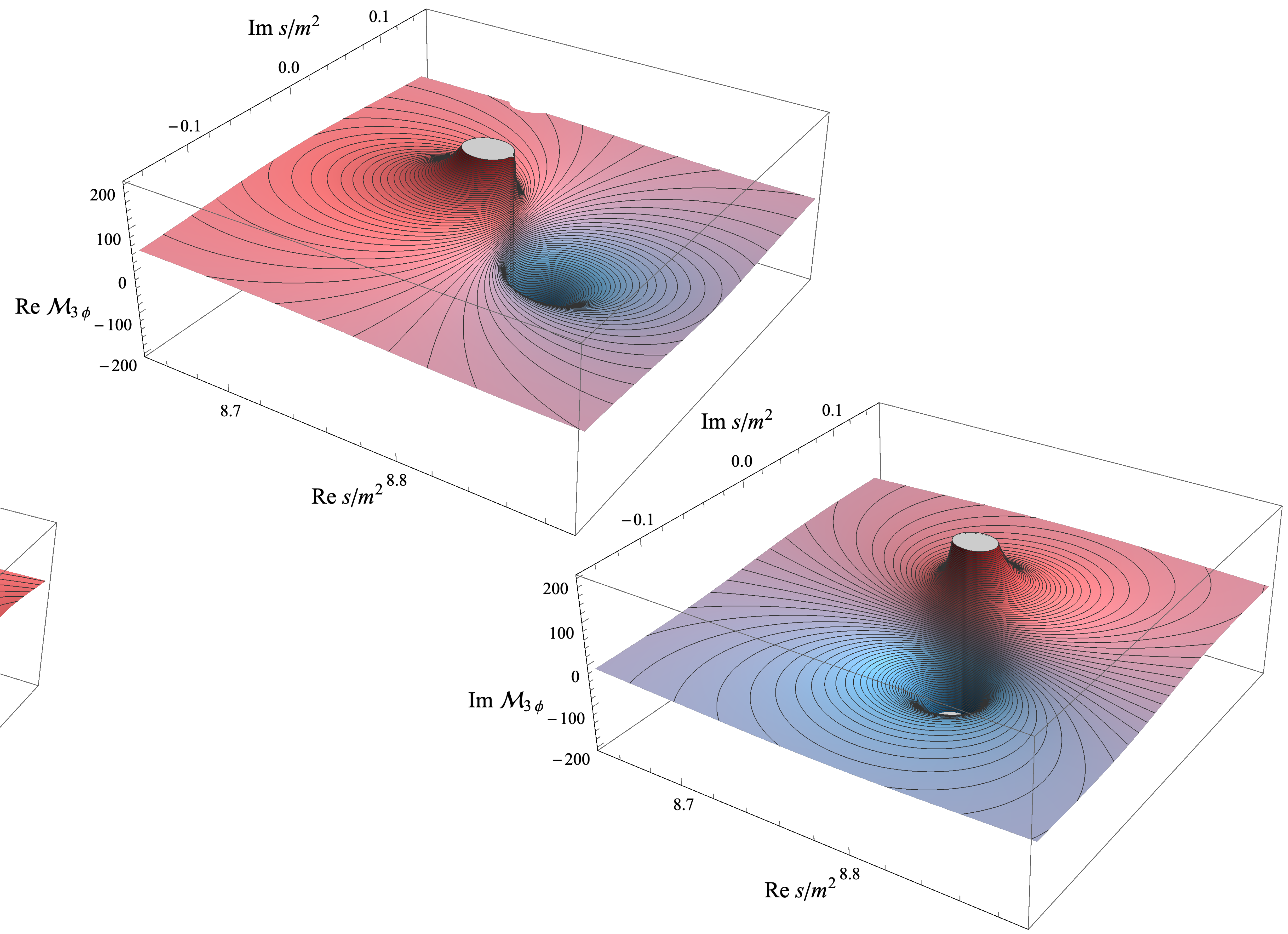
$$= \mathcal{M}_{\varphi b}(s)$$

No LSZ factorization

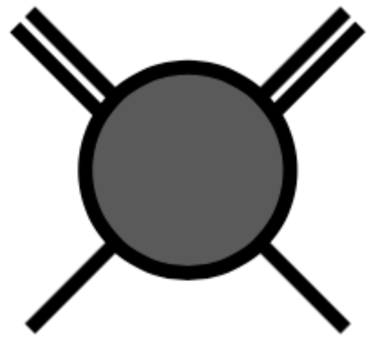


$$= d(p', s, p)$$



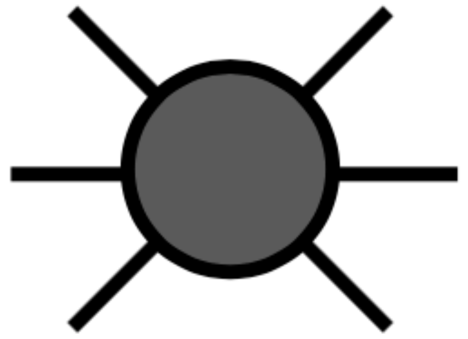


Complex-plane amplitude (ma=16)



$$= \mathcal{M}_{\varphi b}(s)$$

No LSZ factorization



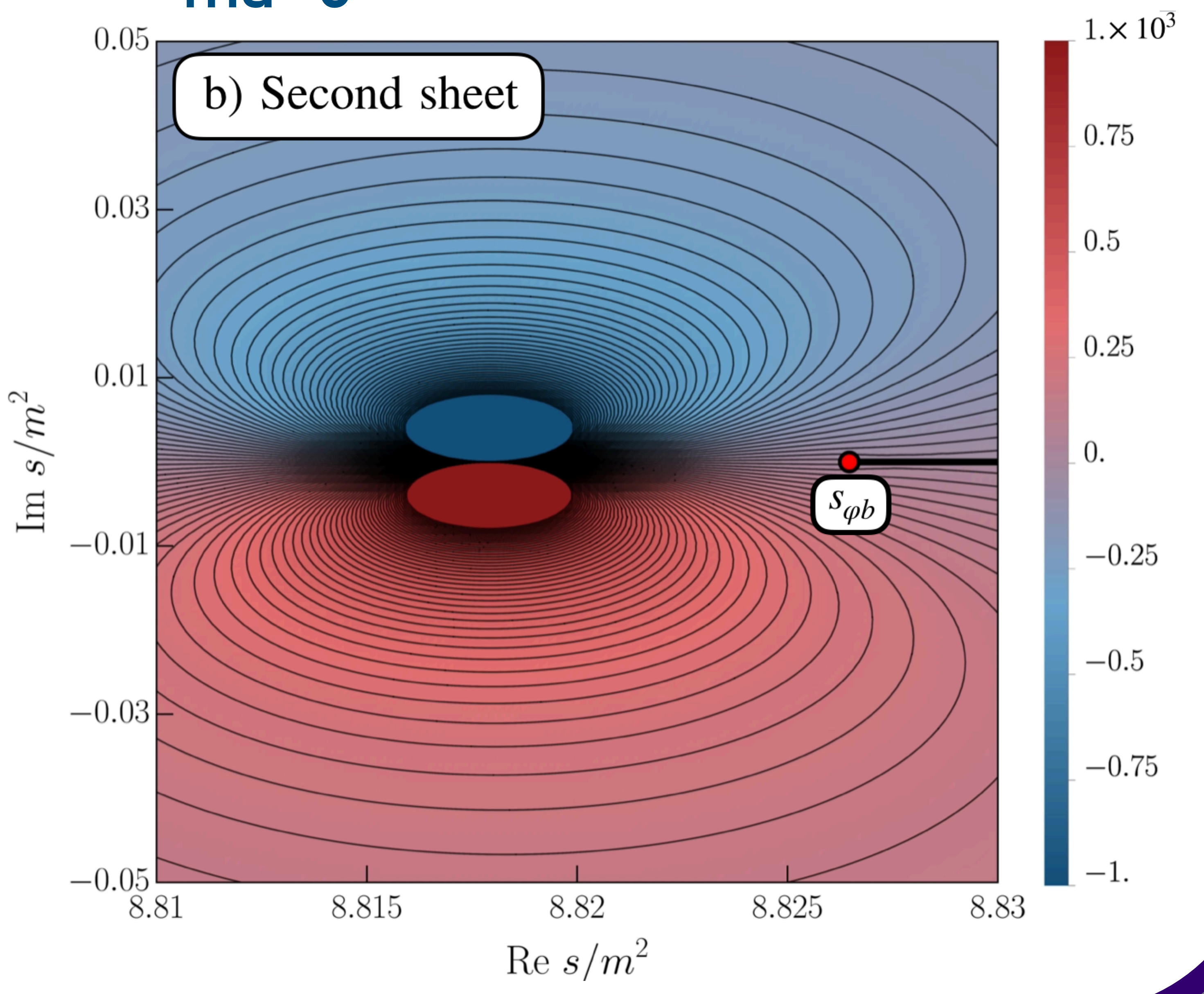
$$= d(p', s, p)$$

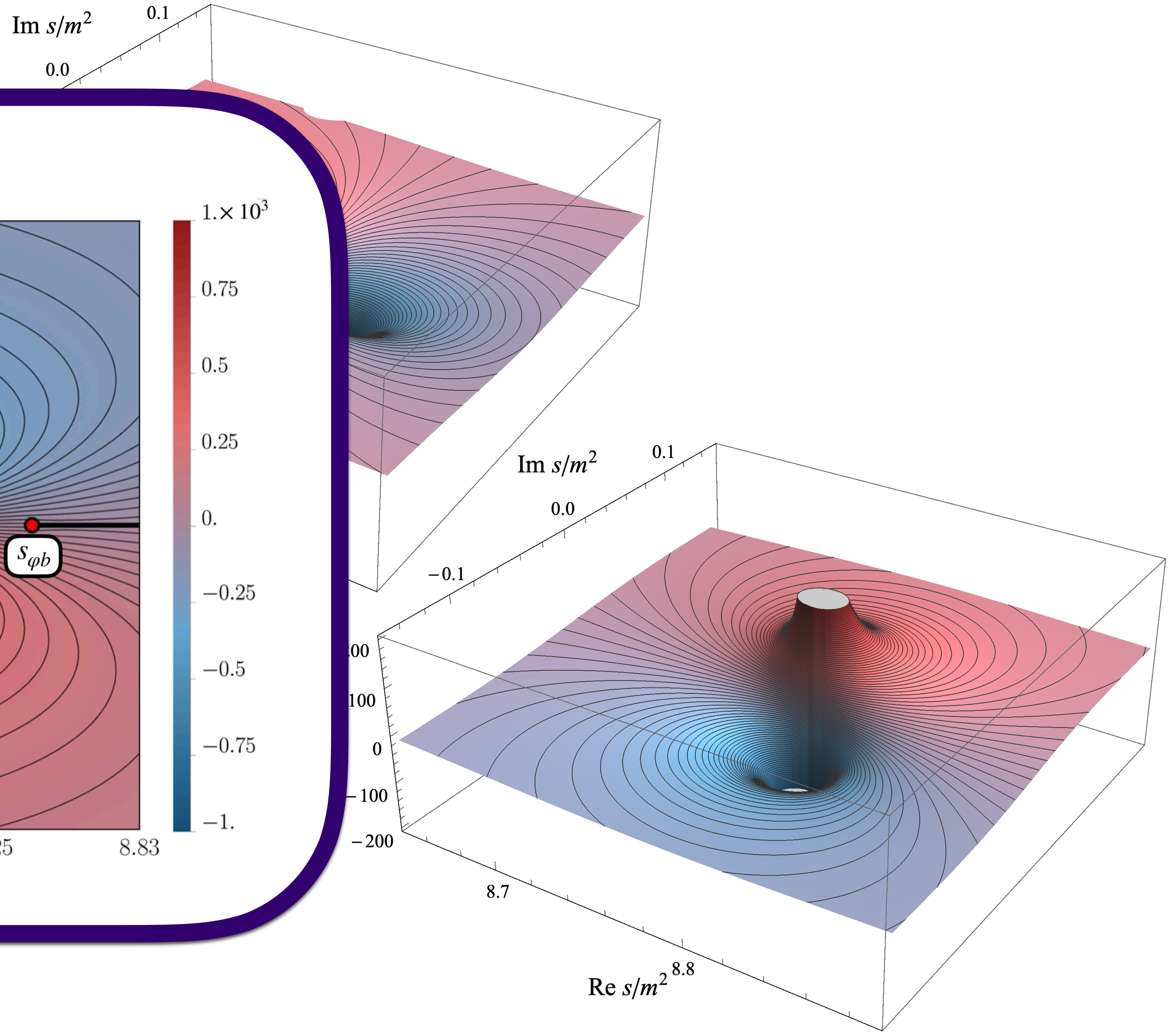
Amplitude on the second sheet

$$\mathcal{M}^{\text{II}}_{\varphi b}(s) = \frac{\mathcal{M}_{\varphi b}(s)}{1 + 2i\rho_{\varphi b}(s)\mathcal{M}_{\varphi b}(s)}$$

