Hadronic Structure, Conformal Maps, & Analytic Continuation

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Based on arXiv:2305.16190

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Hadronic Structure

The role of spectral functions

$e^+e^- \to \text{hadrons}$

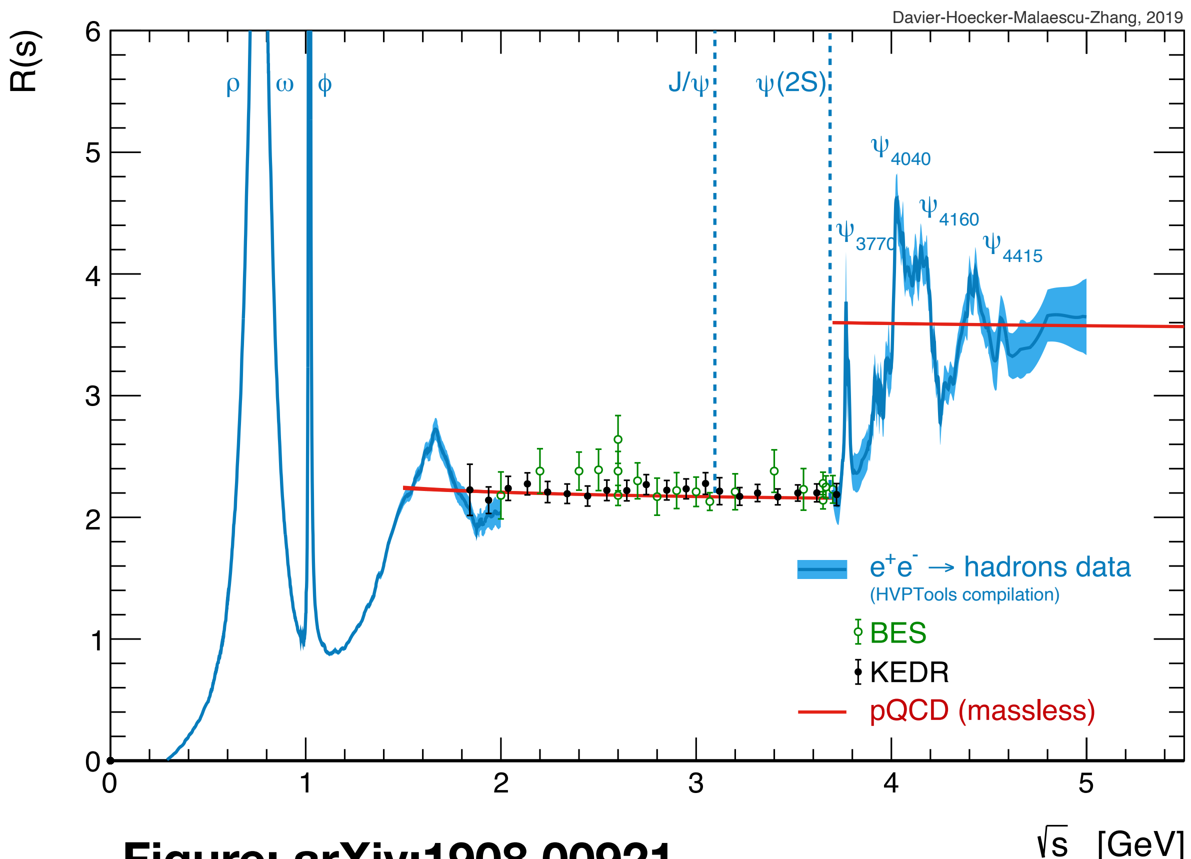
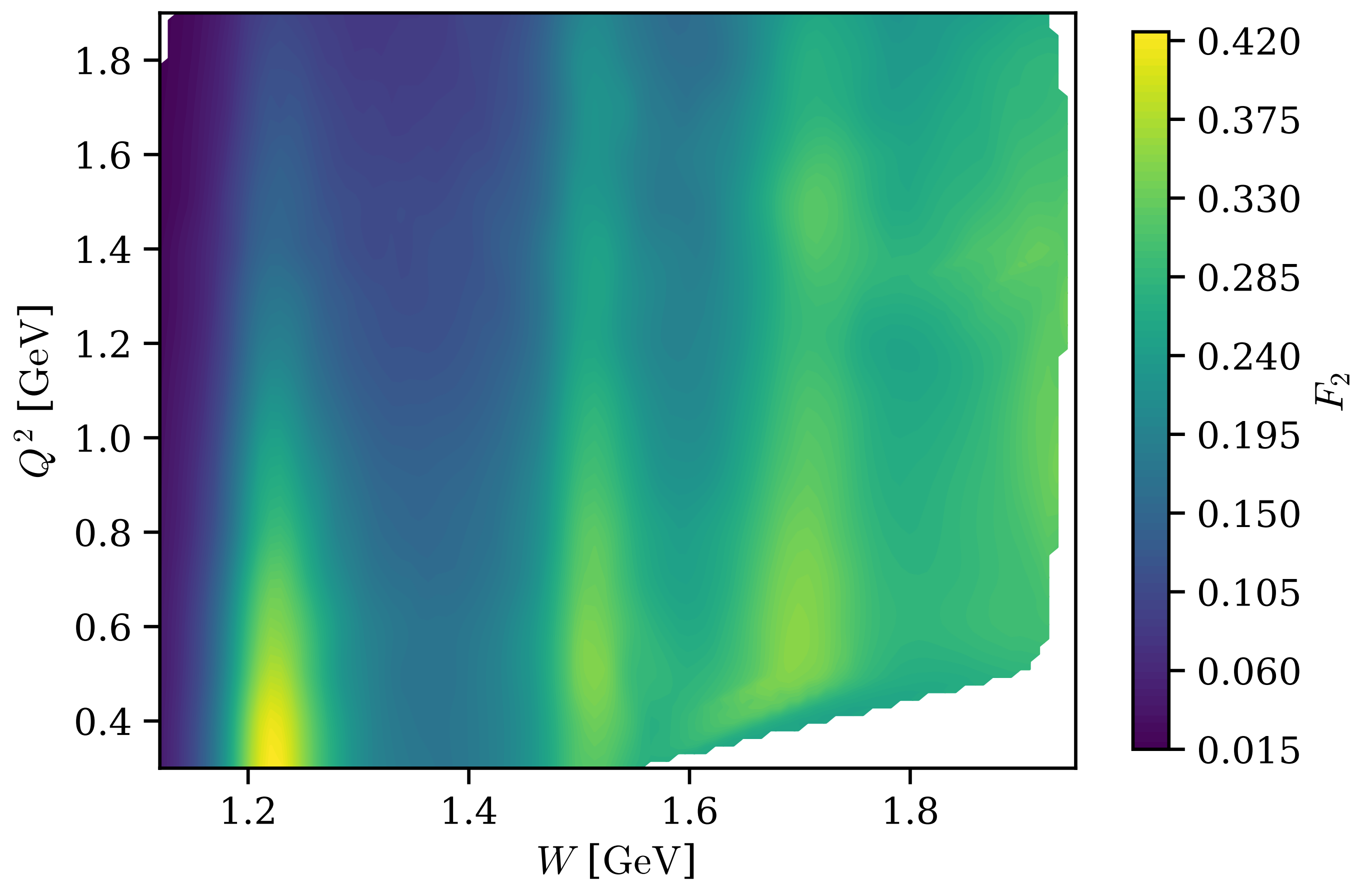


