$K_{\mathrm{L}} \to \mu^+\mu^-$ from lattice QCD

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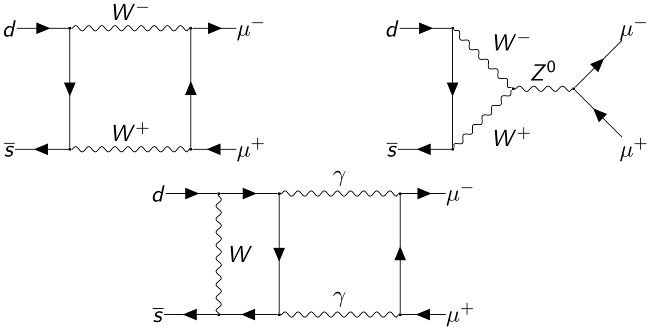
Introduction

- In the Standard Model, $K_{\mathrm{L}} \to \mu^+\mu^-$ comes in at one-loop level with exchange of two W-bosons or two W- and a Z-boson (short-distance contribution, SD).
- Precisely measured $\mathrm{Br}(K_{\mathrm{L}} \to \mu^+\mu^-) = 6.84 \pm 0.11 \times 10^{-9} \Rightarrow$ good test for the SM and potential interest for the physics beyond the SM. [BNL E871 Collab., PRL '00]
- Current theory limitation is the long-distance contribution (LD) involving two-photon exchange entering at $\mathrm{O}(G_{\mathrm{F}}\alpha_{\mathrm{QED}}^2)$, parametrically comparable to the SD contribution: the real part of the amplitude is not well understood.

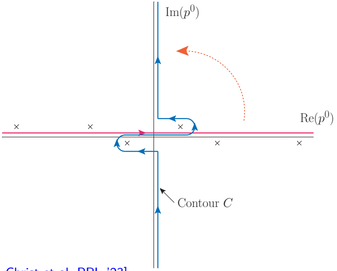


Formalism

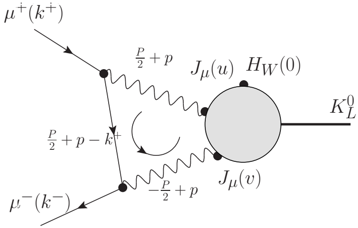
- Strategy: perturbatively expanded kernel function in G_{F} and α_{QED} + hadronic correlation function computed on the lattice.

$$
\begin{aligned}
\mathcal{A}_{ss'}(k^+,k^-) \quad = \quad & e^4 \int d^4p \int d^4u \int d^4v \; e^{-i\left(\frac{P}{2}+p\right)u} e^{-i\left(\frac{P}{2}-p\right)v} \frac{1}{(\frac{P}{2}-p)^2+m_\gamma^2-i\varepsilon} \cdot \frac{1}{(\frac{P}{2}+p)^2+m_\gamma^2-i\varepsilon} \\
& \times \frac{\overline{u}_s(k^-)\gamma_\nu\{\gamma\cdot(\frac{P}{2}+p-k^+)+m_\mu\}\gamma_\mu v_{s'}(k^+)}{(\frac{P}{2}+p-k^+)^2+m_\mu^2-i\varepsilon} \cdot \left\langle 0 \left| T\left\{J_\mu(u)J_\nu(v)\mathcal{H}_W(0)\right\}\right| K_L \right\rangle .
\end{aligned}
$$

- Analytic continuation of the kernel: $\Rightarrow$ unphysical exponentially growing contribution from states lighter than the kaon at rest.
- Finite number of such states on a finite lattice $\Rightarrow$ explicit, precise subtraction of such is possible.