(or contour deformation)

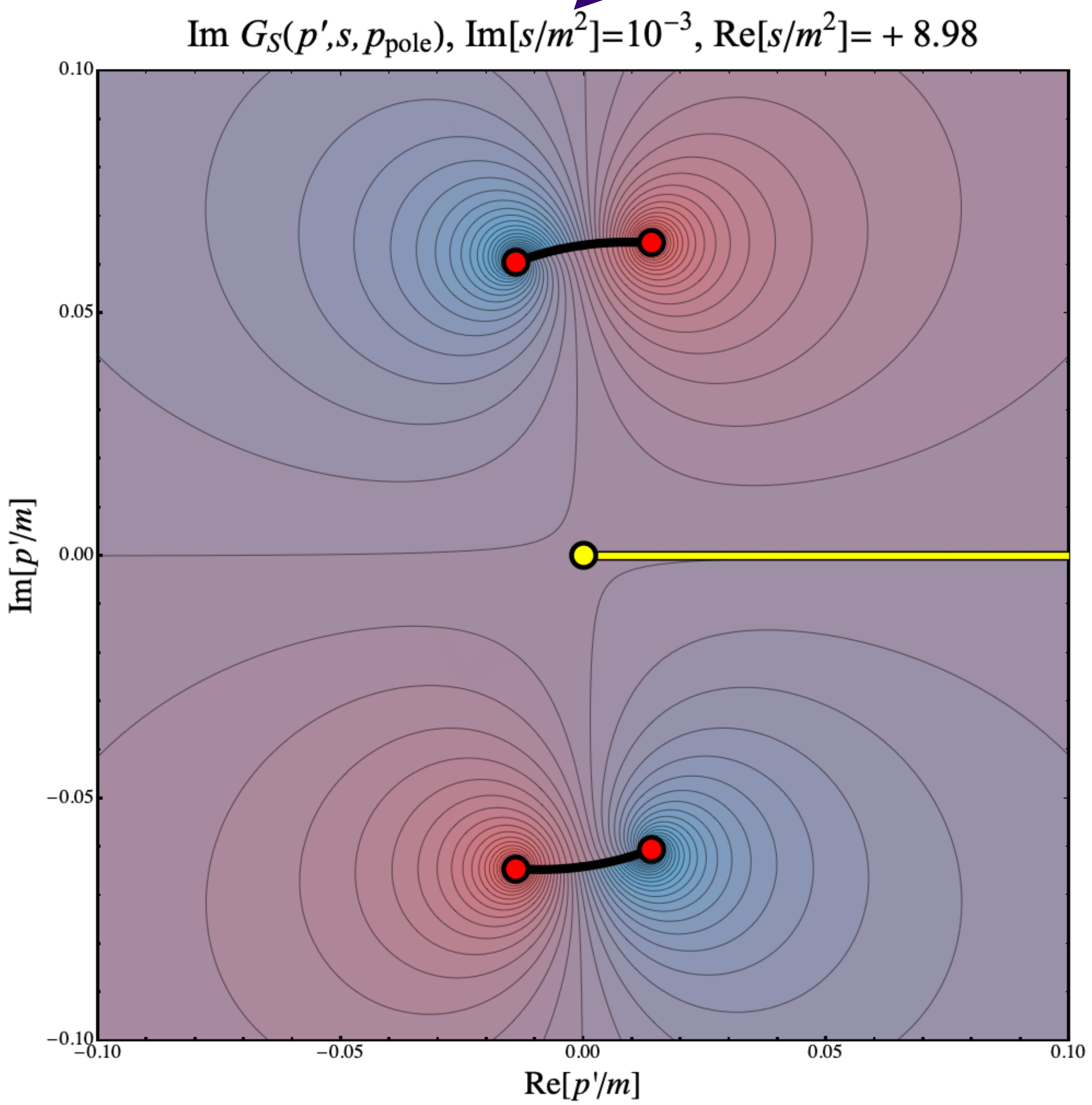
ma=6





Circular cut

Positive Im(s)



- For real s the cut closes and forms a circle
- The S-wave projection

$$G(p', s, p) \propto \int_{-1}^{1} dx \, \frac{1}{z(p', s, p) + x}$$

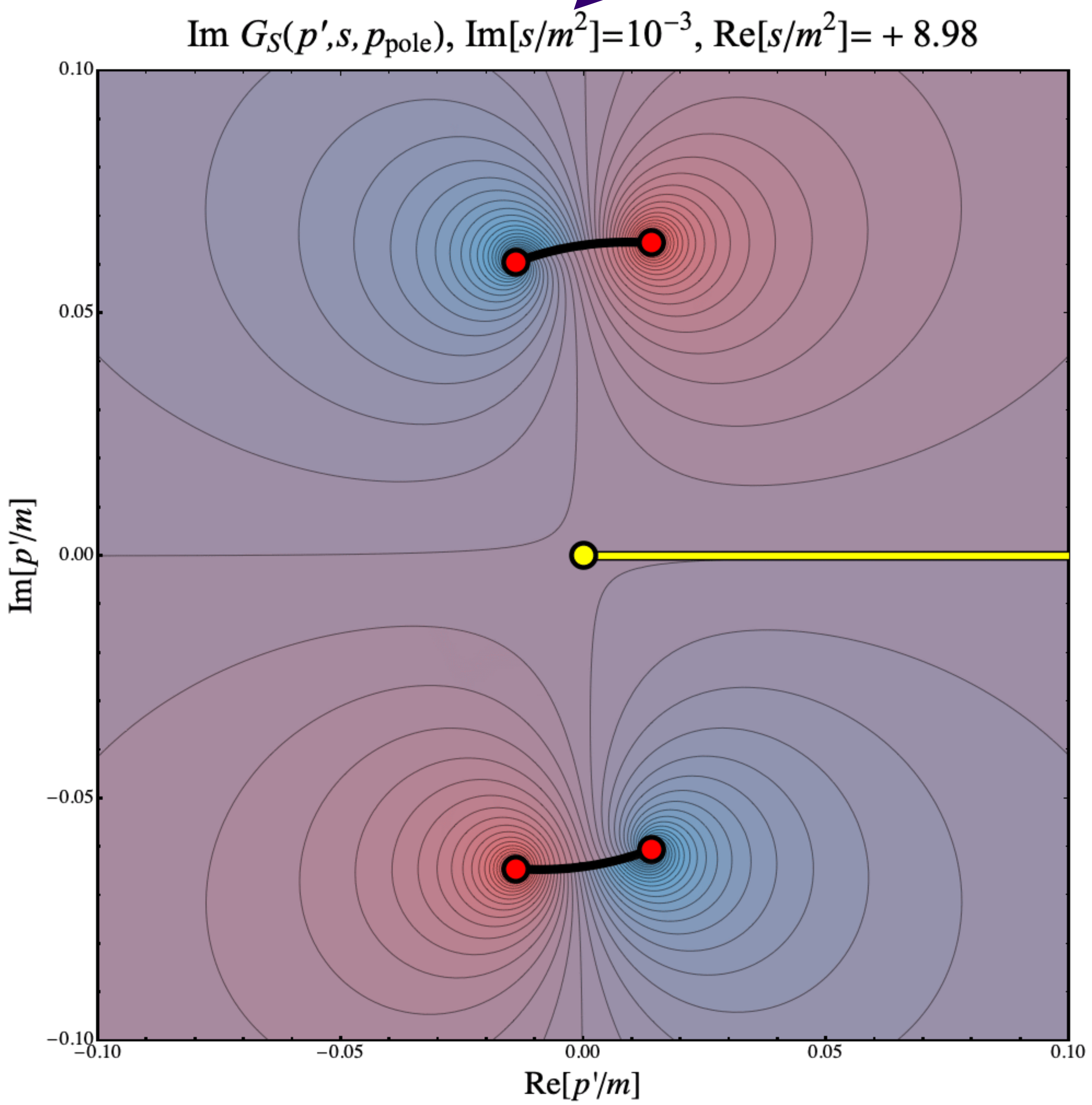
Contour deformation in *x* opens the circle
(Deformation of the cuts in the OPE)

$$G(p', s, p) \propto \log\left(\frac{z(p', s, p) + 1}{z(p', s, p) - 1}\right)$$

Position of the cuts is arbitrary
Position of branch points is not

Circular cut

Positive Im(s)



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Position of the cuts is arbitrary
Position of branch points is not

Analytic continuation of the integral equation

- Reflection of the cuts in p'

