Analytic continuation of the three-particle amplitudes

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Outline

• Need for amplitude analysis
• Three-body integral equations
• **Analytic continuation**
• Scattering observables in a toy model

*Solving relativistic three-body integral equations in the presence of bound states*

*Analytic continuation of the relativistic three-body amplitudes*
Dawid, Islam, Briceño, arXiv:2303.04394

*Evolution of Efimov states in relativistic scattering theory*
Dawid, Islam, Briceño, Jackura, in preparation

**Coupled 3ϕ and ϕb system**
Need for amplitude analysis

Experiment — $Z_c(3900)$ pole

Lattice — $f_0$ isoscalar pole

Amplitude analysis and the nature of the $Z_c(3900)$

Isoscalar $\pi\pi$, KK, $\eta\eta$ scattering and the $\sigma$, $f_0$, $f_2$ mesons from QCD
Briceño et al. (HadSpec), Phys. Rev. D 97, (2018) 054513
Three-body program

Finite Volume

Lattice QCD ✓ Spectrum ✓ Quantization Condition ✓

Infinite Volume

Amplitude ✓ Resonances ✓

Relativistic, model-independent, three-particle quantization condition
Hansen, Sharpe, Phys. Rev. D 90 (2014) 11, 116003

Three-body unitarity in finite volume

Relativistic-invariant formulation of the NREFT three-particle quantization condition
Müller, Pang, Rusetsky, Wu, JHEP 02 (2022), 158

Talk by S. Sharpe
Talk by Z. Draper
Talk by M. Sjö
S-matrix parametrization

All diagrams by Andrew Jackura

One Particle Exchange  Short Range Interactions

Unitarity

Three-body amplitude

$$ [M_3]^J_{\ell' m'_{\ell'} ; \ell m_{\ell}} (p', s, p) $$

- pair-spectator
- partial waves
- symmetrization

$$ M_3 = M_2 B M_2 + M_2 \int B \rho_3 M_3 $$
S-matrix parametrization

All diagrams by Andrew Jackura

Unitarity

One Particle Exchange

Short Range Interactions

Three-body amplitude

$$[\mathcal{M}_3]^J_{\ell',m';\ell m} (p', s, p)$$

- pair-spectator
- partial waves
- symmetrization

$$\tilde{\mathcal{M}}_3 = \mathcal{B} + \int \mathcal{B} \, \mathcal{M}_2 \, \rho_3 \, \tilde{\mathcal{M}}_3$$

$$\mathcal{B} \rho_3$$

$$\mathcal{M}_2 = \mathcal{K}_2 + \mathcal{K}_2 i \rho_2 \mathcal{M}_2$$
Ladder equation

Ladder amplitude \( \tilde{M}_3 = d(p', s, p) \)

Bound state of mass 
\[ m_b = 1.732m \ (ma=2) \]
\[ m_b = 1.996m \ (ma=16) \]

Three-body amplitude

Bound-state–spectator amplitude \( M_{\varphi b} \)

LSZ

Bound state of mass 
\[ m_b = 1.732m \ (ma=2) \]
\[ m_b = 1.996m \ (ma=16) \]
The Born term

One-particle exchange propagator

\[ G(p', s, p) \propto \log \left( \frac{1 + z(p', s, p)}{1 - z(p', s, p)} \right) \]

Partial-wave projection

\[ P_{p'}^2 = P_p^2 = m_b^2 < (2m)^2 \]

Analytic continuation of the relativistic three-body amplitudes
Dawid, Islam, Briceño, arXiv:2303.04394
Analytic continuation of the integral equation

**In a nutshell**
- Avoid crossing the singularities in the integration
- Deform the contour, add discontinuity, deform, ...