[cf. Christ et al, PRL '23]



Numerical implementation

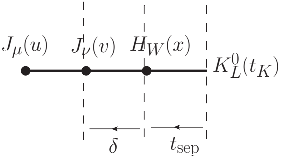
Parameter	Value
$L^3 \times T \times L_s$	$24^3 \times 64 \times 24$
m_π [MeV]	142
M_K [Mev]	515
a^{-1} [GeV]	1.023

- Lattice setup: Möbius Domain Wall fermion ensemble 24ID from the RBC/UKQCD collaboration.
- Master formula:

$$\mathcal{A}(t_{\rm sep}, \delta, x) \equiv \sum_{d\leq\delta} \sum_{u,v\in\Lambda} \delta_{v_0-x_0,d}\; e^{M_K(v_0-t_K)}\; K_{\mu\nu}(u-v)\, \langle J_\mu(u) J_\nu(v) \mathcal{H}_{\rm W}(x) K_{\rm L}(t_K)\rangle\,,$$

$$\mathcal{H}_{\rm W}(x) = \frac{G_{\rm F}}{\sqrt{2}} V_{us}^* V_{ud} (C_1 Q_1 + C_2 Q_2)\,,$$

$$Q_1 \equiv (\bar{s}_a \Gamma^L_\mu d_a)(\bar{u}_b \Gamma^L_\mu u_b)\,, \quad Q_2 \equiv (\bar{s}_a \Gamma^L_\mu d_b)(\bar{u}_b \Gamma^L_\mu u_a)\,.$$



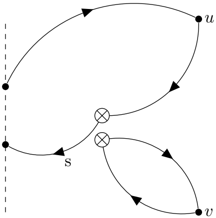
- Control of the contaminations from π^0 and low-energy $\pi\pi\gamma$ states:
  - The unphysical π^0 contribution can be measured and subtracted exactly

$$\frac{1}{2m_\pi} \sum_{\delta\geq 0} \sum_{u\in\Lambda} e^{(M_K-m_\pi)\delta} \langle 0|J_\mu(u) J_\nu(v)|\pi^0\rangle\, K_{\mu\nu}(u-v)\, \langle \pi^0|\mathcal{H}_{\rm W}(v)|K_{\rm L}\rangle\,.$$

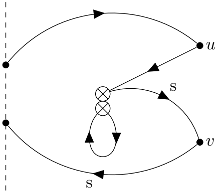
  - Control of the $\pi\pi\gamma$-intermediate state: use several kernels with different $|u-v| \leq R_{\rm max}$.

Contractions

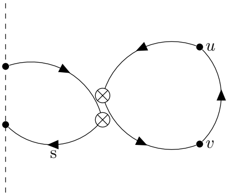
- Quark-connected Wick contractions for $\langle J_\mu(u) J_\nu(v) \mathcal{H}_{\mathrm{W}}(x) K_{\mathrm{L}}(t_K) \rangle$.
  Dashed line: $K_{\mathrm{L}}(t_K)$, crosses: $\mathcal{H}_{\mathrm{W}}(x)$, solid dots: $J_\mu(u)$ and $J_\nu(v)$



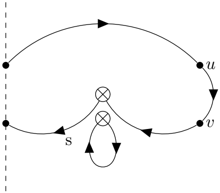
Type 1 diagram 1a



Type 2 diagram 1a



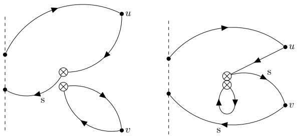
Type 3 diagram 1a



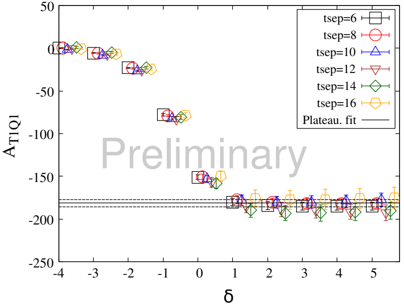
Type 4 diagram 1a

Preliminary results

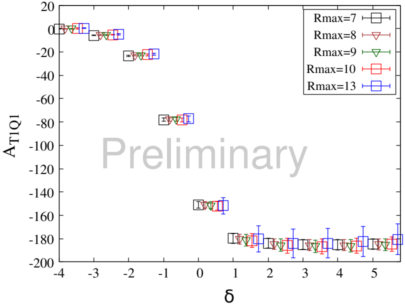
Type 1 and 2



- Plateaux are formed at rather small value of δ.
- Consistency between results obtained with different $t_{\rm sep}$'s, allowing for an error-weighted average.
- Stable central values from different choices of $R_{\rm max}$, evidence of the absence of sizeable unphysical contribution from the $\pi\pi\gamma$ state.