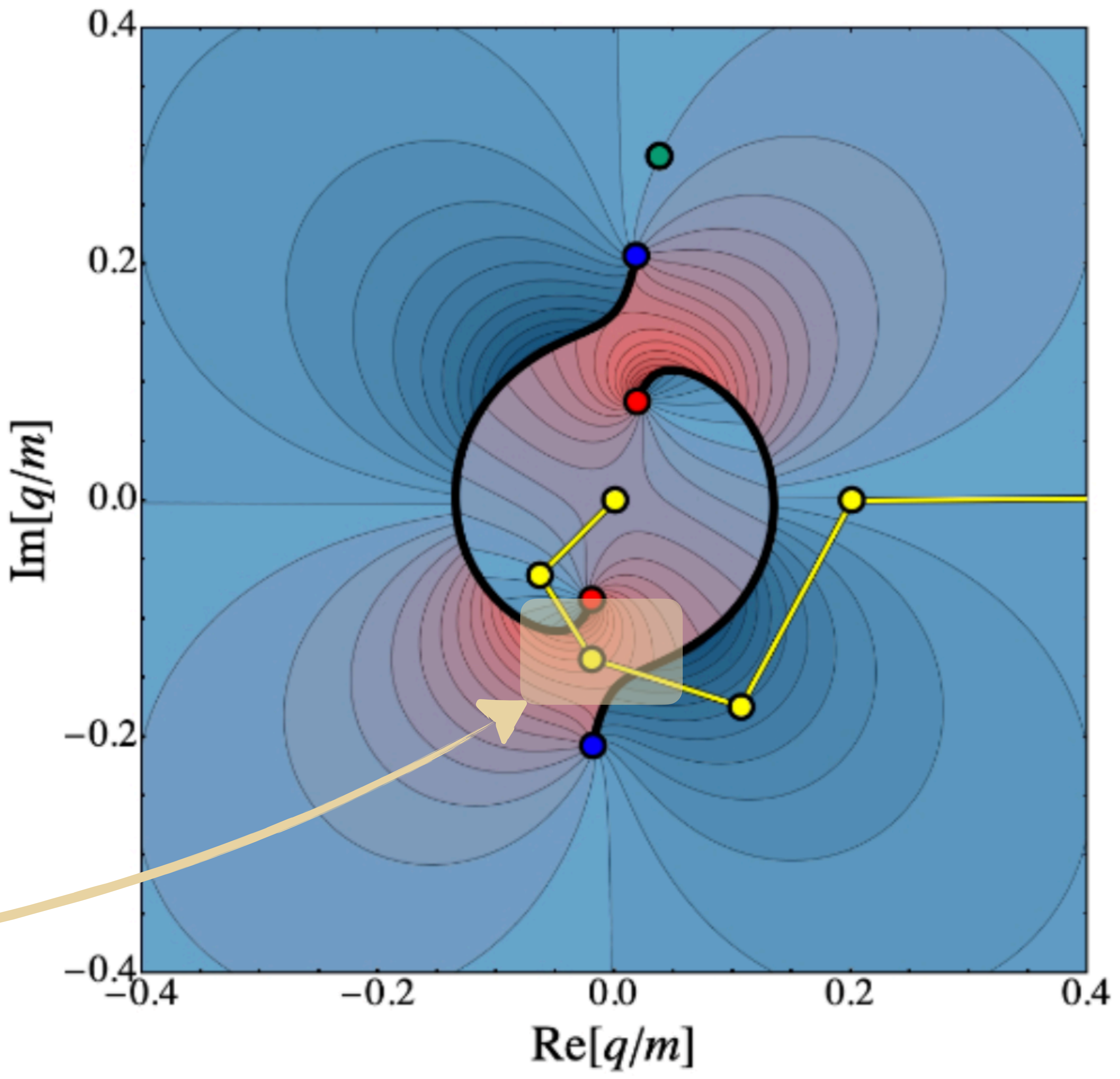
$$G(p', s, p) = G^*(p'^*, s^*, p^*)$$

$$\Downarrow$$

$$G(p', s^*, p_{\text{pole}}) = G^*(p'^*, s, -p^*_{\text{pole}})$$

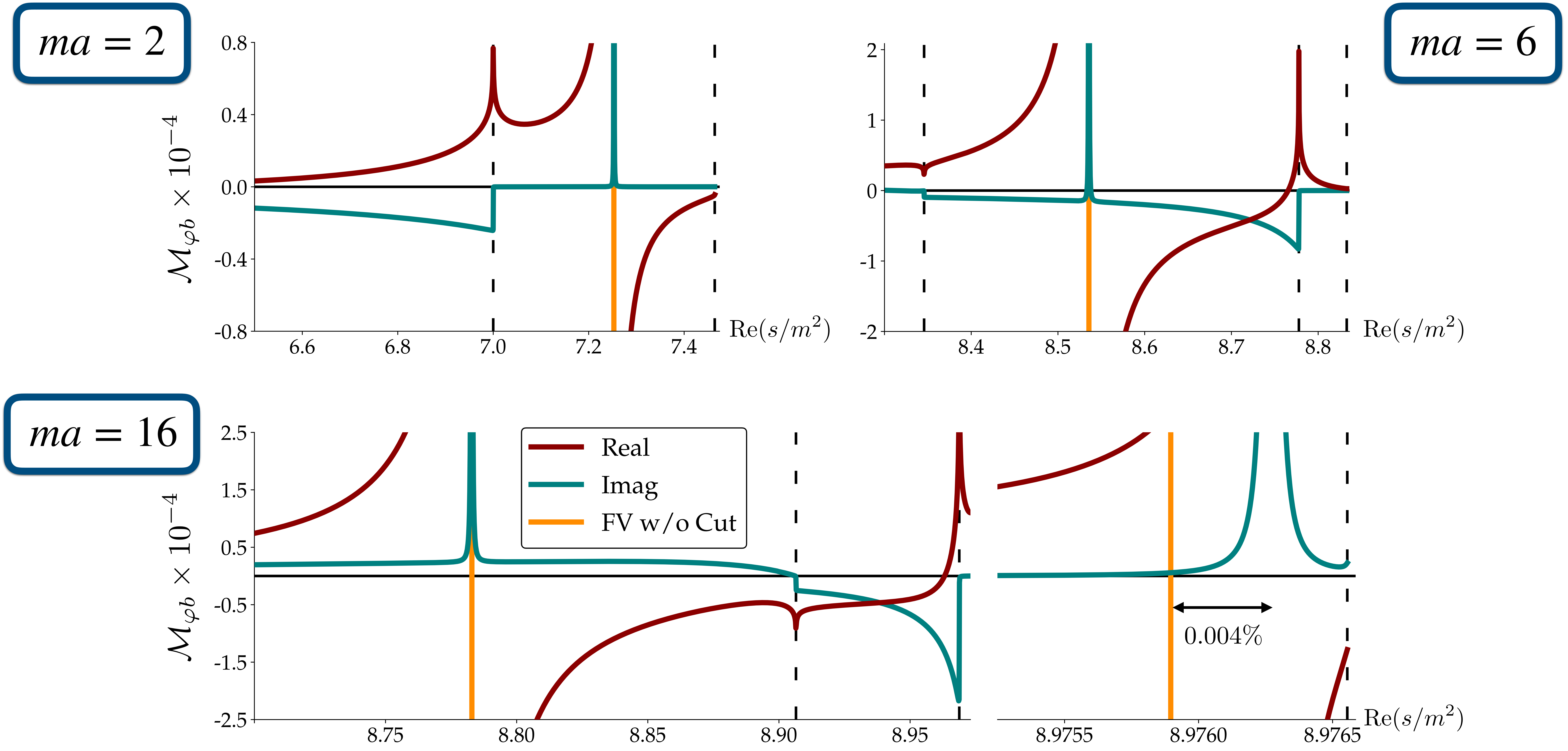
- **Adding discontinuity to OPE**

$$\Delta(p', s, p) = -\frac{H(p'p)}{4p'p}(2\pi i)$$



Amplitude below the threshold

Analytic continuation of the relativistic three-body amplitudes
Dawid, Islam, Briceño, arXiv:2303.04394



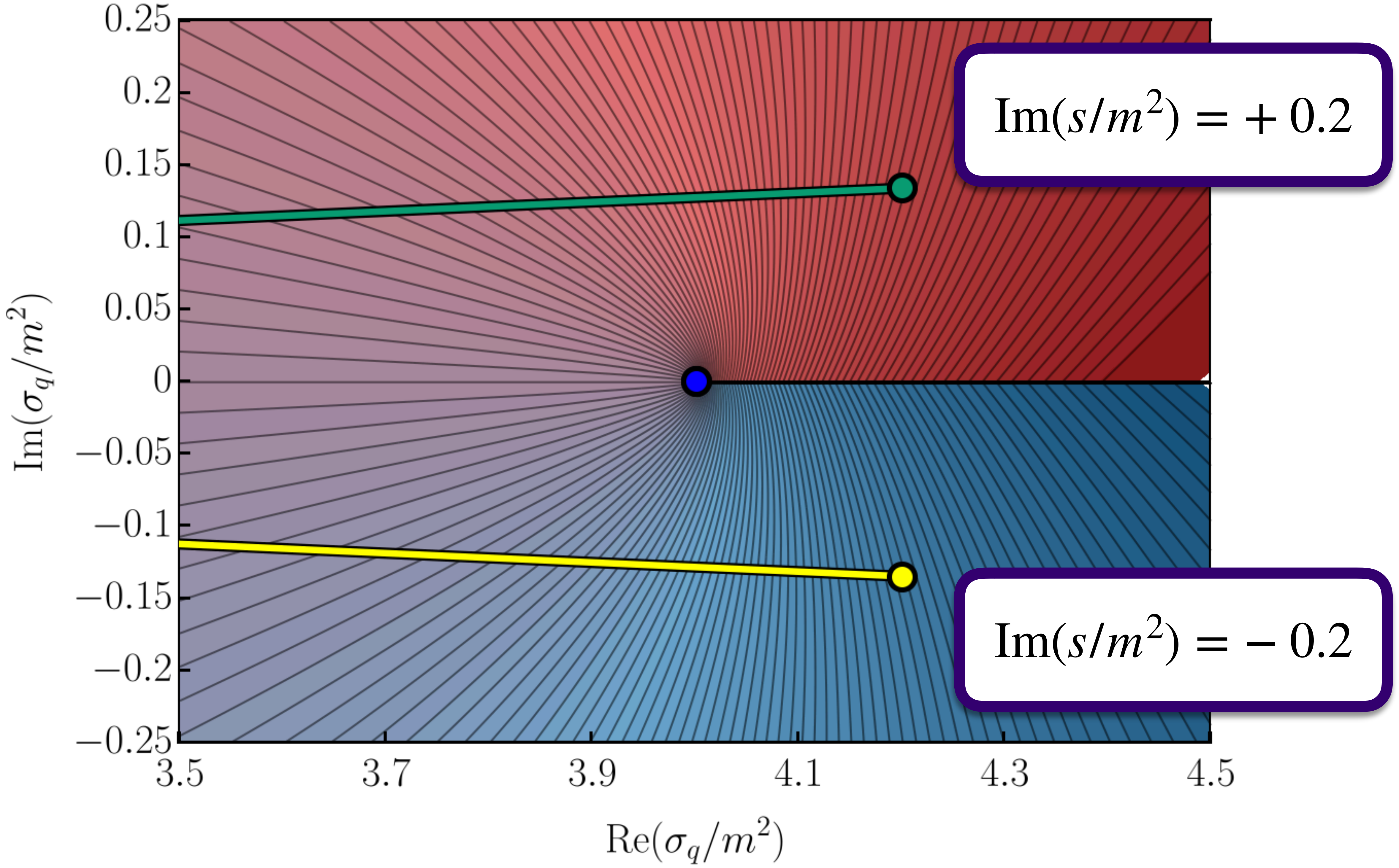
Analytic continuation through the three-body cut

Collision of the integration contour with:

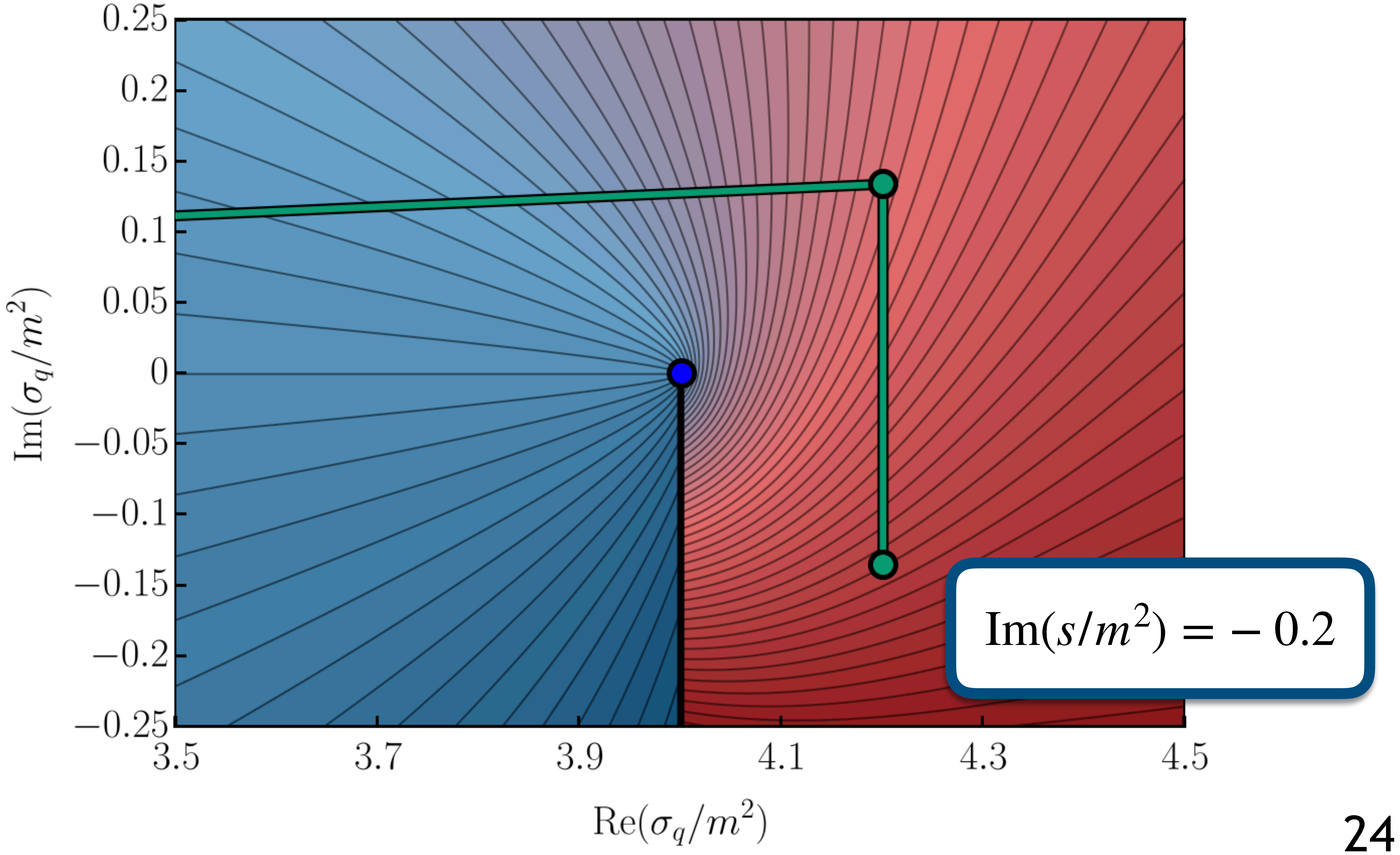
- pole of the pair amplitude ⟶ dimer-particle cut
- unitarity branch cut of the pair ⟶ three-body cut

$$\dots \; - \int_0^{q_{\max}} \frac{dq\, q^2}{(2\pi)^2 \omega_q} G(p', s, q) \mathcal{M}_2(q, s)\, d(q, s, p)$$

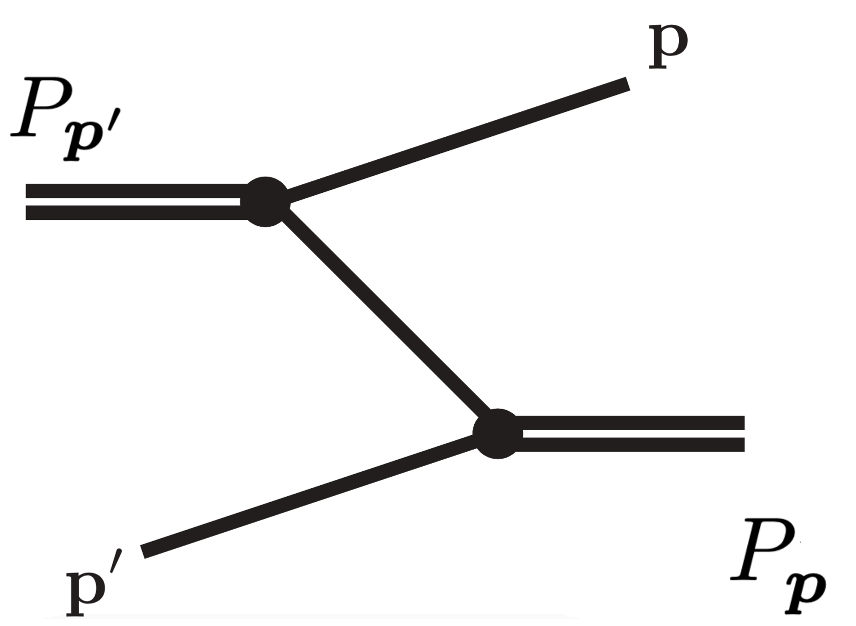
$\mathcal{M}_2$



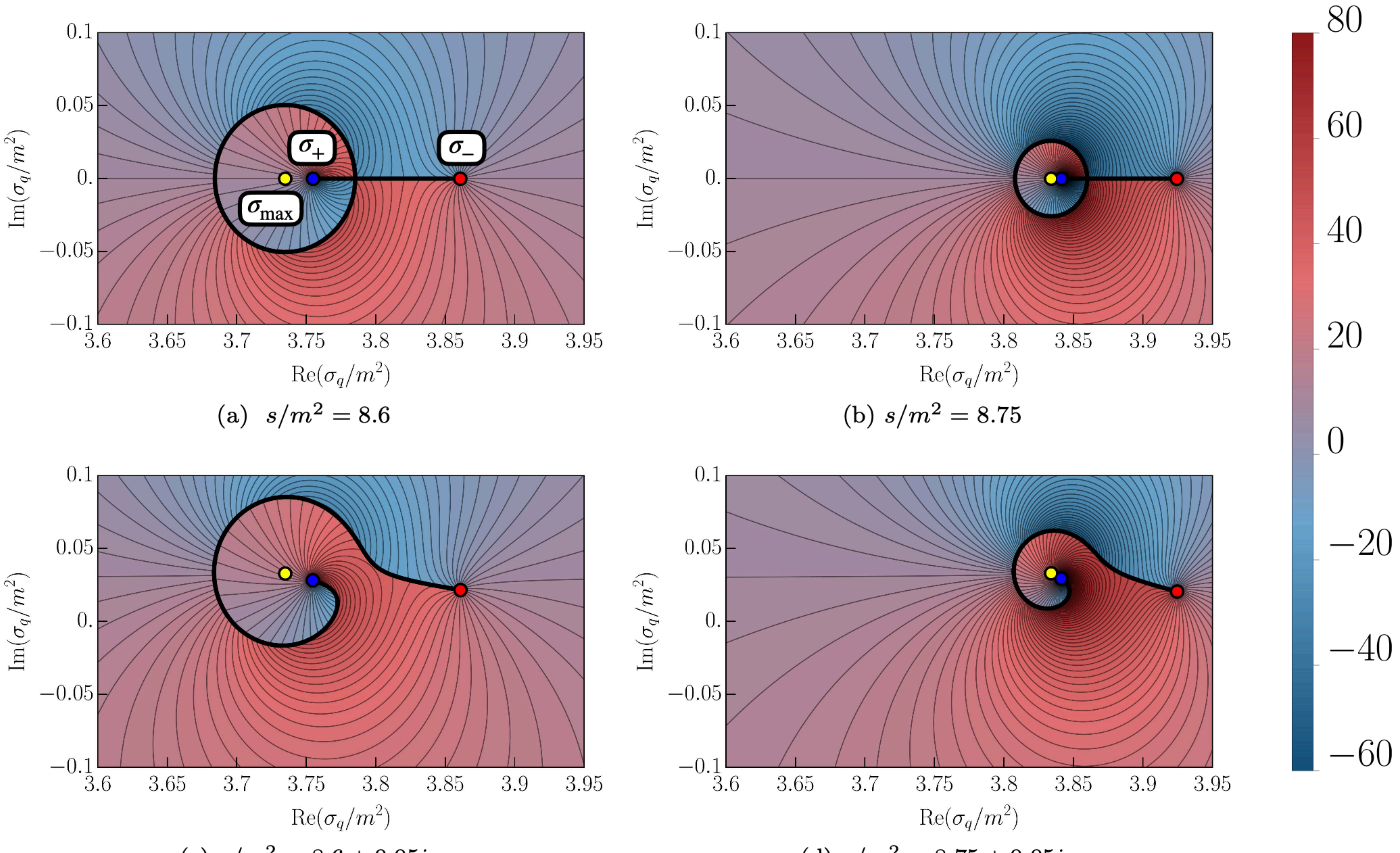
$\mathcal{M}_2$ (on the second sheet)



Kernel singularities in the invariants plane



$$P_{\boldsymbol{p}'}^2 = \sigma_{p'}\,,\ P_{\boldsymbol{p}} = \sigma_p = m_b^2$$



