Canonical Momenta in Digitized SU(2) Lattice Gauge Theory

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Hamiltonian of lattice gauge theories

- [Kogut and Susskind, 10.1103/PhysRevD.11.395 (1975)]

$$H = \frac{g_0^2}{4} \sum_{\boldsymbol{x},c,k} \left(L_{c,k}^2(\boldsymbol{x}) + R_{c,k}^2(\boldsymbol{x}) \right) - \frac{1}{2g_0^2} \sum_{\boldsymbol{x},k<l} \mathrm{Tr}\,\mathrm{Re} P_{kl}(\boldsymbol{x})$$

- g_0 bare gauge coupling
- $\boldsymbol{x}$ coordinate on the spacial lattice, k direction, c color index
- Plaquette operator

$$P_{kl}(\boldsymbol{x}) = U_k(\boldsymbol{x}) U_l(\boldsymbol{x} + \hat{k}) U_k^\dagger(\boldsymbol{x} + \hat{l}) U_l^\dagger(\boldsymbol{x})$$

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- We are investigating the formulation where the magnetic part is diagonal [Jakobs et al., 10.1140/epjc/s10052-023-11829-9, arXiv:2304.02322 (2023)]

SU(2) construction

- We need a discretization of SU(2)
- Choose a finite set of elements
- For each point we define a state $|U\rangle \in \mathcal{H}$ and an SU(2) matrix

$$U = \begin{pmatrix} \hat{y}_0 + i\hat{y}_1 & \hat{y}_2 + i\hat{y}_3 \\ -\hat{y}_2 + i\hat{y}_3 & \hat{y}_0 - i\hat{y}_1 \end{pmatrix}, \qquad \hat{y}_3^2 = 1 - \sum_{i=0}^{2} \hat{y}_i^2$$

- The elements y_i are operators on $\mathcal{H}$

$$\hat{y}_i |U\rangle = y_i |U\rangle$$

- This defines the action of $U : \mathcal{H} \to \mathcal{H}$
- Alternatively we can work with the parametrization $U = e^{i\vec{\alpha}\cdot\vec{\tau}}$

Commutation relations

- Given U the momenta are defined via the commutation relations

$$[L_c, U_{mn}] = (\tau_c)_{mj} U_{jn} \qquad [R_c, U_{mn}] = U_{mj} (\tau_c)_{jn}$$

- τ_c the generator of SU(2)
- Moreover the group structure (similar for R)

$$[L_a, L_b] = f_{abc} L_c$$

- f_{abc} the structure constants
- In the full continuum SU(2) these are fulfilled by

$$L_c f(U) = -i \frac{d}{d\beta} \left(e^{i\beta\tau_c} U \right) \Big|_{\beta=0}$$

$$R_c f(U) = -i \frac{d}{d\beta} \left(U e^{i\beta\tau_c} \right) \Big|_{\beta=0}$$

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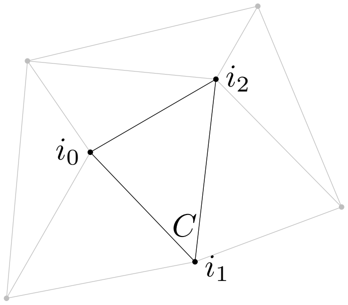
- How to define L and R in a finite subset of SU(2)?

Discretised L construction

- Perform a Delaunay triangulation of the points in SU(2) [B. N. Delaunay (1934)]
- The result is a set of simplices $\mathcal{C} = \{i_0, i_1, i_2, i_3\}$



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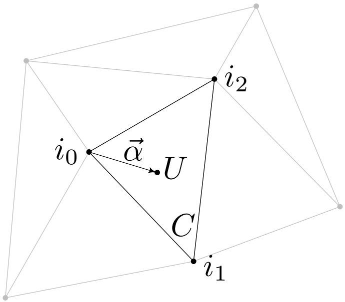
- Every U can be written as $U = e^{i\vec{\alpha}\cdot\vec{\tau}} U_{i_0}$
- Every function can be approximated within $\mathcal{C}$ as

$$f(U) = f(U_{i_0}) + \vec{\nabla} f_{i_0} \cdot \vec{\alpha} + \mathcal{O}(\alpha^2)$$