Figure: arXiv:1908.00921
Davier, Hoecker, Malaescu, Zhang

$e^-p^+ \to e^-X$



$$\rho_{\mu\nu}(q) = \frac{1}{2\pi}\int d^4x\, e^{iq\cdot x}\langle \varnothing \,|\, [j_\mu^{\text{EM}}(x), j_\nu^{\text{EM}}(0)] \,|\, \varnothing \rangle$$

$$\rho_{\mu\nu}(q) = (q_\mu q_\nu - q^2 g_{\mu\nu})\rho(q^2)$$

$$W_{\mu\nu}(p,q) = \int \frac{d^4x}{4\pi} e^{iq\cdot x}\langle p \,|\, [j_\mu^{\text{EM}}(x), j_\nu^{\text{EM}}(0)] \,|\, p \rangle$$

$W_{\mu\nu} = F_1 \times$ (Lorentz projectors)

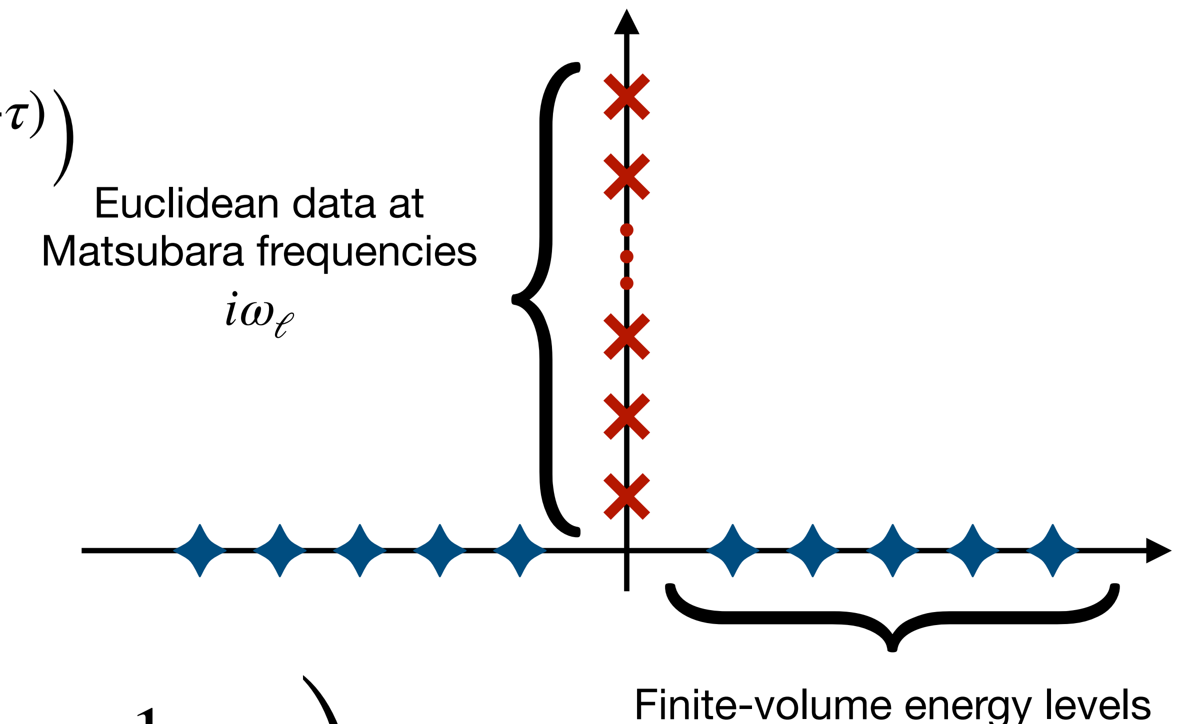
$+F_2 \times$ (Lorentz projectors)

Hadronic Structure

The role of spectral functions

- Lattice QCD calculations occur in Euclidean time:

$$G(\tau) = \sum_n \left| \langle 0 | \mathcal{O} | n \rangle \right|^2 \left(e^{-E_n \tau} + e^{-E_n(\beta-\tau)} \right)$$



- In frequency space (take $a \ll 1$):

$$G(i\omega_\ell) = \int d\tau \, e^{i\omega_\ell \tau} G(\tau)$$

$$= \sum_n \left| \langle 0 | \mathcal{O} | n \rangle \right|^2 \left(\frac{1}{E_n + i\omega_\ell} + \frac{1}{E_n - i\omega_\ell} \right)$$

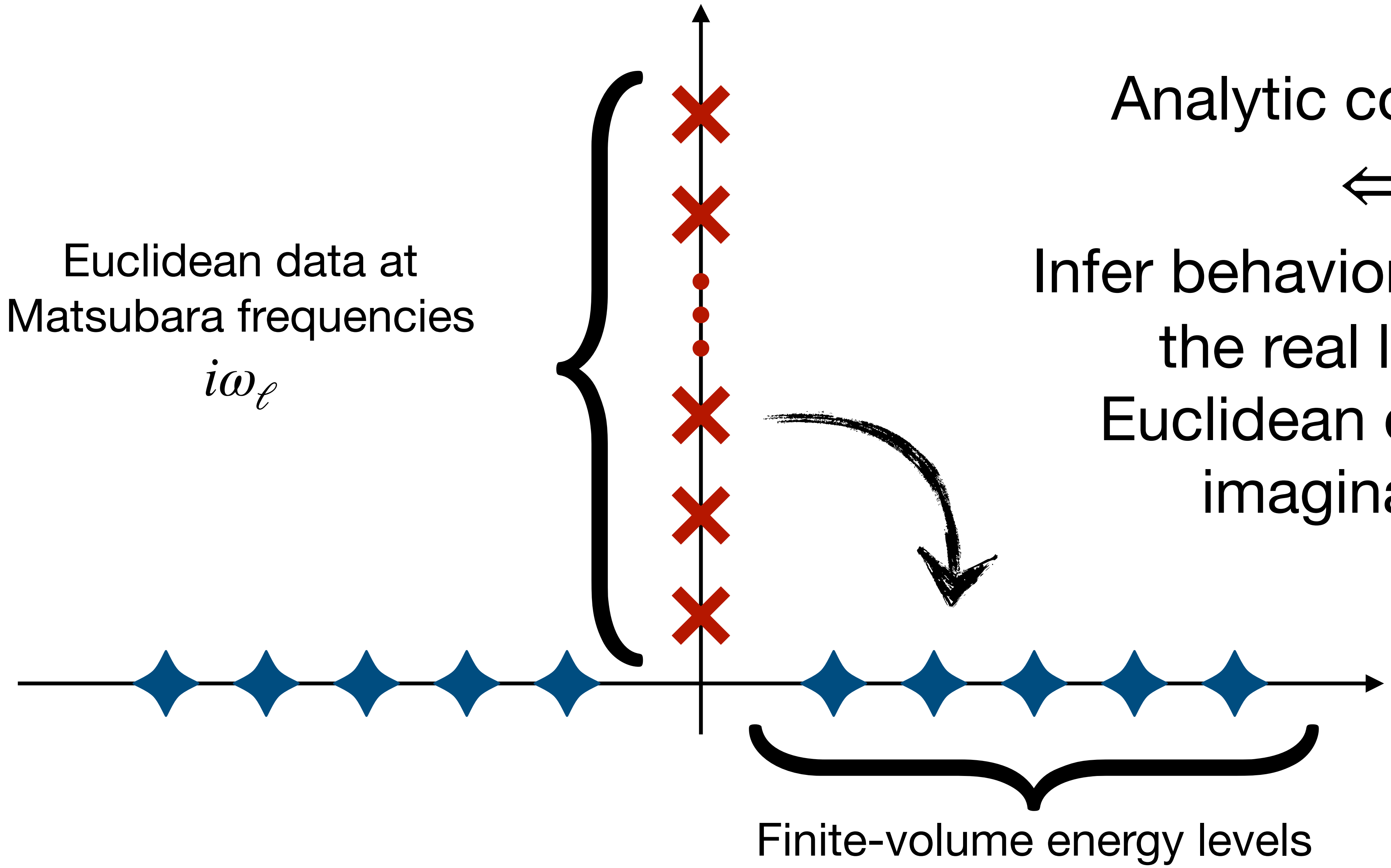
Spectral weight $\Longleftrightarrow$ Residue of pole(s)

