Monte Carlo study of Schwinger model without the sign problem

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Based on H. Ohata, arXiv:2303.05481 [hep-lat]

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Outline

- Bosonized Schwinger model on a lattice
- Analytical and numerical results
- Summary and future study

Schwinger model = QED in 1 + 1 dims

Euclidean action of $N_f = 1$ Schwinger model Schwinger, ’62

$$S_E[A_\mu, \psi, \overline{\psi}]_{g,m,\theta} = \int d^2x \left[\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \overline{\psi}(\not{\partial} + g\not{A} + m)\psi \right] + i\theta Q,$$

$$F_{01} := \partial_0 A_1 - \partial_1 A_0 = -F_{10} = E,$$

$$Q := \int d^2x \frac{g}{4\pi} \epsilon_{\mu\nu} F_{\mu\nu} = \int d^2x \frac{g}{2\pi} E \in \mathbb{Z}.$$

Similarities with QCD

- Chiral anomaly
- Confinement and screening
- Chiral symmetry breaking (due to chiral anomaly)
- Sign problem at non-vanishing CP-breaking θ term

Properties from 1 + 1 dimensional nature

- Gauge coupling g has the mass dimension $[g] = 1$.
- Equivalent bosonized form Coleman, ’75; Mandelstam, ’75
- Analytically solvable at $m = 0$ Schwinger, ’62, Sachs and Wipf, ’92.

In formalism analogous to lattice QCD, sign problem appears. $\Longrightarrow$ Many studies based on the Hamilton formalism

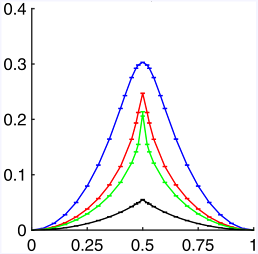
$$H = \int dx \left[-i\psi^\dagger(\partial_x - igA^1)\gamma^5\psi + \frac{g^2}{2}\left(\frac{E}{g} + \frac{\theta}{2\pi}\right)^2 + m\bar{\psi}\psi \right],$$

Gauss law constraint: $\partial_x E/g = \psi^\dagger\psi$.

Using the Kogut-Susskind fermion, solving the Gauss law constraint in open B.C., and using the Jordan-Wigner transformation $\Longrightarrow$ 2^{L_x} dimensional spin Hamiltonian.

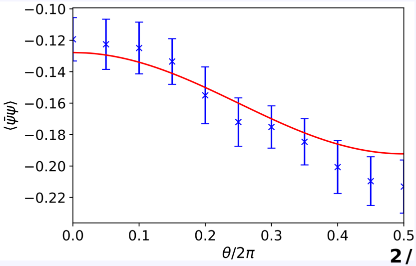
tensor network, $m/g = 0.125, 0.25, 0.3, 0.5$

Buyens et al., ’17



quantum algorithm, $m/g = 0.1$

Chakraborty et al., ’22



Bosonized Schwinger model

Bosonized Schwinger model [Coleman, Jackiw, and Susskind, '75]

$$H = \int dx \, \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x \phi) + \frac{g^2}{2\pi}\phi^2 - \frac{e^\gamma mg}{2\pi^{3/2}} \mathcal{N}_{g/\sqrt{\pi}} \cos\left(2\sqrt{\pi}\phi - \theta\right), \quad \phi/\sqrt{\pi} \longleftrightarrow E/g$$

Here, $\mathcal{N}_\mu$ denotes the normal ordering for creation and annihilation operators defined by

$$\phi(x) =: \int_{-\infty}^{\infty} \frac{dk}{2\pi}\left[\frac{1}{2\omega(k,\mu)}\right]^{1/2}\left(a(k,\mu)e^{-ikx} + a^\dagger(k,\mu)e^{ikx}\right),$$

$$\pi(x) =: -i\int_{-\infty}^{\infty} \frac{dk}{2\pi}\left[\frac{\omega(k,\mu)}{2}\right]^{1/2}\left(a(k,\mu)e^{-ikx} - a^\dagger(k,\mu)e^{ikx}\right),$$