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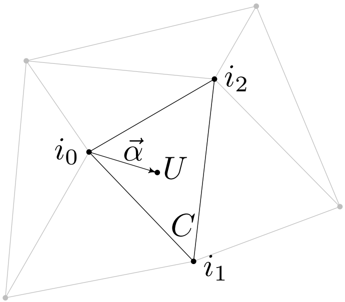
- Imposing that the linear approximation reproduces the function at the vertices

$$\begin{pmatrix} \vec{\alpha}_1^T \\ \vec{\alpha}_2^T \\ \vec{\alpha}_3^T \end{pmatrix} \vec{\nabla} f_{i_0} = \begin{pmatrix} \tilde{f}(i_1) - \tilde{f}(i_0) \\ \tilde{f}(i_2) - \tilde{f}(i_0) \\ \tilde{f}(i_3) - \tilde{f}(i_0) \end{pmatrix}$$

that defines $\vec{L} = -i\vec{\nabla}$

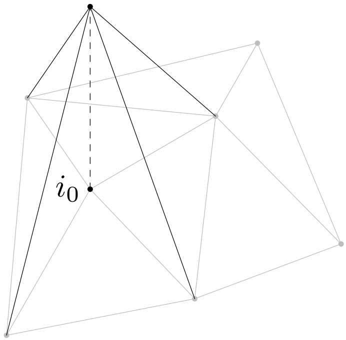


- We average overall simplices containing the point i to have a better estimate
- A similar construction can be done for R writing the link as $U = U_{i_0} e^{i\vec{\alpha}\cdot\vec{\tau}}$
- In the Hamiltonian we need L^2
- Simply taking the square is unlikely to give good approximation since we taking linear aproximation to construct L
- We use the Finite Element method to approximate L^2

Discretised L^2 construction

- Define hat functions on the triangulated lattice: $\phi_{i_0}(U_i) = \delta_{i_0,i}$ and linear piece-wise in between points
- Project the Laplace equation on those function $\Delta u = f \implies \langle \Delta u, \phi_i \rangle = \langle f, \phi_i \rangle$

$$\langle \Delta u, \phi_i \rangle = \sum_C \int_C \Delta u \, \phi_i dV$$



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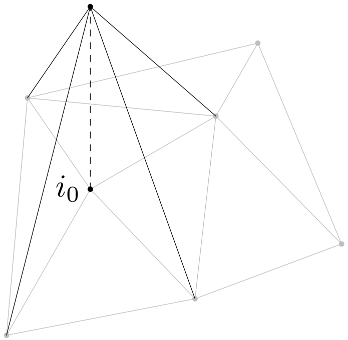

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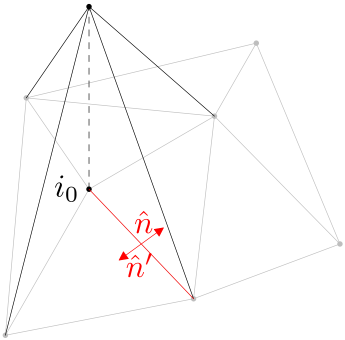

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- Expand $u = \sum u_j \phi_j$

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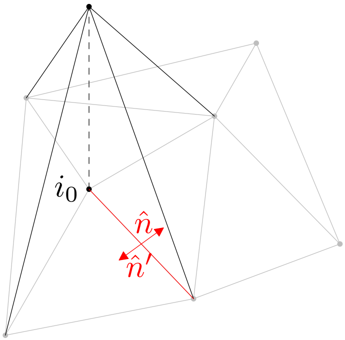

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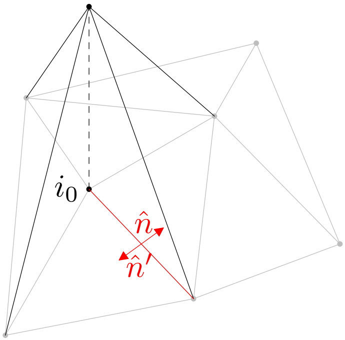

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- if i and j are not connected by a simplex $S_{ij} = 0 \implies S$ is a sparse matrix