- General relation: $\rho(\omega) = \frac{1}{\pi} \mathrm{Im}\, G(\omega)$

$$\frac{1}{\pi} \mathrm{Im} \left(\frac{1}{x + i\epsilon} \right) \longrightarrow \delta(x)$$

Analytic Continuation

Problem description



Analytic continuation

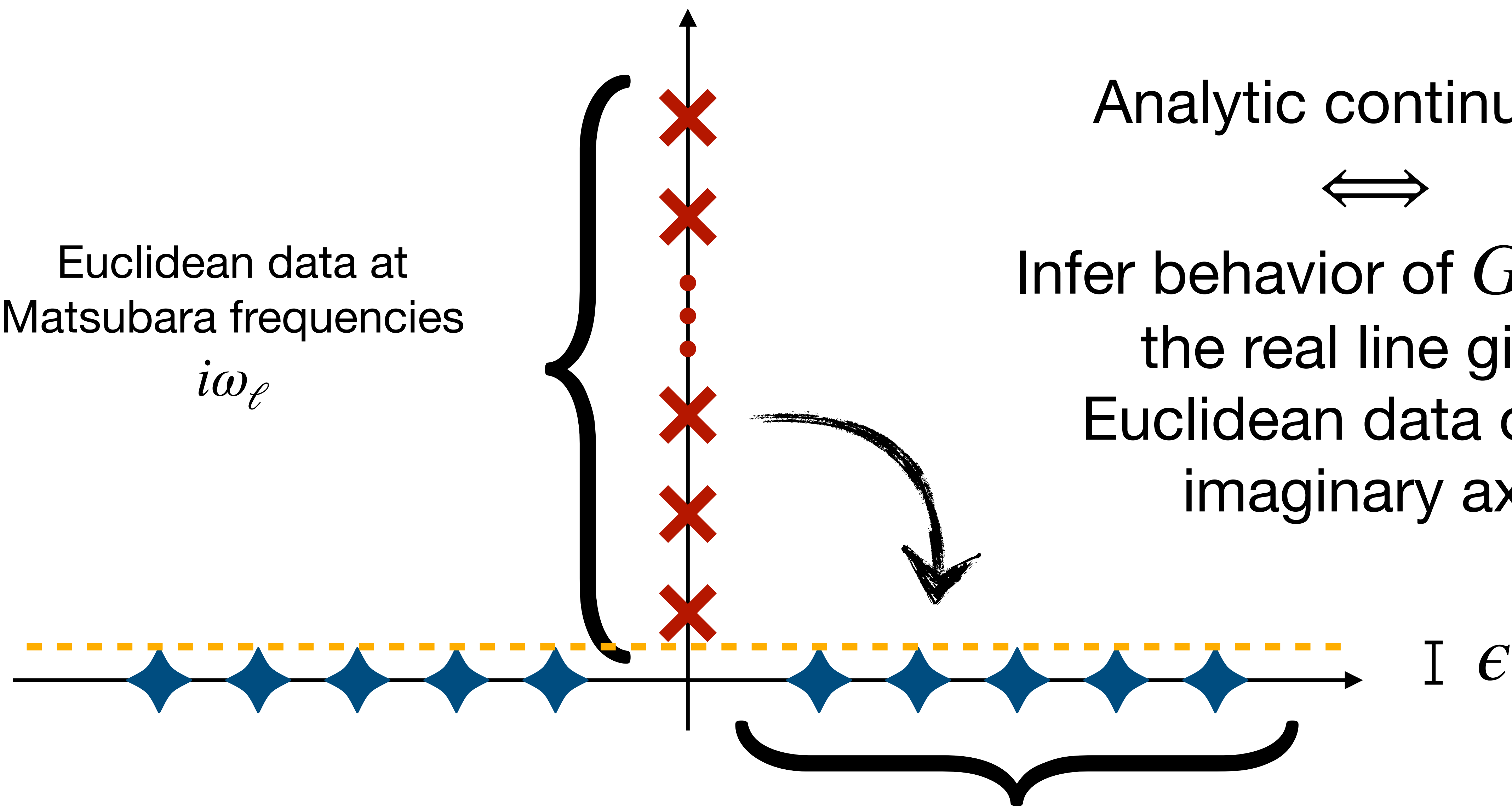
$\Longleftrightarrow$

Infer behavior of $G(z)$ near the real line given Euclidean data on the imaginary axis

Analytic Continuation

Problem description

Euclidean data at
Matsubara frequencies
$i\omega_\ell$



Finite-volume energy levels

Analytic continuation

$\Longleftrightarrow$

Infer behavior of $G(z)$ near
the real line given
Euclidean data on the
imaginary axis

$$\rho^\epsilon(\omega) \equiv \frac{1}{\pi} \mathrm{Im}\, G(\omega + i\epsilon)$$ can be viewed as smeared spectral function in the spirit of Hansen, Meyer, and Robaina [arXiv:1704.08993].