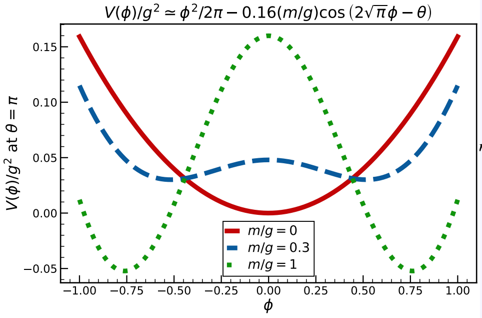
$$\omega(k,\mu) := \sqrt{k^2+\mu^2}.$$

Analytical study on Schwinger model Coleman, '75, '76

Bosonization provides a unique perspective:

$$V(\phi) = \frac{g^2}{2\pi}\phi^2 - \frac{e^\gamma}{2\pi^{3/2}} mg\cos\left(2\sqrt{\pi}\phi - \theta\right), \quad \phi/\sqrt{\pi} \longleftrightarrow E/g.$$

At $\theta = \pi$, $V(\phi)$ is invariant under the CP tr. $\phi \to -\phi$.
A double-well structure appears at large m, suggesting SSB.
The SSB indeed occurs for $m/g \gtrsim 0.33$ at $T = 0$. Byrnes, '02



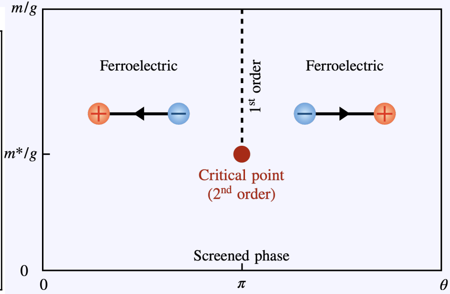


Figure taken from Ikeda et al., '21

Lattice bosonized Schwinger model

In this talk, I present a lattice formulation of the bosonized Schwinger model

$$S_E = \sum_{\tau=0}^{L_\tau-1}\sum_{x=0}^{L_x-1} \frac{1}{2}(\partial_\tau \phi_{x,\tau})^2 + \frac{1}{2}(\partial_x \phi_{x,\tau})^2 + \frac{(ag)^2}{2\pi}\left(\phi_{x,\tau} + \frac{\theta}{2\sqrt{\pi}}\right)^2 - \frac{e^\gamma}{2\pi^{3/2}}\frac{m}{g}(ag)^2 \mathcal{O}(1/ag) \cos\left(2\sqrt{\pi}\phi_{x,\tau}\right) \in \mathbb{R},$$

which enables us to study the Schwinger model without encountering the sign problem.

For that, the normal ordering appearing cosine term

$$\frac{e^\gamma}{2\pi^{3/2}} mg \mathcal{N}_{g/\sqrt{\pi}} \cos\left(2\sqrt{\pi}\hat{\phi} - \theta\right),$$

must be removed properly.

Normal ordering in the continuum Coleman, ’75

Coleman presented the answer in 1975. Using Wick’s theorem, he obtained

$$\mathcal{N}_\mu \exp(i\beta\phi) = \exp\left(\frac{\beta^2}{2}\Delta(x=0;\mu)\right)\exp(i\beta\phi),$$

where $\Delta(x;\mu)$ is the Feynman propagator for the scalar fields of mass μ, and β is an arbitrary real number.

The regularized Feynman propagator with a UV cutoff Λ

$$\Delta(x;\mu;\Lambda) := \Delta(x;\mu) - \Delta(x;\Lambda) = \frac{1}{2\pi}\ln\frac{\Lambda}{\mu} + \mathcal{O}(x^2),$$

leads to the formula to remove the normal ordering

$$\mathcal{N}_\mu \exp(i\beta\phi(x)) = (\Lambda/\mu)^{\beta^2/4\pi}\exp(\beta\phi(x)).$$

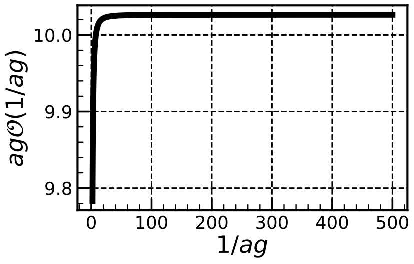
Normal ordering on a lattice

On a lattice, the Feynman propagator is naturally regularized with the lattice spacing a

$$\Delta(ag) = \int_{-\pi}^{\pi} \frac{d^2k}{(2\pi)^2} \left(4 \sum_{\mu} \sin^2\left(\frac{k_\mu}{2}\right) + \left(\frac{ag}{\sqrt{\pi}}\right)^2 \right)^{-1}.$$