- The r.h.s approximated as

$$\langle f, \phi_i \rangle = \sum_C \int f \phi_i = v_i f_i \quad \text{with} \quad v_i = \sum_{C|i \in C} \frac{Vol(C)}{4}$$

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- With this construction $L^2 = R^2$ and local
- L^2 a local operator

Linear Discretization

- We try a list of partitioning of SU(2) [Jakobs et al., 10.1140/epjc/s10052-023-11829-9, arXiv:2304.02322 (2023)]
  - generalizable to SU(3)
  - Asymptotically isotropic in the group
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- Here we present the so-called linear partitioning

$$L_m := \left\{ \frac{1}{M}(y_0, y_1, y_2, y_3) \,\middle|\, \sum_{i=0}^{3} |y_i| = m,\ y_i \in \mathbb{Z} \right\}, \qquad U = \frac{1}{M}\begin{pmatrix} y_0 + iy_1 & y_2 + iy_3 \\ -y_2 + iy_3 & y_0 - iy_1 \end{pmatrix}$$

$$M := \sqrt{\sum_{i=0}^{3} j_i^2}\,.$$

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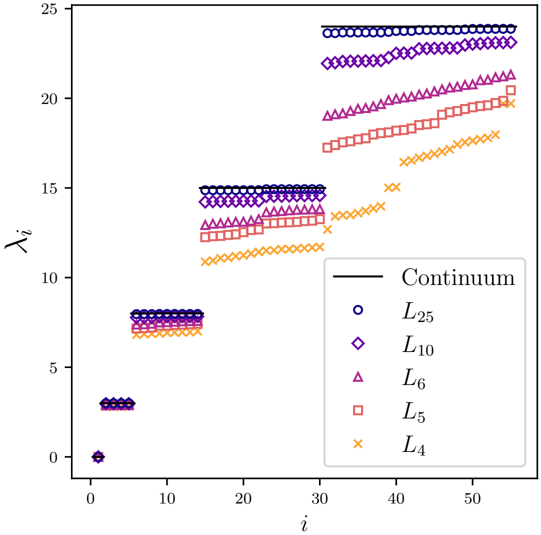
- Number of elements $\propto m^3$
- Mean distance $\propto \frac{1}{m}$

Spectrum of L^2



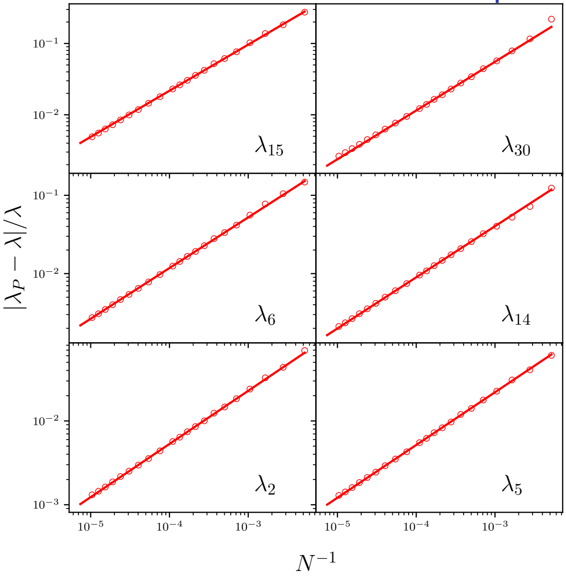
- In the continuum

$$\lambda = \{J(J+2)\,, \quad J = 0, 1, 2, \ldots\}$$

  multiplicity $(J+1)^2$

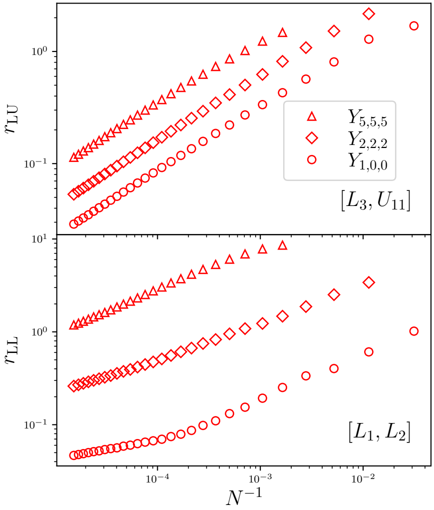
- The low energy spectrum is approaching the continuum with $m \to \infty$