Analytic Continuation

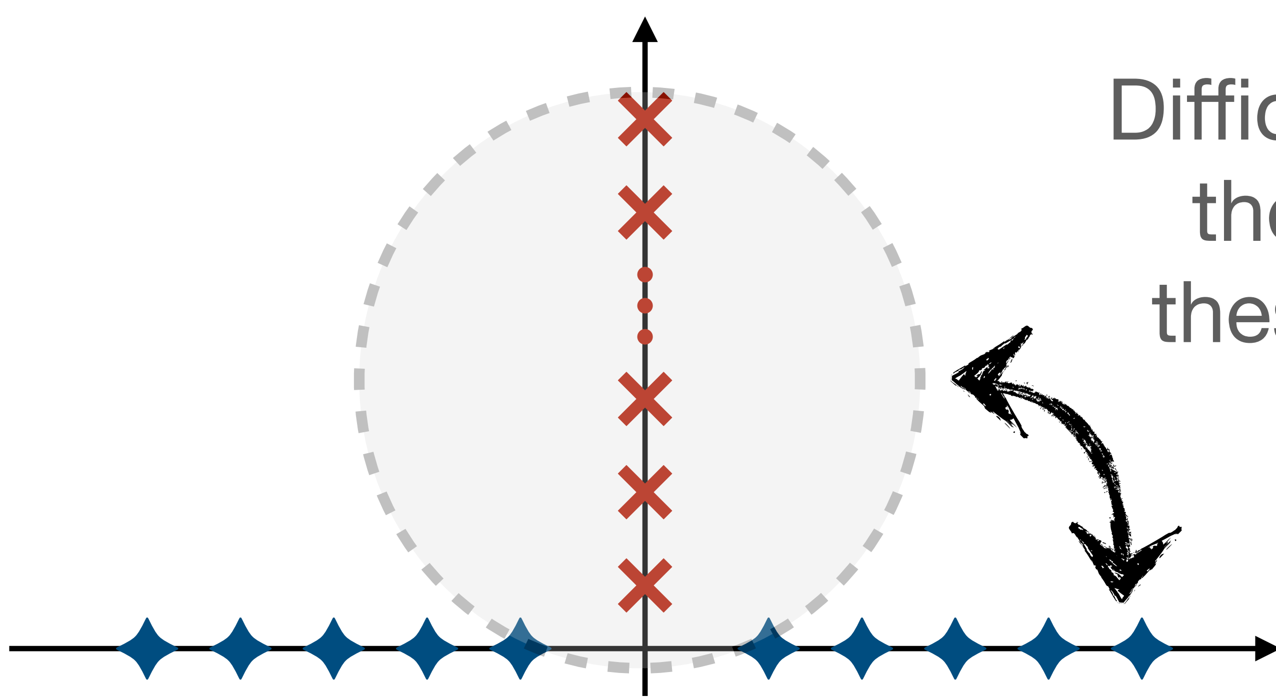
Challenges and hopes

- The received wisdom:
  - **“Excited states decay away exponentially.”** Analytic continuation *to the real line* amounts to an inverse Laplace transform. This problem is ill-defined.
  - **“Little can be said from a finite set of points.”** Complex functions are uniquely defined by analytic continuation given data for the function on a *full open set,* e.g., an open interval on the imaginary axis.
- Causes for cautious optimism:
  - **“Causal Green functions are remarkably rigid.”** Causality dictates that Green functions in QFT must be analytic in the upper half plane, with poles and branch cuts appearing only on the real line.
  - **Carlson’s theorem.** Roughly, *complex functions which do not grow quickly at infinity are uniquely specified by their value on the integers.*

Analytic Continuation

Conformal maps

- Recall: analytic functions are defined by convergent power series in an open set around each nonsingular point
- Radius of convergence is determined by the location of the nearest pole



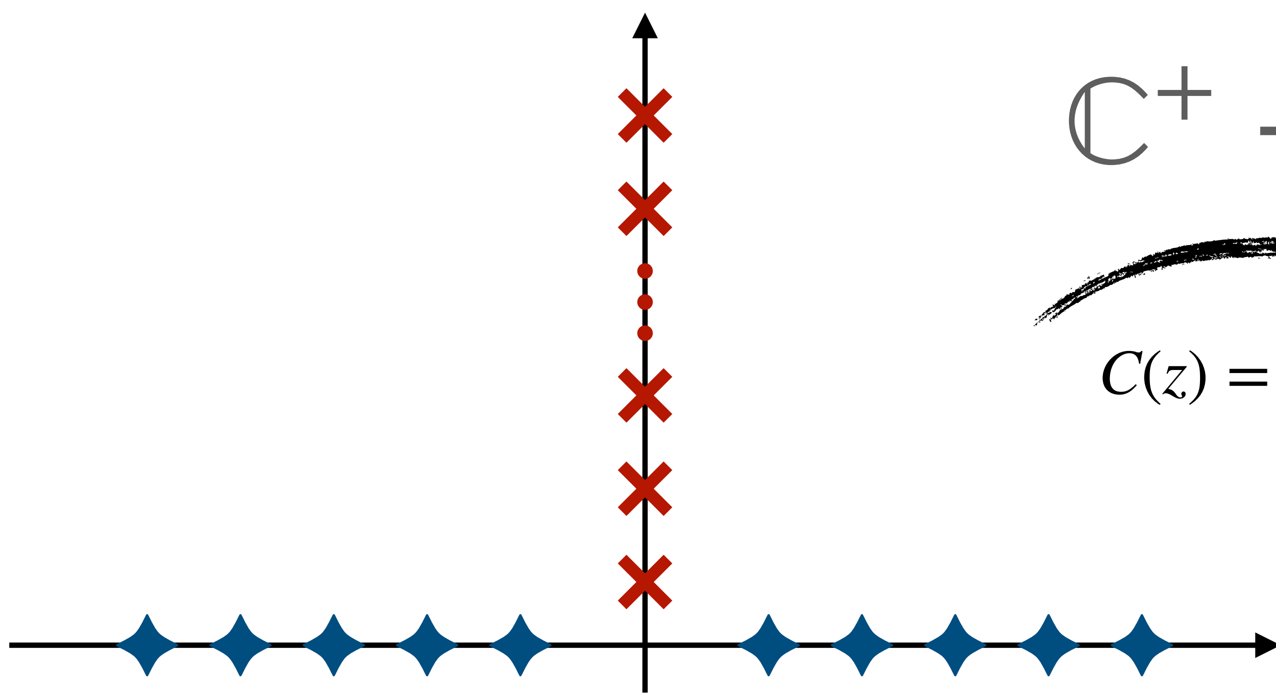
Difficult to "see past the first pole" in these coordinates

Analytic Continuation

Conformal maps

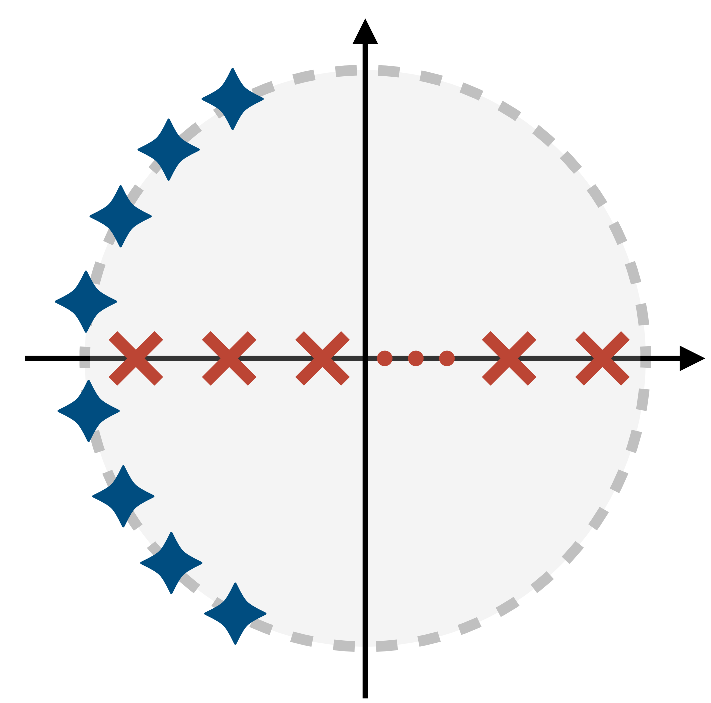
- Recall: analytic functions are defined by convergent power series in an open set around each nonsingular point
- Radius of convergence is determined by the location of the nearest pole