Hence, the normal ordering on a lattice can be removed as

$$\mathcal{N}_{g/\sqrt{\pi}} \exp(i\beta\phi_x) = \mathcal{O}(1/ag)^{\beta^2/4\pi} \exp(i\beta\phi_x),$$
$$\mathcal{O}(1/ag) := \exp(2\pi\Delta(ag)).$$



$$\mathcal{O}(1/ag) \simeq 10/ag$$

for $ag \lesssim 0.05$

We can now define a lattice Hamiltonian without using the normal ordering

$$Ha = \sum_{x=0}^{L_x-1} \frac{1}{2}(a\pi_x)^2 + \frac{1}{2}(\partial_x \phi_x)^2 + \frac{(ag)^2}{2\pi}\phi_x^2 - \frac{e^\gamma}{2\pi^{3/2}}\frac{m}{g}(ag)^2 \mathcal{O}(1/ag) \cos\left(2\sqrt{\pi}\phi_x - \theta\right).$$

The thermal expectation value at temperature T

$$\langle O(\phi)\rangle := \mathrm{tr}\, O(\phi) e^{-H/T} \Big/ \mathrm{tr}\, e^{-H/T} = \int D\phi\, O(\phi) e^{-S_E} \Big/ \int D\phi\, e^{-S_E},$$

where S_E is the real lattice action (No sign problem!)

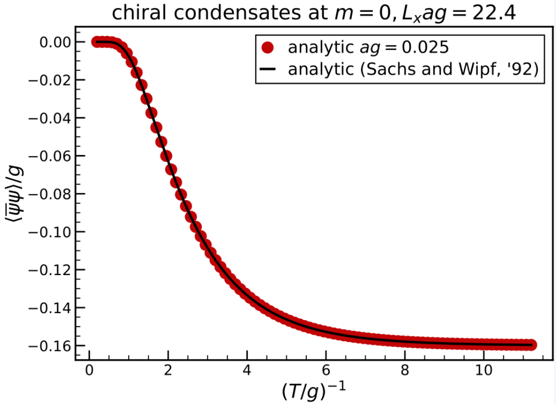
$$S_E = \sum_{\tau=0}^{L_\tau-1} \sum_{x=0}^{L_x-1} \frac{1}{2}(\partial_\tau \phi_{x,\tau})^2 + \frac{1}{2}(\partial_x \phi_{x,\tau})^2 + \frac{(ag)^2}{2\pi}(\phi_{x,\tau})^2 - \frac{e^\gamma}{2\pi^{3/2}}\frac{m}{g}(ag)^2 \mathcal{O}(1/ag) \cos\left(2\sqrt{\pi}\phi_{x,\tau} - \theta\right)$$

Analytical and numerical results

Applying $\mathcal{N}_\mu \exp(i\beta\phi) = \exp\left(\frac{\beta^2}{2}\Delta(x=0;\mu)\right)\exp(i\beta\phi)$ to the lattice system with $L_x \times L_\tau$ sites, we find

$$\left\langle \overline{\psi}\psi \right\rangle_{\text{lattice}} \Big/ g = -\frac{e^\gamma}{2\pi^{3/2}} \mathcal{O}(1/ag) \left\langle \cos\left(2\sqrt{\pi}\right)\right\rangle_{L_x,L_\tau}$$

$$= -\frac{e^\gamma}{2\pi^{3/2}} \exp\left[-2\pi\left\{\Delta(ag)_{L_x,L_\tau} - \Delta(ag)_{L_x,L_\tau\to\infty}\right\}\right].$$



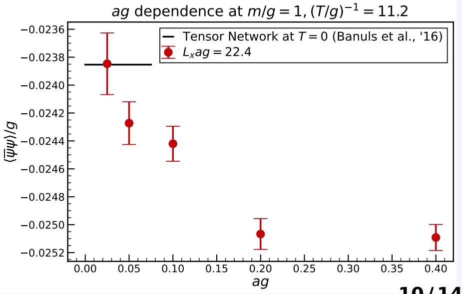
Chiral condensate at $m > 0, T = 0$

The chiral condensates at $T/g = (0.02 \times 672)^{-1}$ in this work, compared with that obtained by the tensor network method

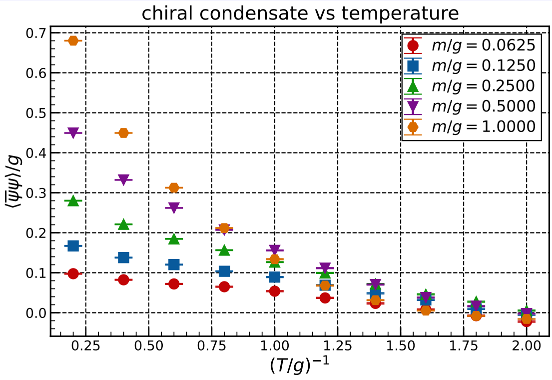
m/g	This work	Bañuls et al.	This work / Bañuls et al.
0.0625	0.1138(10)	0.1139657(8)	0.9989(90)
0.125	0.09214(88)	0.0920205(5)	1.0013(95)
0.25	0.06629(67)	0.0666457(3)	0.995(10)
0.5	0.04191(40)	0.0423492(20)	0.9896(95)
1	0.02399(24)	0.0238535(28)	1.006(10)