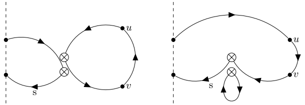
$\mathcal{A}_{\rm T1Q1}$ at fixed $R_{\rm max} = 7$



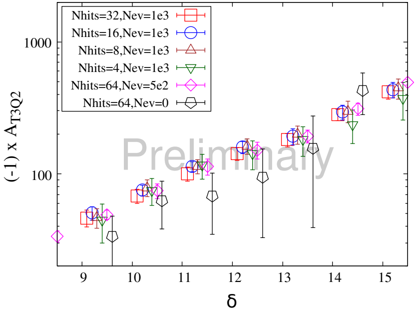
$\mathcal{A}_{\rm T1Q1}$ at fixed $t_{\rm sep} = 6$

Preliminary results

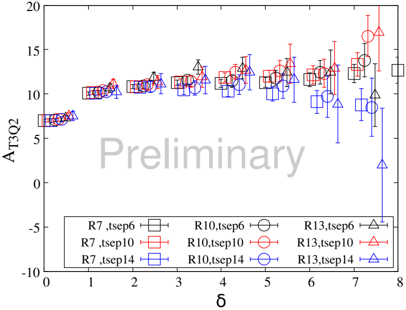
Type 3 and 4



- Stochastic all-to-all propagator with Z-Möbius low modes allowing computing with multiple $R_{\max}$.
- Expected exponentially-growing behavior due to the unphysical π^0 intermediate state.
- Plateau after subtracting the π^0 contamination. No strong sign of the $\pi\pi\gamma$ contamination by increasing $R_{\max}$.



$\mathcal{A}_{\mathrm{T3Q2}}$ in log scale with different hits and low modes.



$\mathcal{A}_{\mathrm{T3Q2}}$ after subtracting π^0

Conclusions and outlook

- A coordinate-space based lattice-QCD formalism for the $K_{\mathrm{L}} \to \mu^+\mu^-$ decay is proposed, enabling the determination of the phenomenologically inaccessible real part of the decay amplitude.
- Numerical strategies allowing to deal with different connected topologies have been developed, with possibility of keeping the $\pi\pi\gamma$ intermediate state under control.
- The so-far ignored disconnected part might not be negligible and can be much noisier due to the η intermediate state. More efficient sampling strategies will be needed.
- Possible finite-volume effects to worry about due to the $\pi\pi\gamma$ state.

Back-up slides

Numerical implementation

Some technical details

- Use of the (z-)Möbius accelerated Domain Wall Fermion solver: two-level solve where the loose inner solver solves the Dirac equation with a low-mode deflated z-Möbius operator.
- Coulomb-gauge-fixed wall sources to better overlap with the pseudoscalar meson ground states at large time-separation t_{sep}.
- Randomly distributed reference points to sample the volume.

Preliminary results

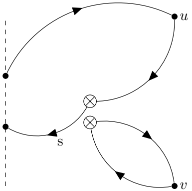
Type 1 and 2

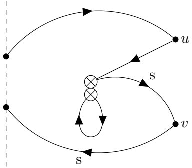
- No unphysical π^0 state expected.
- Construct blocks according to the $t_{\rm sep}$ dependence.
  E.g. Type 1 (a).

$$\mathcal{A}_1^{\rm T1D1a}(t_{\rm sep},\delta,x) = \sum_{d\leq\delta}\sum_{\vec{z}\in\Lambda_0} \delta_{z_0-x_0,d}\, e^{M_K(z-t_K)}\,{\rm Tr}_C\left[\hat{F}^1_{\nu\rho}(z,x,t_{\rm sep})\right]{\rm Tr}_C\left[G_{\nu\rho}(z,x)\right] .$$

- Convolution with the kernel performed with Fast Fourier Transform

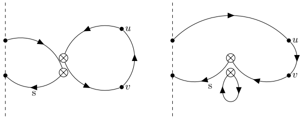
$$\hat{F}_{\nu\rho}(z,x,t_{\rm sep}) \equiv \sum_u K_{\mu\nu}(u-v)F_{\mu\rho}(u,t_{\rm sep}) = \mathcal{F}^{-1}\left[\tilde{K}_{\mu\nu}(-p)\tilde{F}_{\mu\rho}(p,t_{\rm sep})\right] .$$