So change coordinates!



$$\mathbb{C}^{+} \to \mathbb{D}$$

$$C(z) = \frac{z - i}{z + i}$$



Analytic Continuation

The technical problem

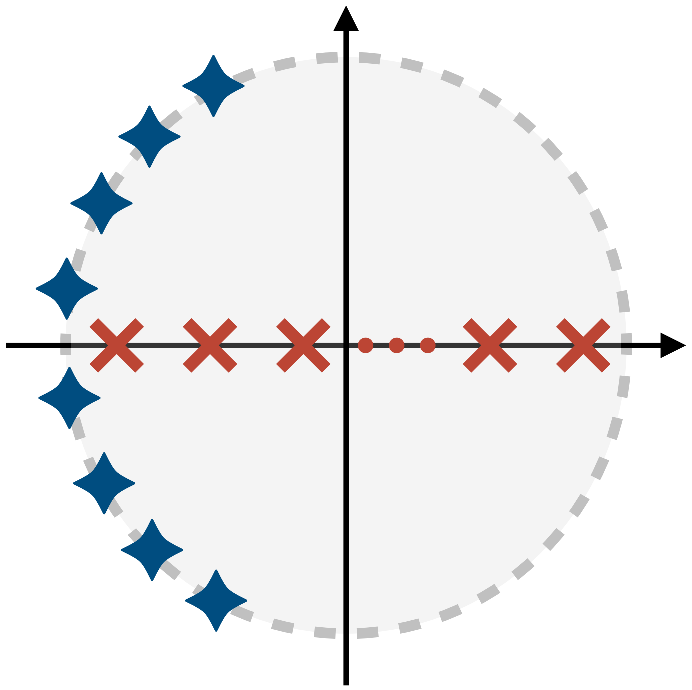
- Recall: analytic functions are defined by convergent power series in an open set around each nonsingular point
- Radius of convergence is determined by the location of the nearest pole
- The Cayley transform maps the problem to the unit disk.
- **Given Euclidean data**

$$\{i\omega_\ell\} \to \zeta_\ell \subset \mathbb{D},$$

$$\{G(i\omega_\ell)\} \mapsto w_\ell \subset \mathbb{D},$$

construct an analytic function $f(\zeta)$

on the disk such that $f(\zeta_\ell) = w_l$**.**



Analytic Continuation

R. Nevanlinna
Ann. Acad. Sci. Fenn. Ser. A 13 (1919)
Ann. Acad. Sci. Fenn. Ser. A 32 (1929)

Nevanlinna's Theorem

- **Theorem (Nevanlinna, 1919/1929):**
  - Any solution to the interpolation problem with N points can be written in the form

$$f(\zeta) = \frac{P_N(\zeta) f_N(\zeta) + Q_N(\zeta)}{R_N(\zeta) f_N(\zeta) + S_N(\zeta)}$$

  where the coefficient functions P_N, Q_N, R_N, S_N are calculable using an inductive formula in terms of the input data $\{\zeta_\ell\}$ and $\{w_\ell\}$and an arbitrary analytic function $f_N(\zeta) : \mathbb{D} \to \mathbb{D}$.

  Derivation: See our preprint [arXiv:2305.16190], which follows modern treatment by mathematician Nicolau [https://mat.uab.cat/~artur/data/nevanlinna-pick.pdf]

- $P_N, Q_N, R_N, S_N \iff$"*Nevanlinna coefficients*"
- Arbitrary function $f_N(\zeta) \iff$ "Freedom to specify further Euclidean data to constrain the interpolating function"
- Applicability to field-theory problems first recognized by condensed-matter theorists Fey, Yeh, and Gull [arXiv:2010.04572]

Analytic Continuation

The full space of solutions

- Question: For fixed N and ζ, **what are the possible values that an interpolating function $f(\zeta)$ can take,** by varying possible values of the arbitrary function $f_N(\zeta) \in \mathbb{D}$?
  - The size of this set $\iff$ ambiguity in the analytic continuation
  - Remarkably, this set can be parameterized explicitly for each, N and $\zeta \in \overline{\mathbb{D}}$

Analytic Continuation

The full space of solutions

- Question: For fixed N and ζ, **what are the possible values that an interpolating function $f(\zeta)$ can take,** by varying possible values of the arbitrary function $f_N(\zeta) \in \mathbb{D}$?

- Answer: The space of possible values is given by the *Wertevorrat* $\Delta_N(\zeta)$, which is the disk of radius $r_N(\zeta)$ and centered at $c_N(\zeta)$.

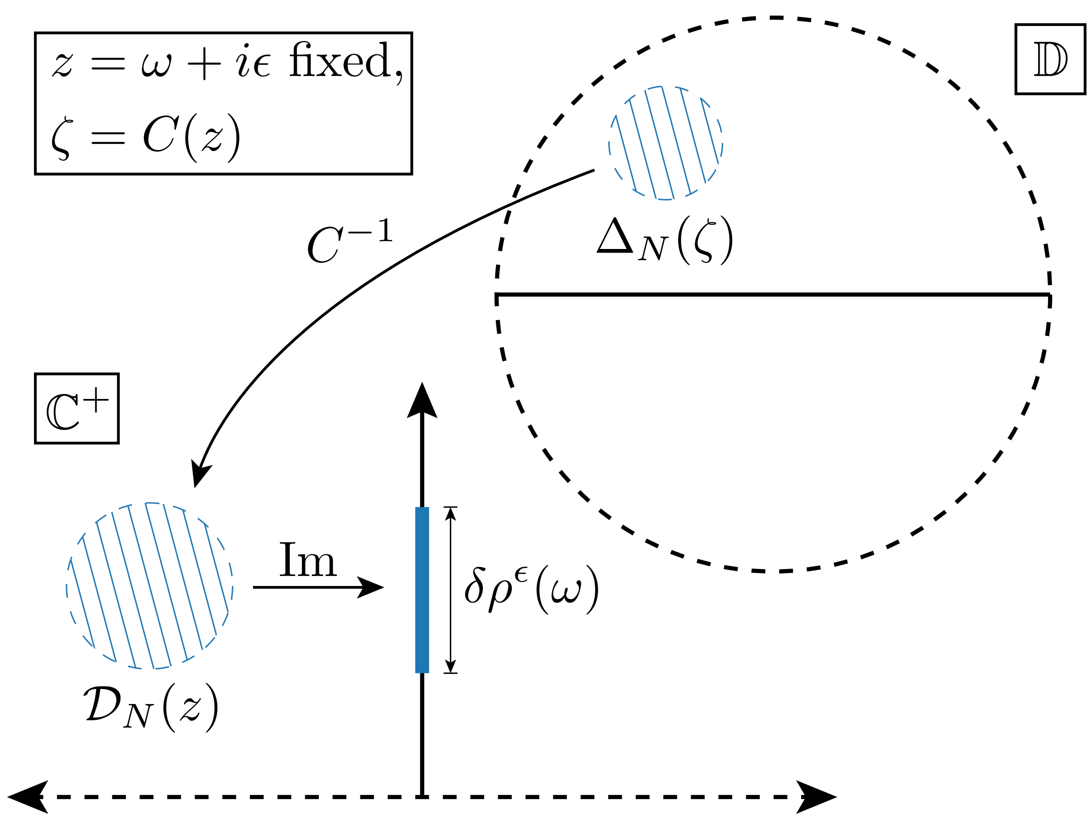
$$c_N = \frac{P_N\overline{(-R_N/S_N)} + Q_N}{R_N\overline{(-R_N/S_N)} + S_N} \qquad r_N = \frac{|P_N S_N - Q_N R_N|}{|S_N|^2 - |R_N|^2}$$