### Singularities of the ladder equation

\[ d(p', s, p) = -G(p', s, p) - \int_0^{q_{\text{max}}} \frac{dq q^2}{(2\pi)^2 \omega_q} G(p', s, q) \mathcal{M}_2(q, s) \, dq, \quad \omega_q = \sqrt{p'^2 - s - m^2} \]

**Integration kernel**

**Solution**

- **Inhomogeneous term**
- **Homogeneous term**

**Integration kernel**

**Solution**

- **Integration kernel**
- **Solution**

**In a nutshell**
- Avoid crossing the singularities in the integration
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**Graphical representation**

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**Trimer?**

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- **Trimer?**
Analytic continuation of the integral equation

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Singularities of the ladder equation

\[ d(p', s, p) = -G(p', s, p) - \int_0^{q_{\text{max}}} \frac{dq q^2}{(2\pi)^2 \omega_q} G(p', s, q) M_2(q, s) d(q, s, p) \]

Integration kernel  

\[ \text{Solution} \]

Inhomogeneous term  

Homogeneous term

\[ \phi \quad \phi b \quad 3\phi \quad \text{S} \]

trimer?
**Self-consistency of the deformed contour**

Singularities of the ladder equation

\[
d(p', s, p) = -G(p', s, p) - \int_0^{q_{\text{max}}} \frac{dq q^2}{(2\pi)^2 \omega_q} G(p', s, q) \mathcal{M}_2(q, s) d(q, s, p)
\]

Extrapolation to the desired momentum \( p' \rightarrow q_{\phi b} \)

Addition of discontinuity to the integration kernel \( \Delta(p', s, p) \propto 2\pi i \)

\[ s/m^2 = 8.6 - 0.05i \]
Efimov phenomenon in relativistic scattering

Trajectories of the first three trimers: binding momentum $\kappa \propto \sqrt{E}$ vs $1/ma$

At unitarity, ratios of binding energies approach Efimov's constant $\lambda^2=515$

- Emergence of virtual state
- Two- and three-body threshold
- Value at unitarity
Beyond the three-body cut

Collision of the integration contour with:
- pole of the pair amplitude \( \rightarrow \) dimer-particle cut
- unitarity branch cut of the pair \( \rightarrow \) three-body cut

\[
- \int_0^{q_{\text{max}}} \frac{dq}{(2\pi)^2 \omega_q} G(p', s, q) M_2(q, s) \, d(q, s, p)
\]

\[
s/m^2 = 9.01 - 0.01i
\]

Accessing the 2nd Riemann sheet

Accessing the 3rd Riemann sheet
Three-body resonances (2nd sheet)

Evolution of Efimov states in relativistic scattering theory
Dawid, Islam, Briceño, Jackura, in preparation

Poles travel on loop-like trajectories and "disappear" (Missing poles problem)
Three-body resonances (2nd sheet)

\[ m_a = -16 \]
\[ m_a = -12 \]
\[ m_a = -10 \]

\[ m_a = -8 \]
\[ m_a = -6 \]
\[ m_a = -4 \]
\[ m_a = -2 \]

Evolution of Efimov states in relativistic scattering theory
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Poles travel on loop-like trajectories and "disappear" (Missing poles problem)
Summary

- Development of the formalism
  - Consistency check of the three-body framework
  - Breakdown of the Lüscher QC
  - Amplitude in the complex plane
  - Trajectories of Efimov trimers

- Future work
  - Higher partial waves
  - Application to realistic states
  - Multi-body, multi-channel formalism

Talk by M. Islam

Talk by S. Sharpe
Thank you
Personal choice of relevant literature

**Three-body scattering and quantization conditions from S matrix unitarity**
A. Jackura, arXiv:2208.10587 (2022)

**Off- and On-Shell Analyticity of Three-Particle Scattering amplitudes**
D. Brayshaw, Phys. Rev. 176, 5 (1968)

**S-matrix pole trajectory in the three-neutron model**

**S-matrix pole trajectory of the three-body system**

**Fate of the Tetraquark Candidate \(Z_{c}(3900)\) from Lattice QCD**
Ikeda et al. (HAL QCD), Phys. Rev. Lett. 117 (2016) 24, 242001

**The Three-Boson System with Short-Range Interactions**

**Energy-Dependent \(\pi\pi\pi\) Scattering Amplitude from QCD**
Hansen et al. (HadSpec), Phys. Rev. Lett. 126 (2021), 012001