Spectrum of L^2



- λ_P eigenvalue of the discretized L^2
- Fit $(\lambda_P - \lambda)/\lambda = a\left(\frac{1}{N}\right)^b$
- a in $[1, 10]$, $b \sim 0.6$

Commutator $[L_i, L_j]$ and $[L_i, U]$



- Eigenfunction of L^2 4d-spherical harmonics Y_{J,l_1,l_2}
- Evaluate the commutator

$$w_{LU} = ([L_a, U_{jl}] - (\tau_a)_{ji} U_{il}) \cdot Y_{J,l_1,l_2}$$
$$w_{LL} = ([L_a, L_b] + 2i\epsilon_{abc} L_c) \cdot Y_{J,l_1,l_2}$$

- mean deviation weighted by barycentric cell volume $v(i) = \sum_{C|i \in C} \mathrm{Vol}(C)/4$

$$r_{LU} = \sum_i v(i) |w_{LU}(i)|$$
$$r_{LL} = \sum_i v(i) |w_{LL}(i)|$$

1+1D SU(2) with fermions

- Hamiltonian

$$\hat{H} = \mu \sum_{\boldsymbol{x}} \sum_{c=1}^{2} (-1)^{\boldsymbol{x}} \phi_{\boldsymbol{x}}^{c\dagger} \phi_{\boldsymbol{x}}^{c} + \frac{1}{2} \sum_{a} \sum_{\boldsymbol{x}} \left[\phi_{\boldsymbol{x}}^{c\dagger} U_{\boldsymbol{x}}^{cc'} \phi_{\boldsymbol{x}+1}^{c'} + \text{H.c.} \right] + \frac{g^2}{2} \sum_{\boldsymbol{x}} L^2(\boldsymbol{x})$$

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$$G_{\boldsymbol{x}}^{a} = L_{\boldsymbol{x}}^{a} - R_{\boldsymbol{x}}^{a} - \frac{1}{2} \phi_{\boldsymbol{x}}^{\dagger} \tau^{a} \phi_{\boldsymbol{x}}$$

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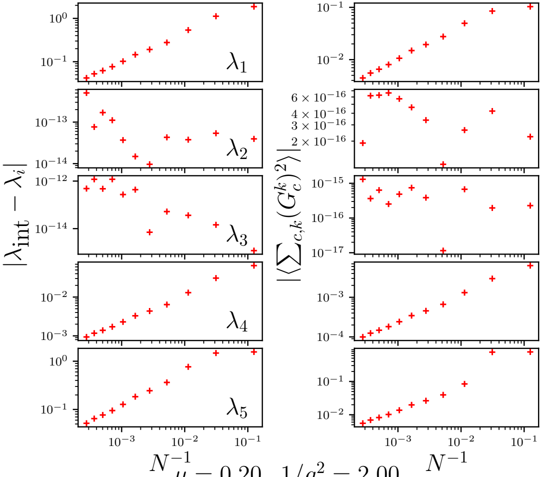
- To ensure that the low spectrum of H satisfy the Gauss's law we add a penalty term in the Hamiltonian

$$\hat{H}_{Penalty} = \kappa \sum_{\boldsymbol{x}} G^2(\boldsymbol{x})$$

Spectrum $\hat{H}$



$\mu = 0.20,\ 1/g^2 = 2.00,$
$\kappa = 10.00,\ 2$ sites

- For N large enough the exact result is recovered
- For finite N also the Gauss's law operator G^a has discretization effects

Summary

- For a given subset of $SU(2)$ we construct the canonical momentum operator L and L^2 in the base where the links U are diagonal
- For the number of point in the $SU(2)$ partition $N \to \infty$:
  - the commutation relations $[L_i, L_j]$ and $[L_i, U]$ are fulfilled
  - The continuum spectrum is recovered in the free and interacting theory
- Construction generalizable to $SU(N)$

Outlook

- Investigation on $1 + 2D$ (ideally $1 + 3D$)
- Investigate the effects of breaking the commutation relation for finite N
- Compare the cost with other approaches
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