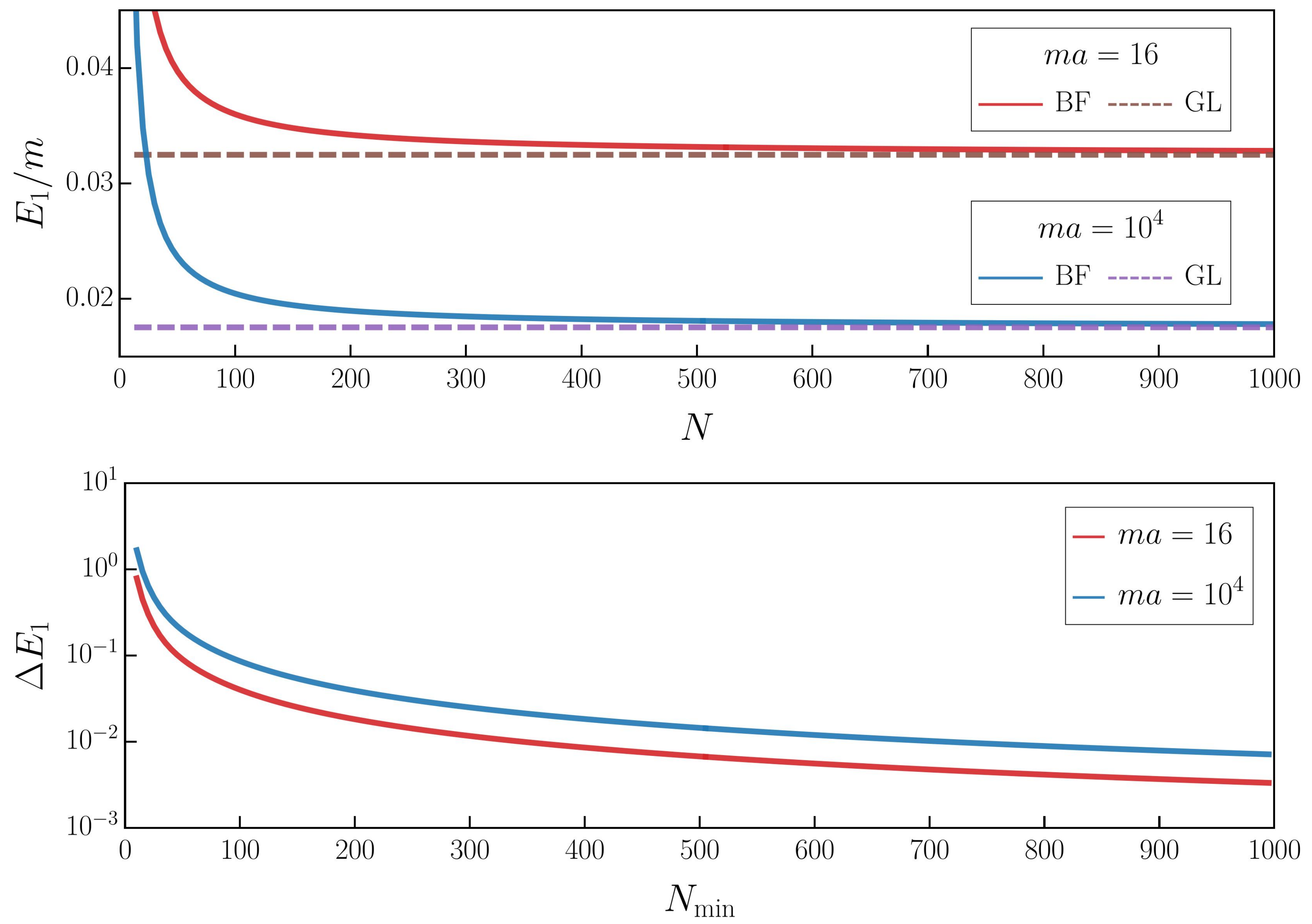
(a) $s/m^2 = 8.6$

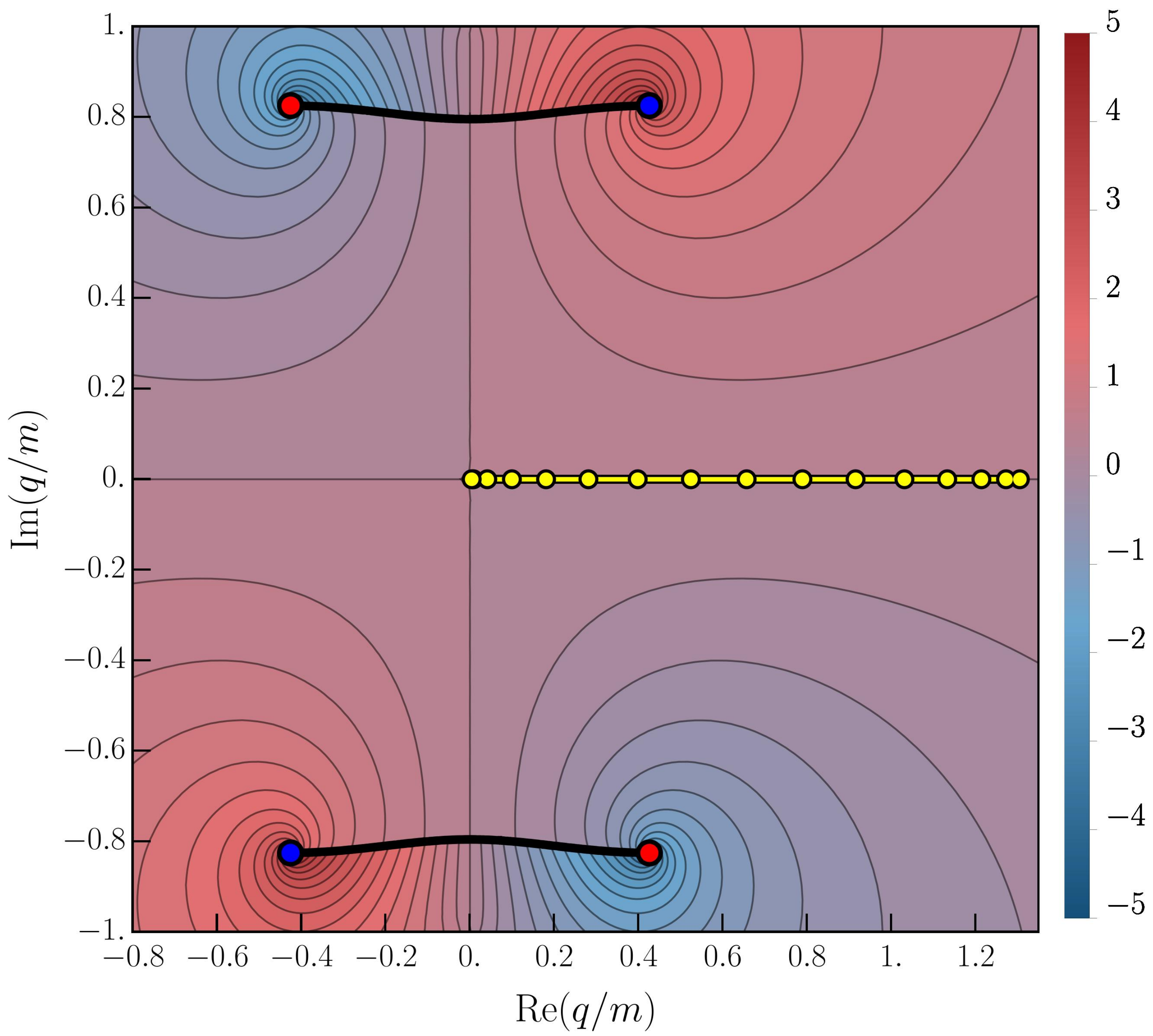
(b) $s/m^2 = 8.75$

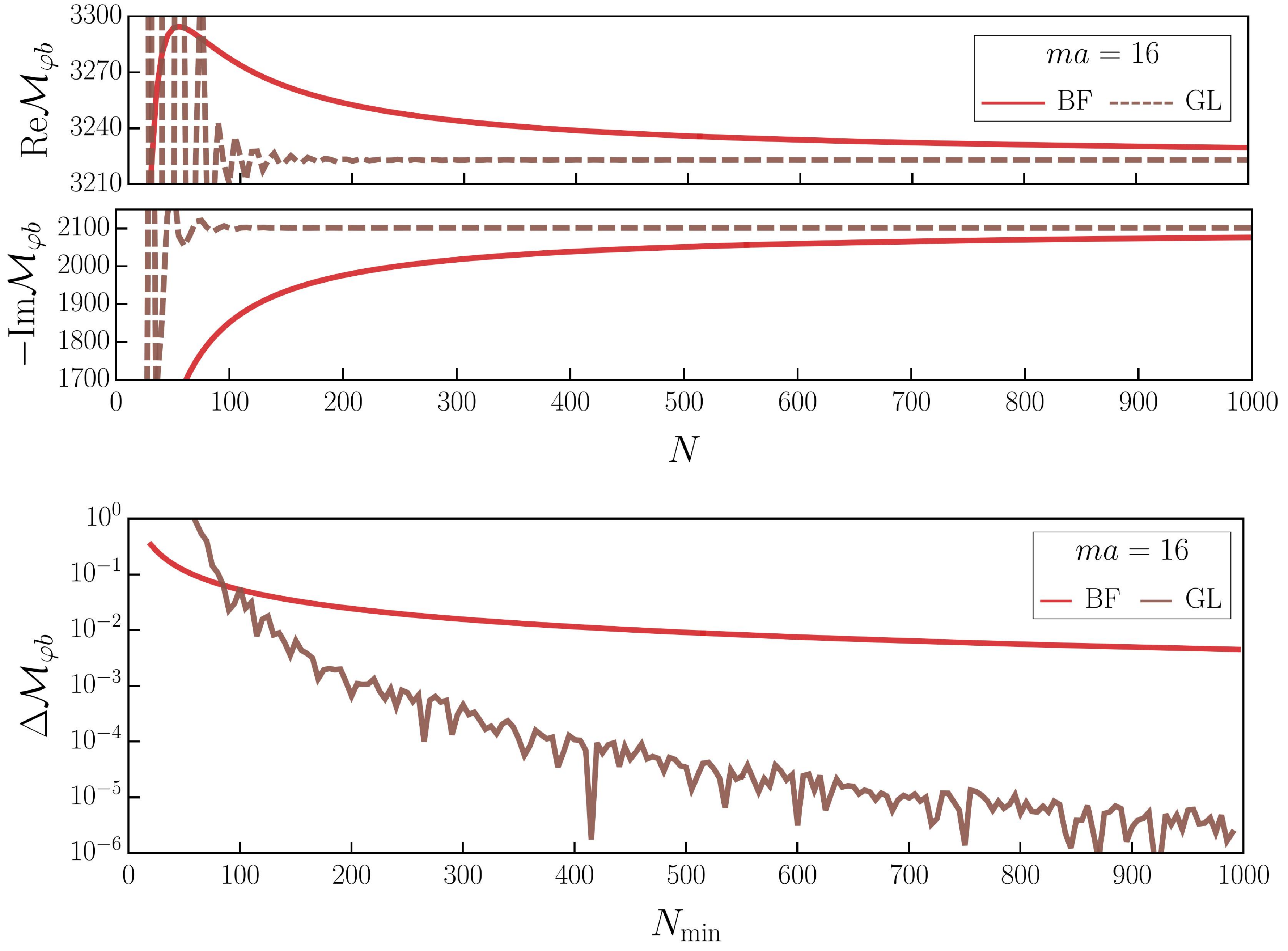
(c) $s/m^2 = 8.6 + 0.05i$

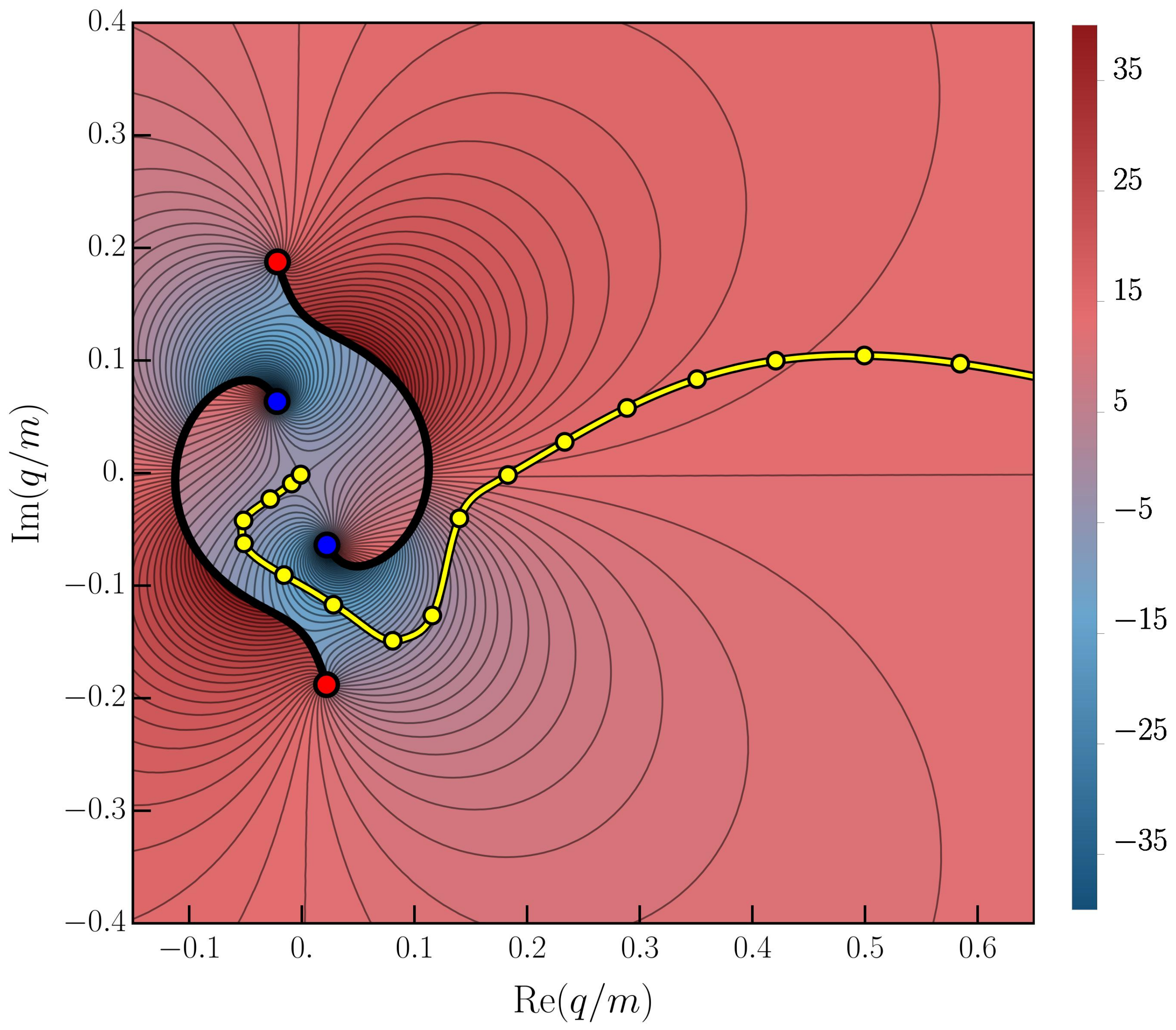
(d) $s/m^2 = 8.75 + 0.05i$

Numerical convergence



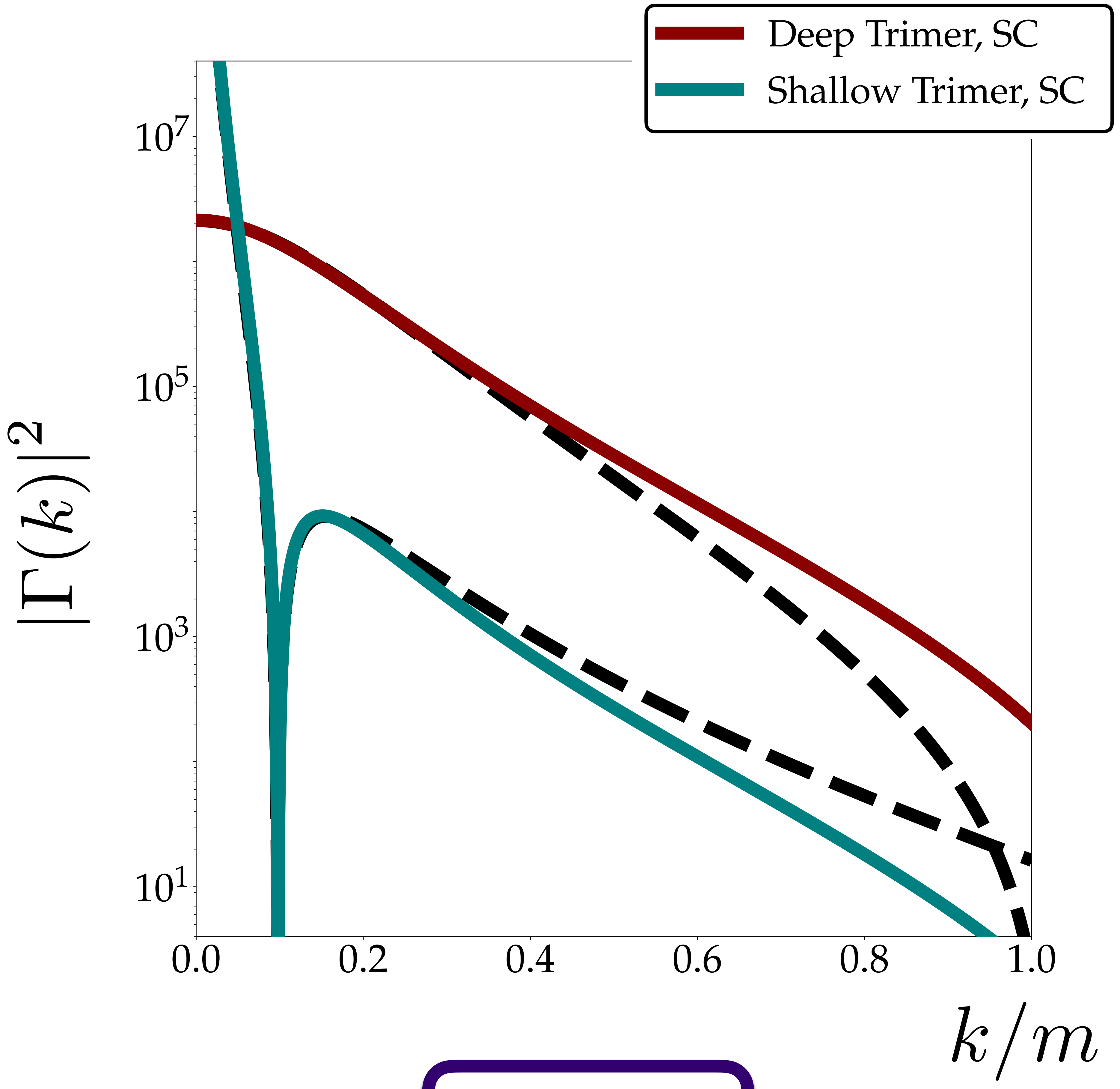






Vertex functions

Hansen, Sharpe, Phys. Rev. D 95, 034501 (2017)
Briceño, Hansen, Sharpe, Phys. Rev. D 98, 014506 (2018)



$ma = 16$

Homogeneous equation

$$(\text{amplitude}) \propto -\frac{\Gamma(p')\Gamma^*(p)}{s - s_b}$$

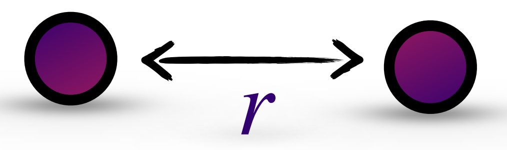
$$\Gamma(p) = -\mathcal{M}_2(p) \int_0^{q_{\max}} \frac{dq\, q^2}{(2\pi)^2 \omega_q} G(p, s, q)\, \Gamma(q)$$

Non-relativistic prediction

$$|\Gamma_{\rm NR}(k)|^2 = |c||A|^2 \frac{256\pi^{5/2}}{3^{1/4}} \frac{m^2 \kappa_{\rm NR}^2}{k^2(\kappa_{\rm NR}^2 + 3k^2/4)} \frac{\sin^2\left(s_0 \sinh^{-1}\left(\sqrt{3}k/2\kappa_{\rm NR}\right)\right)}{\sinh^2(\pi s_0/2)}$$