- The Wertevorrat $\Delta_N(\zeta)$ rigorously contains the full infinite family of all possible analytic continuations at each point $\zeta \in \mathbb{D}$.

Analytic Continuation

The Wertevorrat and rigorous bounds on $\rho^{\epsilon}(\omega)$

- Finally we need to map the Wertevorrat back to the upper half plane. Use the inverse Cayley transform $z = C^{-1}(\zeta)$.



$$\rho^{\epsilon}(\omega) = \frac{1}{\pi} \mathrm{Im}\, G(\omega + i\epsilon)$$

$$\delta\rho^{\epsilon}(\omega) = \frac{1}{\pi} \Big[\max \mathrm{Im}\, \partial D_N(\omega + i\epsilon) - \min \mathrm{Im}\, \partial D_N(\omega + i\epsilon)\Big]$$

Analytic Continuation

The Algorithm

1. Start with a Euclidean correlation function $G(t)$
2. Evaluate the Fourier coefficients to obtain $G(i\omega_\ell)$
3. Map the Euclidean data to the unit disk
4. Solve the interpolation problem
    - Evaluate the Nevanlinna coefficients
    - Compute the Wertevorrat
5. Map the Wertevorrat back to the upper half-plane
6. For each point $\omega + i\epsilon$, evaluate the space of possible smeared spectral densities $\delta\rho^\epsilon(\omega)$

Numerical Example

The R-ratio — reconstructing a parameterization

- Bernecker and Meyer [arXiv:1107.4388] give a useful parameterization of R-ratio data

- This parameterization can serve as input for a spectral reconstruction

- Can easily convert:
  $R(s) \iff \rho(\omega) \iff G(i\omega_\ell)$

Formula from beginning of talk

"Laplace transform"

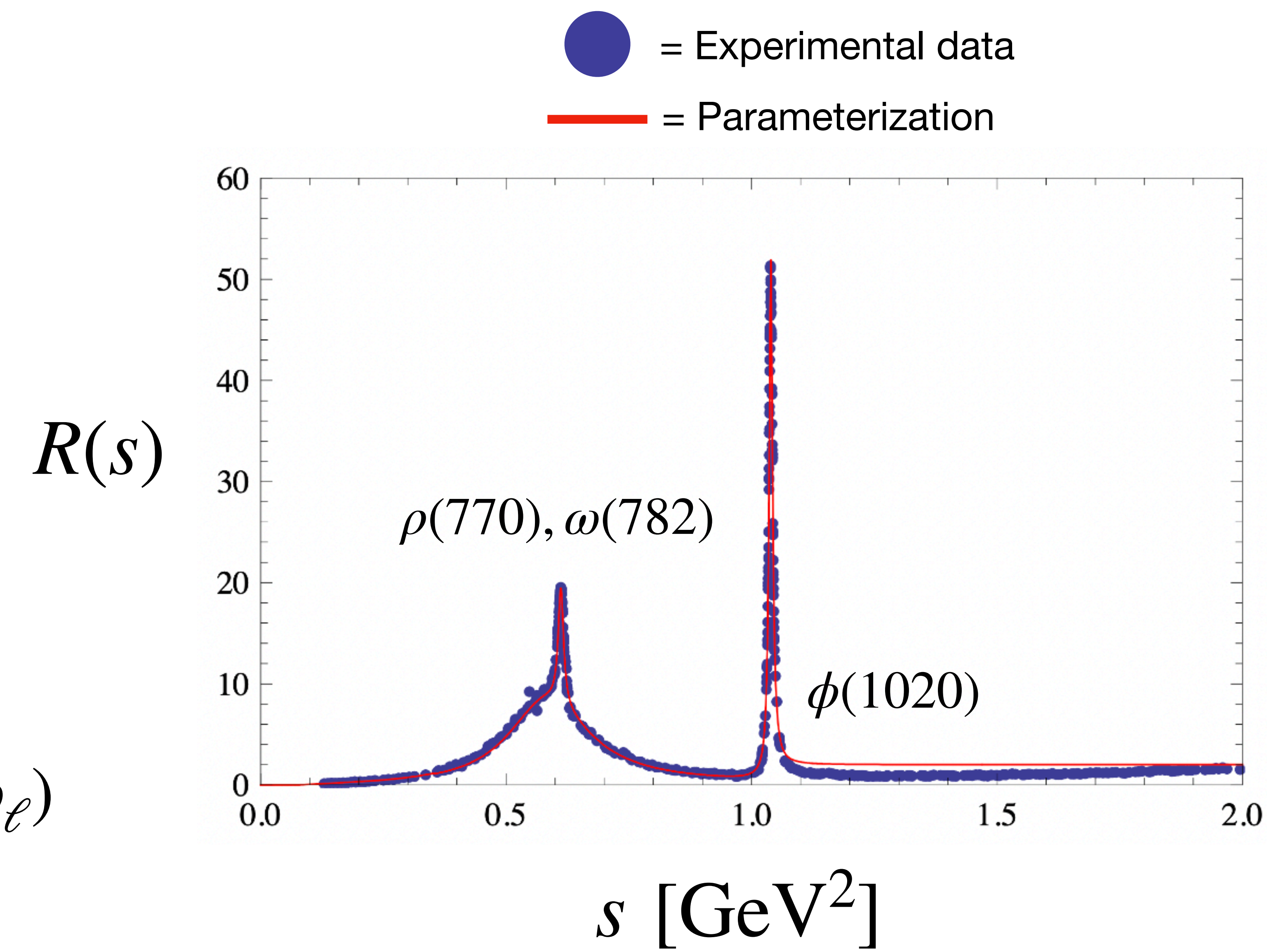


Figure from [arXiv:1107.4388]