Preliminary results

Type 3 and 4



- All-to-all propagator estimator with Z-Möbius low modes h_i

$$\hat{L}(x,y) = \sum_{i=1}^{N_{\mathrm{ev}}} V^{45'} \left[\lambda_i^{-1} h_i h_i^\dagger\right]^{5'} U^{5'4} - \sum_{i=1}^{N_{\mathrm{ev}}} V^{45'} \left[\lambda_i^{-1} h_i h_i^\dagger\right]^{5'} U^{5'4} \sum_{j=1}^{N_{\mathrm{hits}}} \xi_j \xi_j^\dagger + V^{45} \left[D^{-1}\right]^{5} U^{54} \sum_{j=1}^{N_{\mathrm{hits}}} \xi_j \xi_j^\dagger \,,$$

- Choice for the stochastic source ξ_i: $\mathbb{Z}_2$ time-diluted source for Type 3 and Gaussian volume source for Type 4.

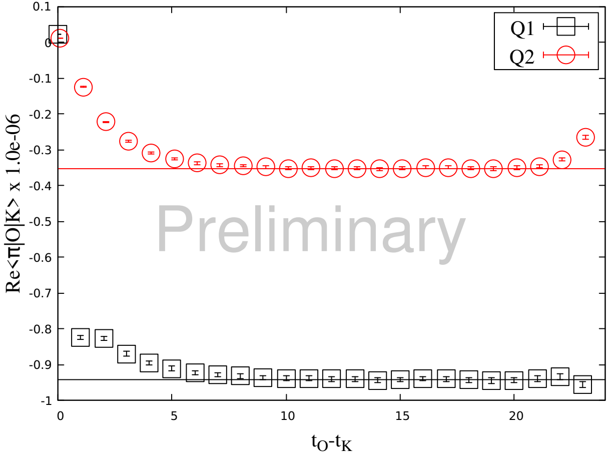
- Reuse of the data for different kernels (only M and P are kernel dependent).
  E.g. building blocks for Type 3:

$$\begin{aligned}\hat{C}_q^a(t_{\mathrm{sep}},\delta,v) = \quad & \sum_{i=1}^{N_{\mathrm{ev}}} \left\langle M_i(v), N_{q,i}^a(t_{\mathrm{sep}},\delta,v)\right\rangle - \frac{1}{N_{\mathrm{hits}}} \sum_{i=1}^{N_{\mathrm{ev}}} \sum_{j=1}^{N_{\mathrm{hits}}} \left\langle w_i^l(z), \xi_j(z)\right\rangle \left\langle P_j(v), N_{q,i}^a(t_{\mathrm{sep}},\delta,v)\right\rangle \\ & + \frac{1}{N_{\mathrm{hits}}} \sum_{j=1}^{N_{\mathrm{hits}}} \left\langle P_j(v), Q_{q,j}^a(t_{\mathrm{sep}},\delta,v)\right\rangle \,.\end{aligned}$$

- Unphysical π^0 intermediate state contamination, expected by inspecting the quark flows, needs to be removed.

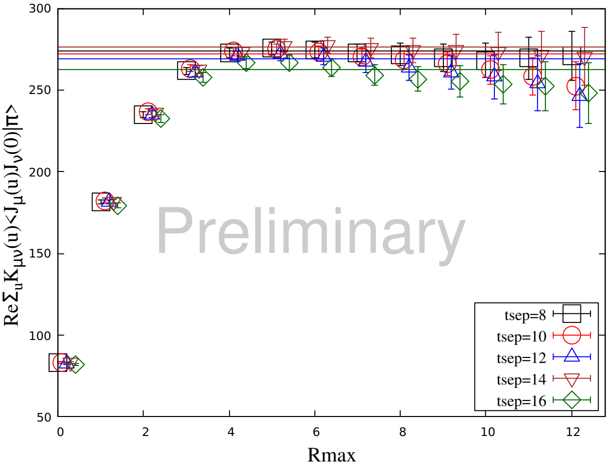
Study of the unphysical π^0 contribution

$$\frac{1}{2m_\pi}\sum_{\delta\geq 0}\sum_{u\in\Lambda} e^{(M_K-m_\pi)\delta}\left\langle 0|J_\mu(u)J_\nu(v)|\pi^0\right\rangle K_{\mu\nu}(u-v)\left\langle \pi^0|\mathcal{H}_{\mathrm{W}}(v)|K_{\mathrm{L}}\right\rangle .$$