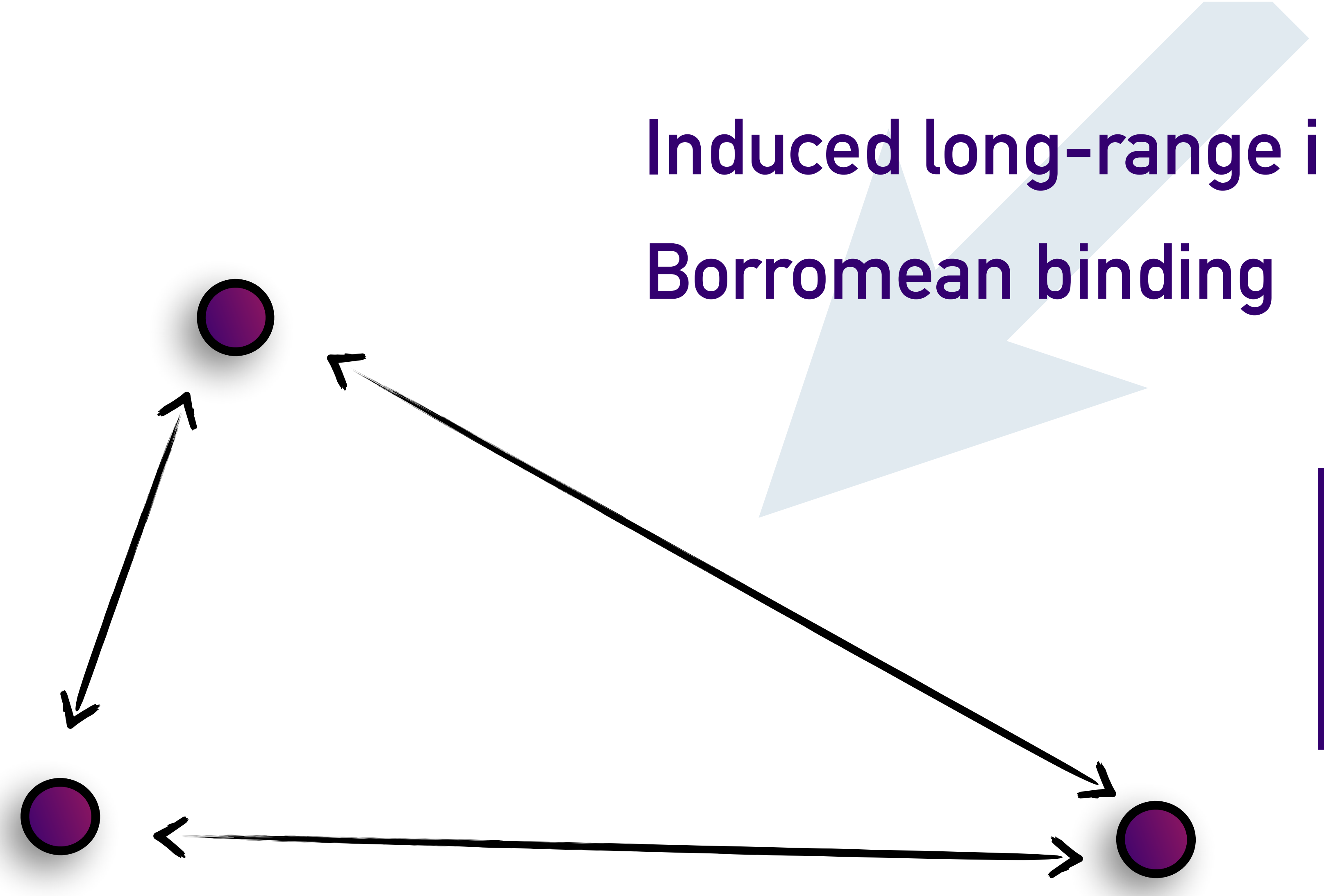
Efimov phenomenon

Short range

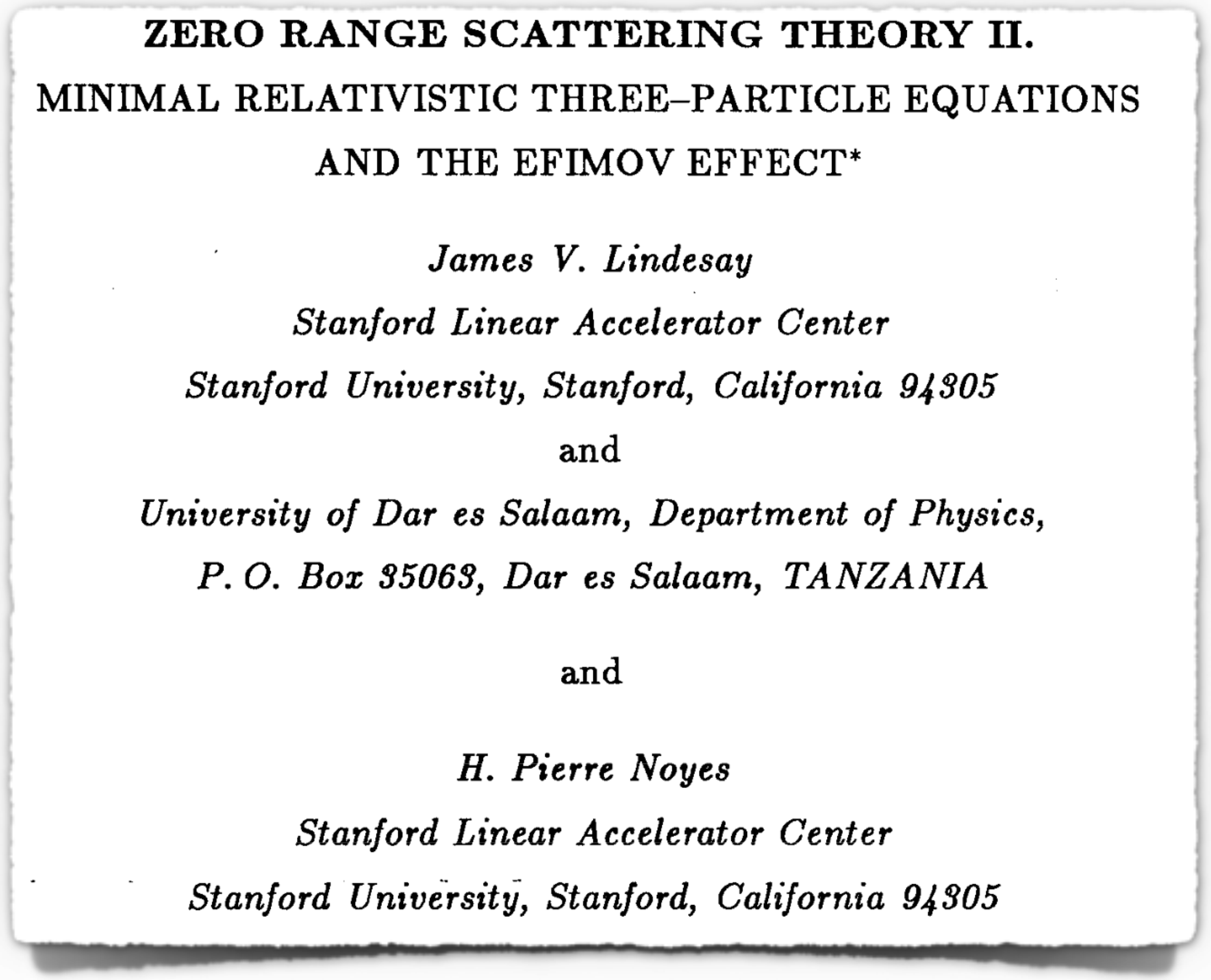


Induced long-range interaction

Borromean binding



$$E_n \propto -\left(e^{-\frac{2\pi}{s_0}}\right)^n$$


ZERO RANGE SCATTERING THEORY II.
MINIMAL RELATIVISTIC THREE–PARTICLE EQUATIONS
AND THE EFIMOV EFFECT*

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Stanford University, Stanford, California 94305
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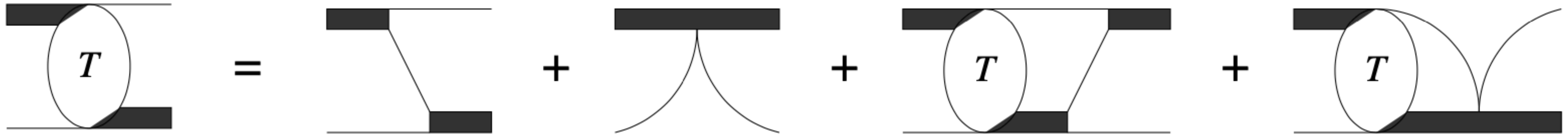
P. O. Box 35063, Dar es Salaam, TANZANIA

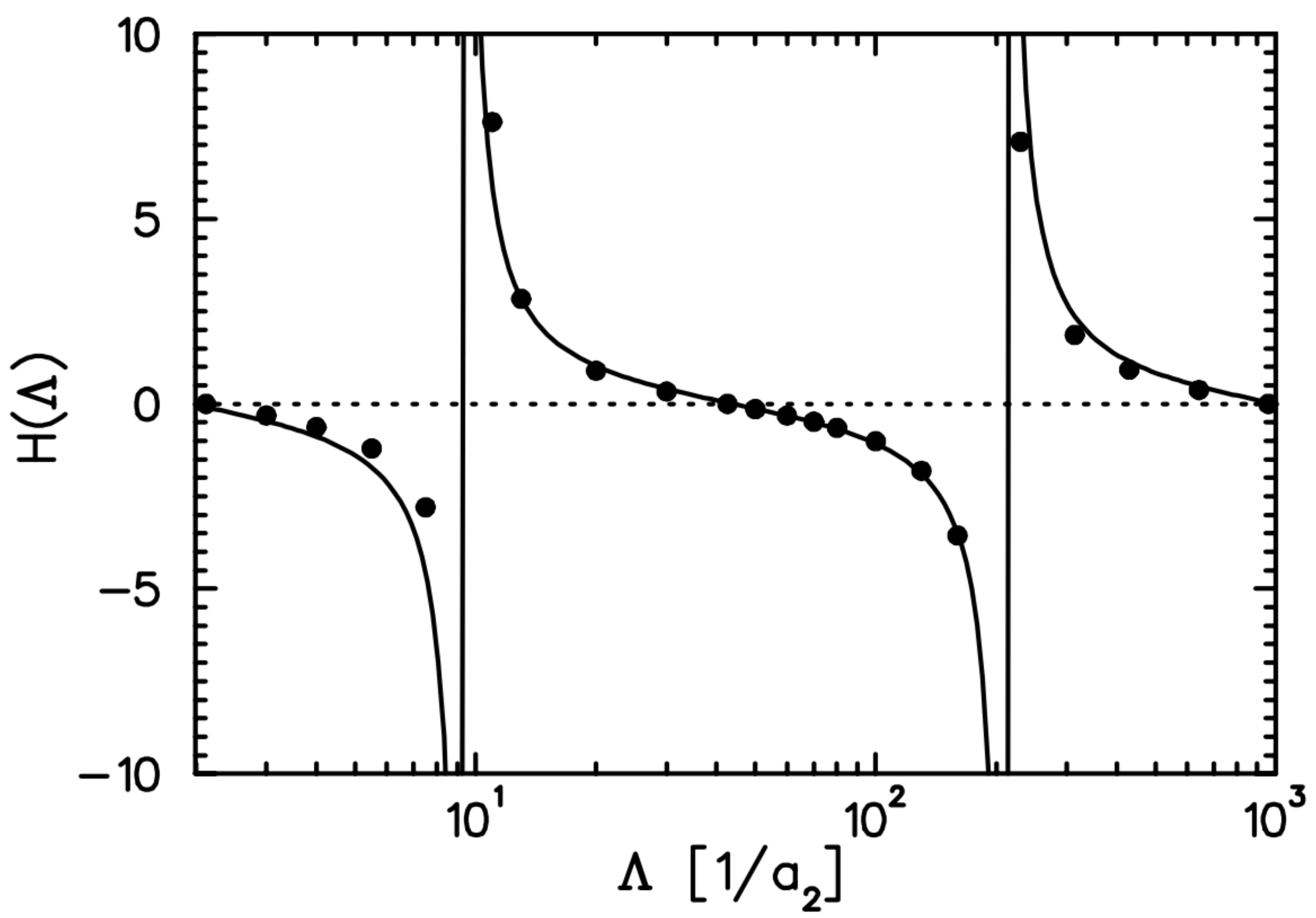
and

H. Pierre Noyes
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

- NREFT three-body equation

$$\mathcal{L} = \psi^\dagger\left(i\partial_0 + \frac{\vec{\nabla}^2}{2m}\right)\psi + \Delta T^\dagger T - \frac{g}{\sqrt{2}}\left(T^\dagger\psi\psi + \text{h.c.}\right) + hT^\dagger T\psi^\dagger\psi + \ldots$$

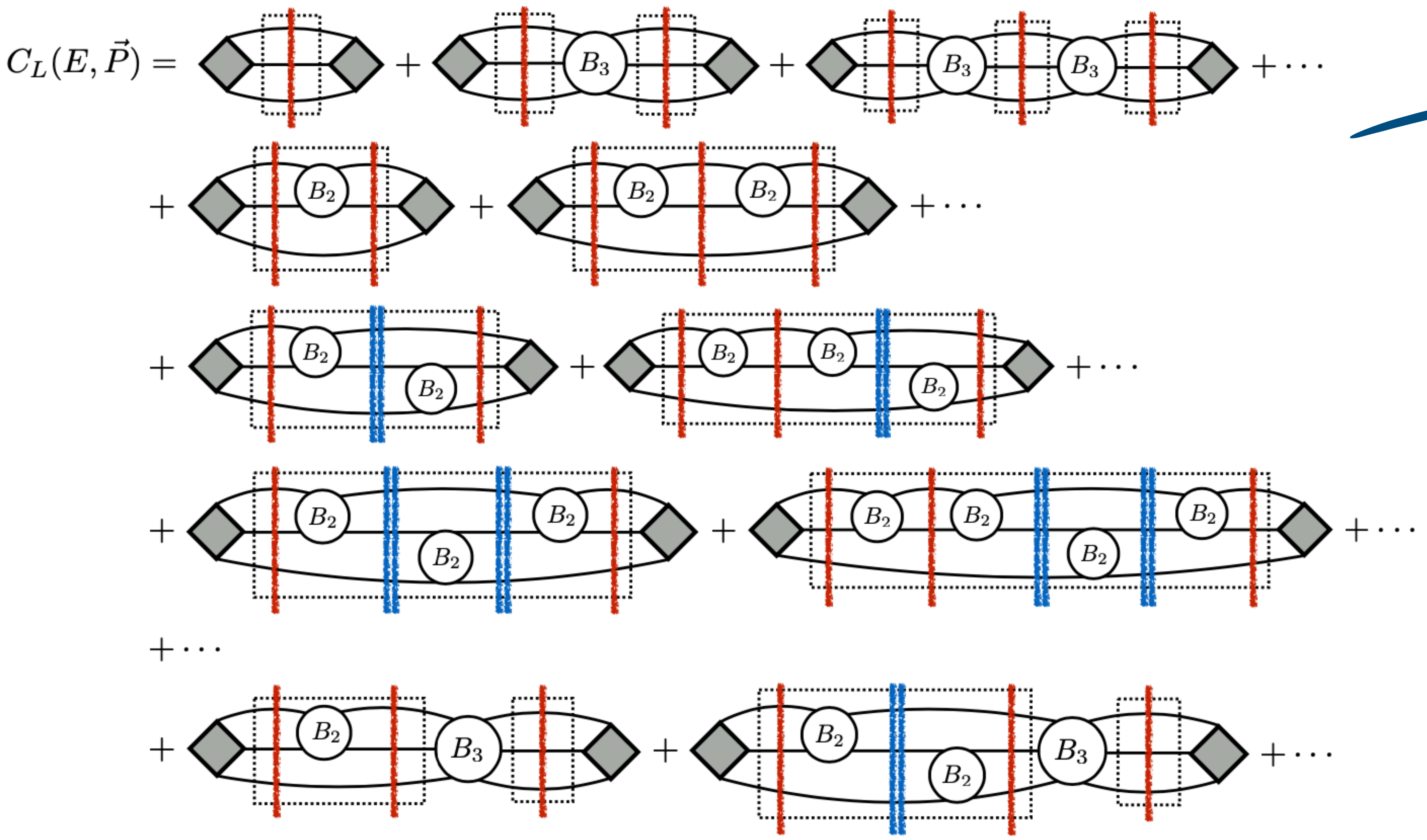




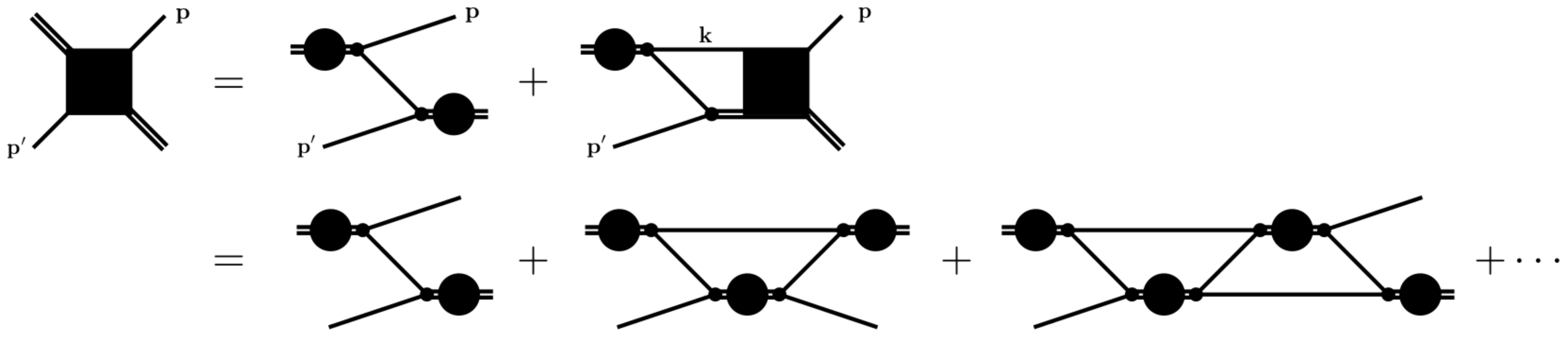
The Three-Boson System with Short-Range Interactions
Bedaque, Hammer, van Kolck, Nucl. Phys. A 646 (1999) 444