Resonance, Ariel Davis for Quanta Magazine
Basics of scattering theory

Properties of the S matrix
- Analyticity (causality)
- Unitarity (probability conservation)
- Poincaré symmetry (frame independence)
- Crossing symmetry (particles—antiparticles)
- Internal symmetries (charge, isospin, G-parity)

Analyticity on the first Riemann sheet
- Bound-states & resonances correspond to poles
- Branch cuts correspond to open channels

K-matrix parametrization
\[ M_\ell(s) = \frac{1}{K_\ell^{-1}(s) - i\rho(s)} \]

Phase shift
\[ K_\ell^{-1}(s) = \frac{q^*}{8\pi\sqrt{s}} \cot(\delta_\ell(s)) \]
Basics of scattering theory

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Phase shift

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Brief introduction to analytic continuation

\[ I(z) = \int_{C(w_1,w_2)} f(w, z) \, dw \]

\[ I(x) = \int_{-1}^{1} \frac{dw}{w - x} = \log \left( \frac{x - 1}{x + 1} \right) \]

\[ x \in (-\infty, -1) \cup (1, \infty) \]
Unphysical left-hand cut

- Complex conjugation of $s$ reflects the cuts in $p'$
- We add discontinuity to the inhomogeneous part
- Check self-consistency of the contour

Analytic continuation of the relativistic three-body amplitudes
Dawid, Islam, Briceño, arXiv:2303.04394
Unphysical left-hand cut

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- Check self-consistency of the contour

Text box with diagrams and equations:

$\frac{s}{m^2} = 8.6 + 0.05i$

Analytic continuation of the relativistic three-body amplitudes
Dawid, Islam, Briceno, arXiv:2303.04394
Rotating away the unphysical cut ($ma=16$)
Complex-plane amplitude \((ma=16)\)

\[
\begin{align*}
\mathcal{M}_\varphi b(s) &= \mathcal{M}_\varphi b(s) \\
\mathcal{M}_\varphi b(s) &= d(p', s, p)
\end{align*}
\]
Complex-plane amplitude \((\text{ma}=16)\)

\[
\mathcal{M}_{\varphi b}(s) = \mathcal{M}_{\varphi b}(s)
\]

No LSZ factorization

Amplitude on the second sheet

\[
\mathcal{M}_{\varphi b}^{\text{II}}(s) = \frac{\mathcal{M}_{\varphi b}(s)}{1 + 2i\rho_{\varphi b}(s)\mathcal{M}_{\varphi b}(s)}
\]

(or contour deformation)
Circular cut

- For real $s$ the cut closes and forms a circle
- The S-wave projection

$$G(p', s, p) \propto \int_{-1}^{1} dx \frac{1}{z(p', s, p) + x}$$

Contour deformation in $x$ opens the circle (Deformation of the cuts in the OPE)

$$G(p', s, p) \propto \log \left( \frac{z(p', s, p) + 1}{z(p', s, p) - 1} \right)$$

Position of the cuts is arbitrary
Position of branch points is not
• For real $s$ the cut closes and forms a circle
• The S-wave projection

\[
G(p', s, p) \propto \int_{-1}^{1} dx \frac{1}{z(p', s, p) + x}
\]

Contour deformation in $x$ opens the circle
(Deformation of the cuts in the OPE)

\[
G(p', s, p) \propto \log \left( \frac{z(p', s, p) + 1}{z(p', s, p) - 1} \right)
\]

Position of the cuts is arbitrary
Position of branch points is not
Analytic continuation of the integral equation

- Reflection of the cuts in $p'$

\[ G(p', s, p) = G^*(p'^*, s^*, p^*) \]

\[ G(p', s^*, p_{\text{pole}}) = G^*(p'^*, s, -p_{\text{pole}}^*) \]

- Adding discontinuity to OPE

\[ \Delta(p', s, p) = -\frac{H(p'p)}{4p'p} (2\pi i) \]
Amplitude below the threshold

$ma = 2$

$ma = 6$

$ma = 16$

Analytic continuation of the relativistic three-body amplitudes
Dawid, Islam, Briceño, arXiv:2303.04394
Analytic continuation through the three-body cut