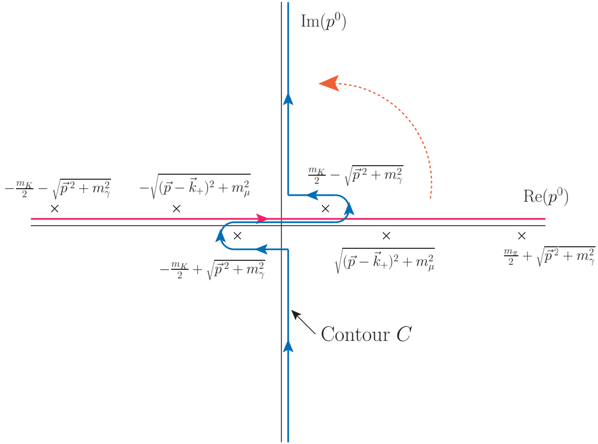
Study of the unphysical π^0 contribution

$$\frac{1}{2m_\pi}\sum_{\delta\geq 0}\sum_{u\in\Lambda} e^{(M_K-m_\pi)\delta}\left\langle 0|J_\mu(u)J_\nu(v)|\pi^0\right\rangle K_{\mu\nu}(u-v)\left\langle \pi^0|\mathcal{H}_{\mathrm{W}}(v)|K_{\mathrm{L}}\right\rangle .$$

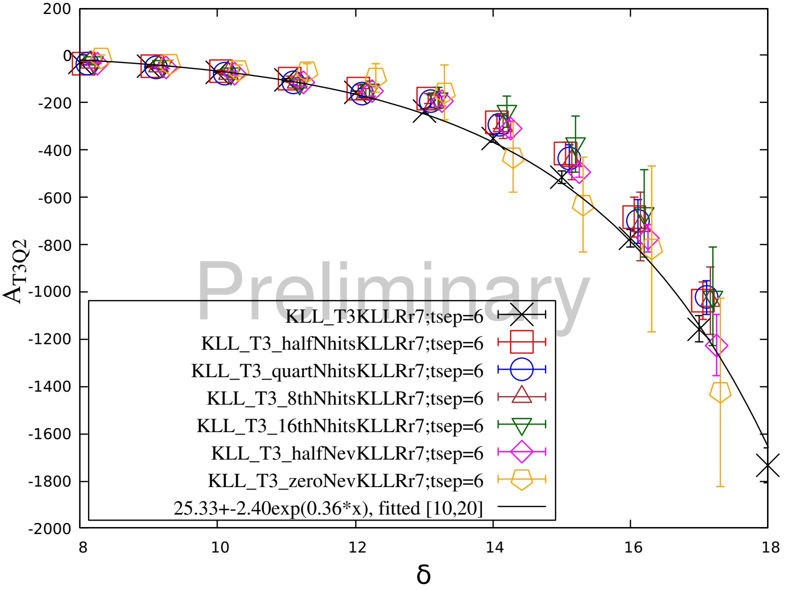


Analytic continuation

$$
\begin{aligned}
\mathcal{A}_{ss'}(k^+,k^-) \;=\;& e^4 \int d^4p \int d^4u \int d^4v\; e^{-i\left(\frac{P}{2}+p\right)u} e^{-i\left(\frac{P}{2}-p\right)v} \frac{1}{\left(\frac{P}{2}-p\right)^2+m_\gamma^2-i\varepsilon}\cdot\frac{1}{\left(\frac{P}{2}+p\right)^2+m_\gamma^2-i\varepsilon} \\
&\times \frac{\overline{u}_s(k^-)\gamma_\nu\{\gamma\cdot\left(\frac{P}{2}+p-k^+\right)+m_\mu\}\gamma_\mu v_{s'}(k^+)}{\left(\frac{P}{2}+p-k^+\right)^2+m_\mu^2-i\varepsilon}\cdot\langle 0|\,T\,\{J_\mu(u)J_\nu(v)\mathcal{H}_W(0)\}|\,K_L\rangle\,.
\end{aligned}
$$



Unphysical π^0 from direct exp. fit



The disconnected diagram