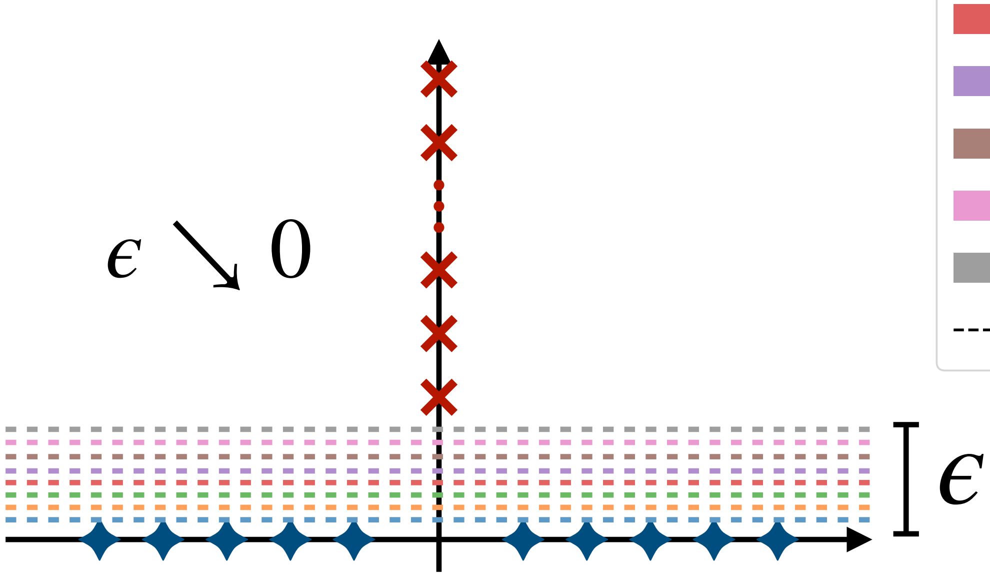
Numerical Example

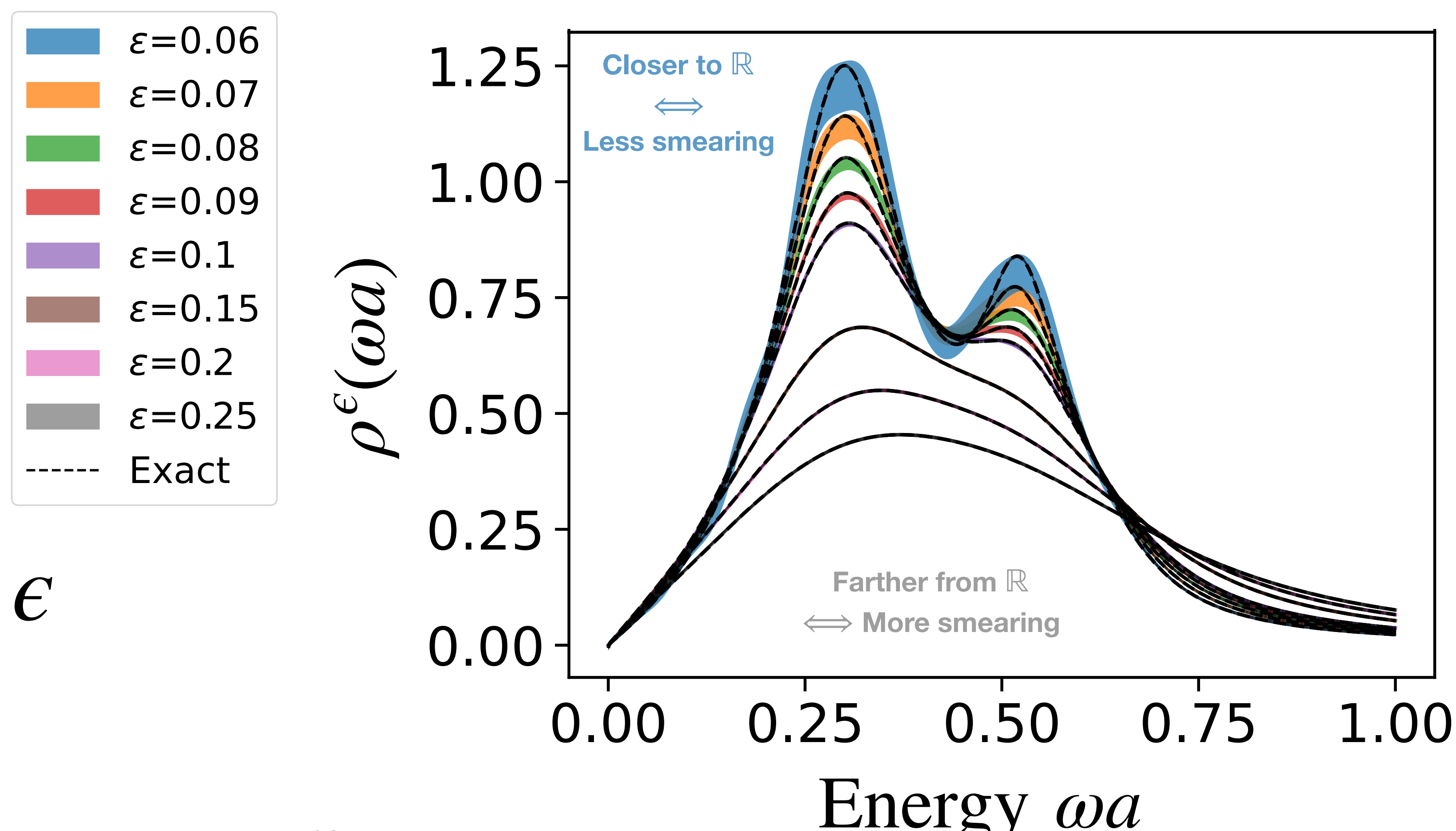
The R-ratio — reconstructing a parameterization

- Energies rescaled to line in unit interval $\Longrightarrow$ lattice units with $a \approx 0.07\,\mathrm{fm}$, so $am_\rho \approx 0.25$
- Euclidean data generated for $\beta = 96$ total points on the imaginary-energy axis

✓ Spectral peaks from $\rho(770)/\omega(782)$ and $\phi(1020)$ cleary visible in reconstructions

✓ Exact answer is contained within the bounding envelope of the Wertevorrat





Summary

- Today's spectral reconstruction algorithm:
  - Leverages a century-old theorem due to Nevanlinna
  - Builds off of work in the condensed matter community by Fey, Yeh, and Gull [arXiv:2010.04572]
  - Applys to diagonal correlation functions of bosonic/fermionic operators
    - Bosonic operators (alternative view by Nogaki and Shinaoka [arXiv:2305.03449])
    - Fermionic operators [arXiv:2010.04572]
- Our method:
  - Admits interpretation as a smeared spectral density in the spirit of Hansen, Meyer, and Robaina [arXiv:1704.08993]
  - Bound errors rigorously with Nevanlinna's *Wertevorrat*
    - Wertevorrat $\Longleftrightarrow$ Full space of functions consistent with the input data and analyticity
- Next steps: understanding interplay with statistical uncertainties
  - Important existing work in this direction by Huang, Gull, and Lin [arXiv:2210.04187]

Backup

The Pick Criterion

G. Pick
Math. Ann. 77, 7 (1915)

Existence of an Interpolating Function

- Question: When does an interpolating function, analytic on $\mathbb{D}$, exist for the input data $\{\zeta_l\},\{w_\ell\}$ such that $f(\zeta_\ell) = w_\ell$ for all ℓ?
- Theorem (Pick, 1915):

  Such a function exists if and only if the Pick matrix

$$\left[\frac{1 - w_i \overline{w}_j}{1 - \zeta_i \overline{\zeta}_j}\right]$$

  is positive semidefinite.