Collision of the integration contour with:
- pole of the pair amplitude → dimer-particle cut
- unitarity branch cut of the pair → three-body cut

\[ \text{Im} \left( \frac{s}{m^2} \right) = -0.2 \]

\[ \text{Im} \left( \frac{s}{m^2} \right) = +0.2 \]
Kernel singularities in the invariants plane

\[ P_{p'}^2 = \sigma_{p'}, \quad P_p = \sigma_p = m_b^2 \]
Numerical convergence
Vertex functions

\[ |\Gamma(k)|^2 \]

Homogeneous equation

\[
(\text{amplitude}) \propto - \frac{\Gamma(p')\Gamma^*(p)}{s - s_b}
\]

\[
\Gamma(p) = -M_2(p) \int_0^{q_{\text{max}}} dq q^2 \frac{G(p, s, q) \Gamma(q)}{(2\pi)^2 \omega_q}
\]

Non-relativistic prediction

\[
|\Gamma_{\text{NR}}(k)|^2 = |c|A^2 \frac{256\pi^{3/2}}{3^{1/4}} \frac{m^2 \kappa_{\text{NR}}^2}{k^2 (\kappa_{\text{NR}}^2 + 3k^2/4)} \frac{\sin^2 \left( s_0 \sinh^{-1} \left( \sqrt{3k/2} \kappa_{\text{NR}} \right) \right)}{\sinh^2 (\pi s_0/2)}
\]

Hansen, Sharpe, Phys. Rev. D 95, 034501 (2017)
Efimov phenomenon

\[ E_n \propto -\left( e^{-\frac{2\pi}{s_0}} \right)^n \]

NREFT three-body equation

\[ \mathcal{L} = \psi^\dagger \left( i\partial_0 + \frac{\vec{v}^2}{2m} \right) \psi + \Delta T^\dagger T - \frac{g}{\sqrt{2}} \left( T^\dagger \psi \psi + \text{h.c.} \right) + hT^\dagger T \psi^\dagger \psi + \ldots \]
REFT three-body formalism

\[ C_{L}(E, \tilde{P}) = \cdots + \frac{D^{u}_{u}}{P^{u}_{u}} + \frac{M^{u}_{df, u}}{P^{u}_{df, u}} + \cdots \]

\[ \text{det} \left[ K_{df, 3}(s) + F_{3}(s, P, L)^{-1} \right] = 0 \]

Infinite volume integral equations:

\[ M_{3}^{(u, u)} = D^{(u, u)} + M_{df, 3}^{(u, u)} \]
Riemann sheets
## Properties of some three-body states

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<tbody>
<tr>
<td>$\omega(782)$</td>
<td>0 (1--)</td>
<td>782.66 ± 0.13</td>
<td>8.68 ± 0.13</td>
<td>$\pi^+\pi^-\pi^0$</td>
<td>(89.2 ± 0.7)%</td>
</tr>
<tr>
<td>$\eta'(958)$</td>
<td>0 (0--)</td>
<td>957.78 ± 0.06</td>
<td>0.188 ± 0.006</td>
<td>$\pi^+\pi^-\eta$</td>
<td>(42.5 ± 0.5)%</td>
</tr>
<tr>
<td>$a(1260)$</td>
<td>1 (1++)</td>
<td>1230 ± 40</td>
<td>420 ± 35</td>
<td>3$\pi$</td>
<td>seen</td>
</tr>
<tr>
<td>$N(1440)$</td>
<td>$\frac{1}{2}$ (1/2)</td>
<td>1440 ± 30</td>
<td>350 ± 100</td>
<td>$N\pi\pi$</td>
<td>(17 − 50)%</td>
</tr>
<tr>
<td>$\pi_1(1600)$</td>
<td>1 (1--)</td>
<td>1661 ± 15</td>
<td>240 ± 50</td>
<td>3$\pi$</td>
<td>seen</td>
</tr>
<tr>
<td>$\chi_c(3872)$</td>
<td>0 (1++)</td>
<td>3871.65 ± 0.06</td>
<td>1.19 ± 0.21</td>
<td>$D^0\bar{D}^0\pi^0$</td>
<td>(40 ± 20)%</td>
</tr>
</tbody>
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