Lattice QCD and three-particle decays of resonances
Hansen, Sharpe, Ann. Rev. Nucl. Part. Sci. 69 (2019) 65-107

REFT three-body formalism

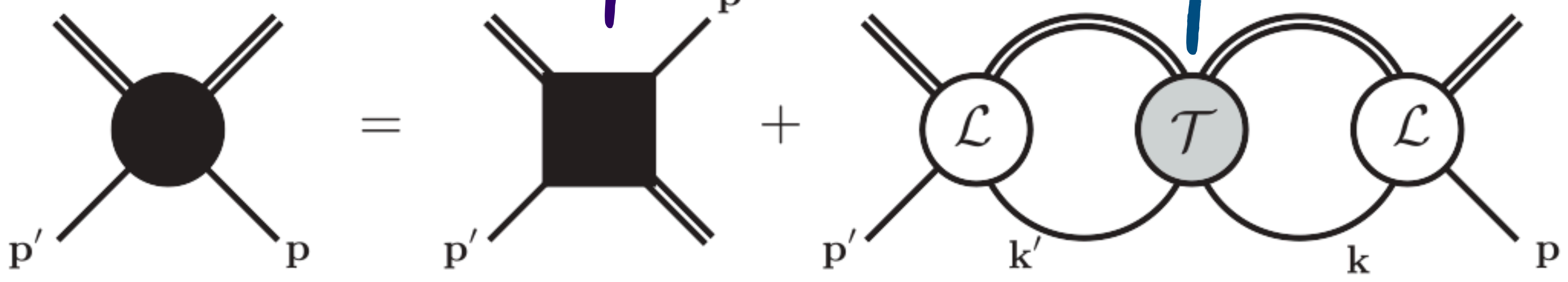


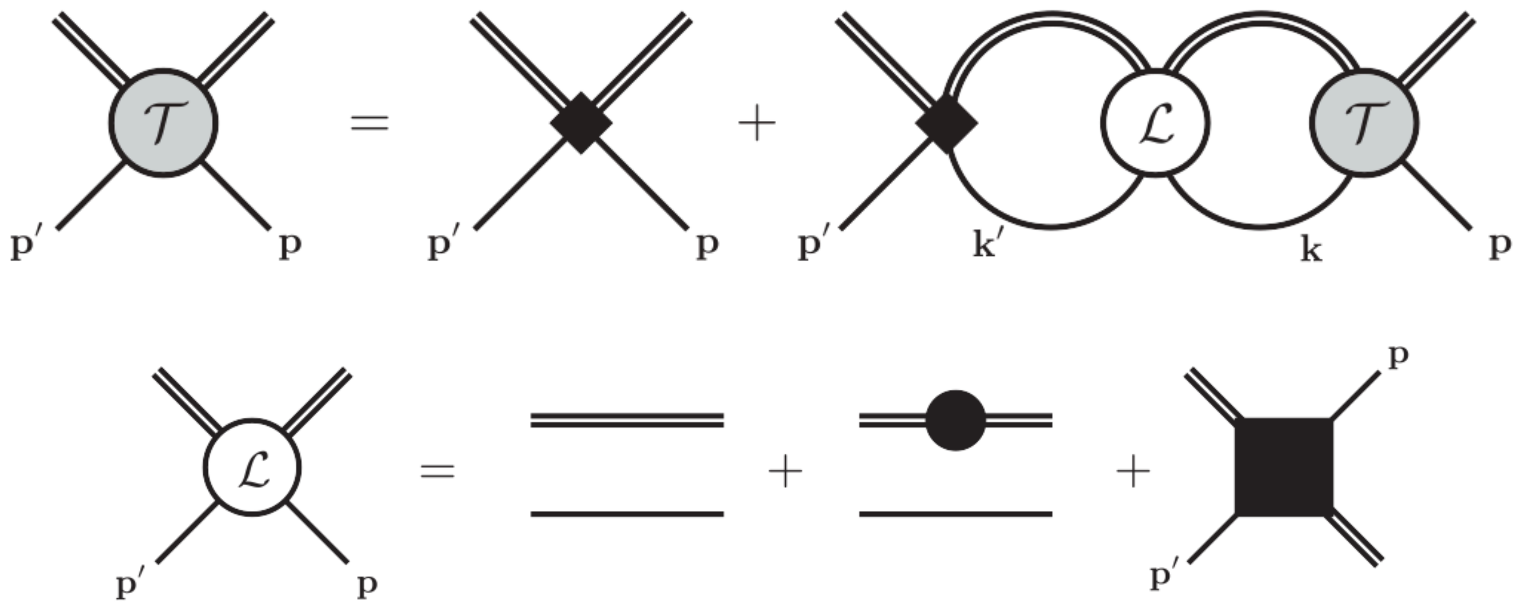
$$\det\left[\mathcal{K}_{\mathrm{df},3}(s) + F_3(s, \boldsymbol{P}, L)^{-1}\right] = 0$$



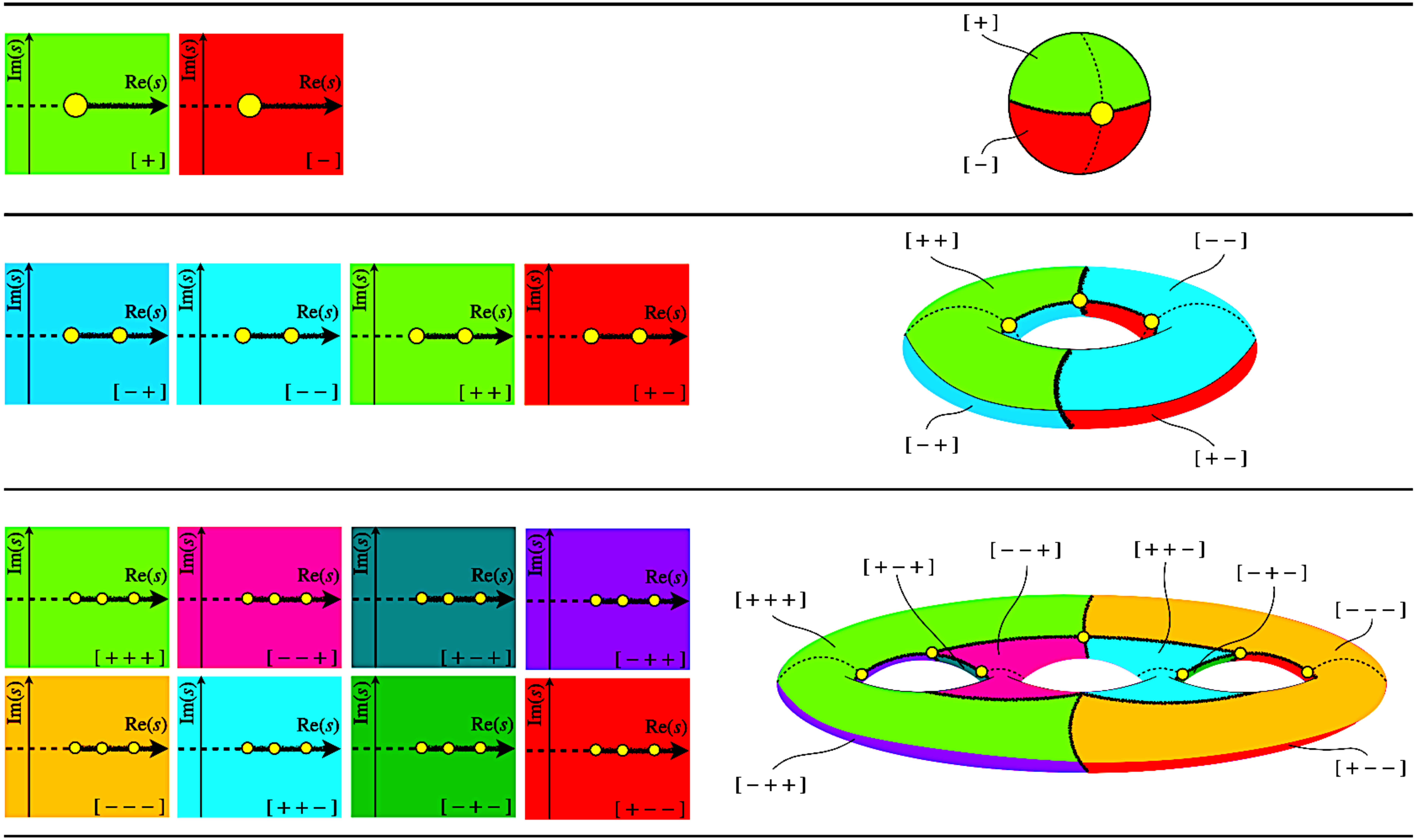
Infinite volume integral equations:

$$\mathcal{M}_3^{(u,u)} = \mathcal{D}^{(u,u)} + \mathcal{M}_{\mathrm{df},3}^{(u,u)}$$





Riemann sheets



Properties of some three-body states

State	$I\ (J^{PC})$	Mass [MeV]	Width [MeV]	Decay	Branching ratio
$\omega(782)$	$0\ (1^{--})$	782.66 ± 0.13	8.68 ± 0.13	$\pi^+\pi^-\pi^0$	$(89.2 \pm 0.7)\%$
$\eta'(958)$	$0\ (0^{-+})$	957.78 ± 0.06	0.188 ± 0.006	$\pi^+\pi^-\eta$	$(42.5 \pm 0.5)\%$
$a(1260)$	$1\ (1^{++})$	1230 ± 40	420 ± 35	3π	seen
$N(1440)$	$\frac{1}{2}\ (\frac{1}{2}^+)$	1440 ± 30	350 ± 100	$N\pi\pi$	$(17 - 50)\%$
$\pi_1(1600)$	$1\ (1^{-+})$	1661 ± 15	240 ± 50	3π	seen
$\chi_{c1}(3872)$	$0\ (1^{++})$	3871.65 ± 0.06	1.19 ± 0.21	$D^0\bar{D}^0\pi^0$	$(40 \pm 20)\%$