- This condition can fail for noisy data (observed already in arXiv:2010.04572). See also arXiv:2210.04187 for work toward robust reconstructions.

Nevanlinna Coefficients

Concrete Formulae

Convenient notation for manipulating Möbius transformations:

$$\begin{pmatrix} a(\zeta) & b(\zeta) \\ c(\zeta) & d(\zeta) \end{pmatrix} h(\zeta) \equiv \frac{a(\zeta)h(\zeta) + b(\zeta)}{c(\zeta)h(\zeta) + d(\zeta)}$$

The Nevanlinna coefficients:

$$f(\zeta) = U_1(\zeta)U_2(\zeta)\cdots U_N(\zeta)f_N(\zeta)$$

$$\equiv \begin{pmatrix} P_N(\zeta) Q_N(\zeta) \\ R_N(\zeta) S_N(\zeta) \end{pmatrix} f_N(\zeta)$$

Inductive blocks:

$$U_n(\zeta) = \frac{1}{\sqrt{1 - \left|w_n^{(n-1)}\right|^2}} \begin{pmatrix} b_{\zeta_n}(\zeta) & w_n^{(n-1)} \\ \overline{w}_n^{(n-1)} b_{\zeta_n}(\zeta) & 1 \end{pmatrix}$$

Blaschke factors:

$$b_a(\zeta) = \frac{|a|}{a} \frac{a - \zeta}{1 - \bar{a}\zeta}$$

"The nth interpolant f_n evaluated at the mth zero"

$$w_m^{(n)} \equiv f_n(\zeta_m)$$

Analytic Continuation

Computing properties of the *Wertevorrat*

- Question: For fixed N and ζ, **what are the possible values that an interpolating function $f(\zeta)$ can take,** by varying possible values of the arbitrary function $f_N(\zeta) \in \mathbb{D}$?

- Answer: For fixed N and ζ, consider the auxiliary function $T : \mathbb{D} \to \mathbb{D}$

$$T(w) = \frac{P_N w + Q_N}{R_N w + S_N}$$

  - We want to evaluate the image of this map.

  - First, observe that $T(\omega)$ is a Möbius transformation, which maps circles to circles. Therefore $T(\mathbb{D})$ must be also be disk $\Delta_N(\zeta)$.

  - Since $T(-S_N/R_N) = \infty$, the reflection property of Möbius transformations says that disk is centered at the point c_N. A calculation gives the radius r_N.

$$c_N = \frac{P_N\overline{(-R_N/S_N)} + Q_N}{R_N\overline{(-R_N/S_N)} + S_N} \qquad r_N = \frac{|P_N S_N - Q_N R_N|}{|S_N|^2 - |R_N|^2}$$

Bosons and fermions

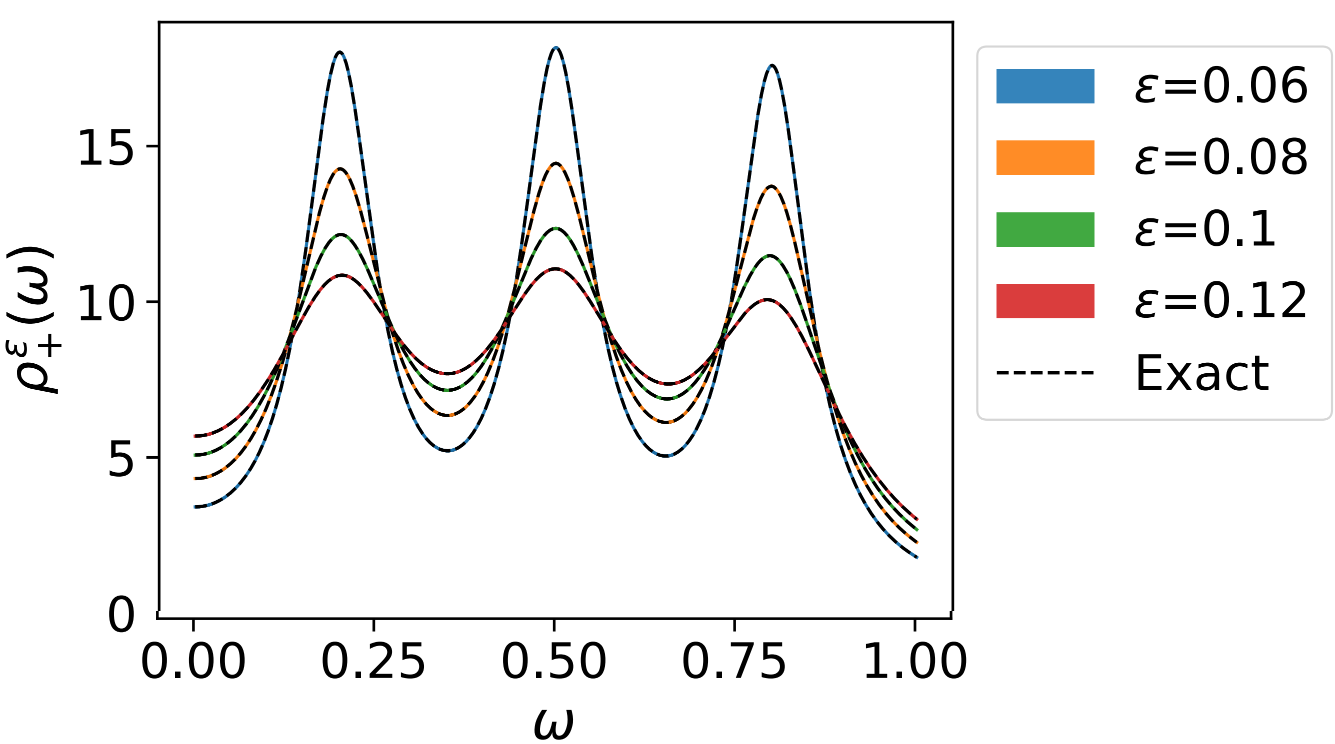
- Can consider two possibilities:
  - Fermionic: $\rho(-\omega) = \rho(\omega)$ [+ from anti-commutator]
  - Bosonic: $\rho(-\omega) = -\,\rho(\omega)$ [- from commutator]
- Technical detail: slightly different conformal maps used to transform the problem to the unit disk. See our arXiv:2305.16190
- For finite smearing ϵ, $\rho_{+}^{\epsilon}(\omega) \neq \rho_{-}^{\epsilon}(\omega)$.
- BUT they converge to the same function: $\lim_{\epsilon \to 0} \rho_{\pm}^{\epsilon}(\omega) = \rho(\omega),\ \omega > 0$

Numerical Examples

Discrete poles

- For $\omega > 0$: $\rho(\omega) = \delta(\omega - 0.2) + \delta(\omega - 0.5) + \delta(\omega - 0.8)$
- Euclidean data generated for $\beta = 64$ total points on the imaginary-energy axis
- The exact answer is always contained with bounding envelope of the Wertevorrat

Fermionic



Bosonic