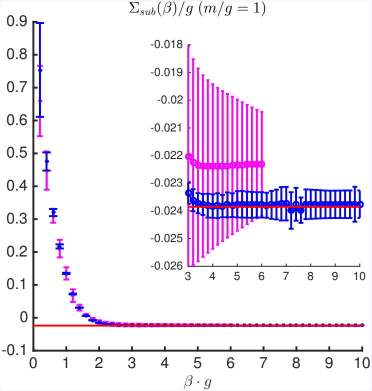
Bañuls et al., Phys. Rev. D 93, 094512 (2016).

Our results at $ag = 0.02$ are consistent with that extrapolated to the continuum limit in the Kogut-Susskind fermion formalism.



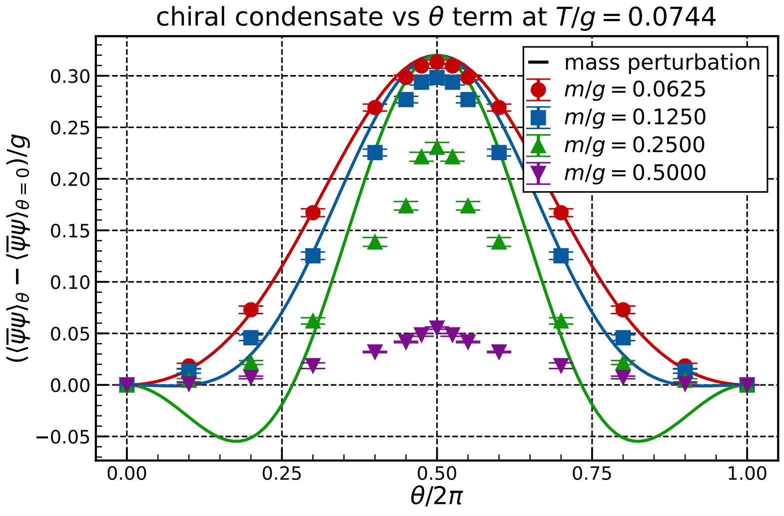
Chiral condensate at $m > 0, T > 0$





- Agreements with two different groups using the tensor network method
  Bañuls et al., Phys. Rev. D 93, 094512 (2016),
  Buyens et al., Phys. Rev. D 94 085018 (2016).
- Better precision at high temperature $T/g \gtrsim 1$

Chiral condensate vs θ term

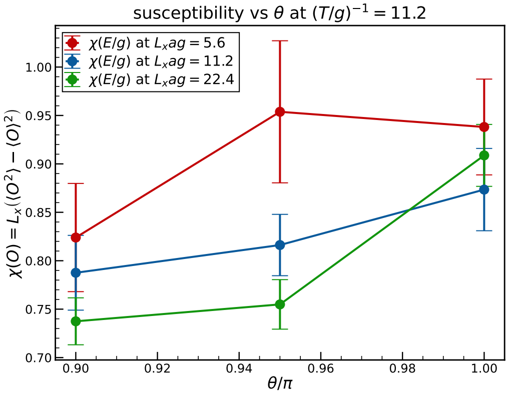


- Mass perturbation Adam, ’97 works well at $m/g \lesssim 0.125$.
- A cusp-like behavior at $m/g = 0.5, \theta = \pi$
  $\Longrightarrow$ phase transition at $T/g = 0.0744$?

CP symmetry at $\theta = \pi$ at finite temperature

The analogy with the one-dimensional transverse Ising model and a tensor network study Buyens et al., '16 suggest that the CP symmetry is restored at any nonzero temperature.

Susceptibility of an order parameter $E/g = \phi/\sqrt{\pi}$ at three spatial lengths:



No phase transition?

Summary and future study

Summary and future study

Summary

- I presented the correct lattice formulation of the bosonized Schwinger model for the first time.
- The lattice formulation enables us to study the model using the Monte Carlo method without encountering the sign problem. Also, the numerical costs are very small.
- I omitted the finite density study from this talk due to the limited time.

Future study

- Establish new results, such as CP restoration at finite temperature, phase diagram in (μ, T) plane.
- Application to more nontrivial models, such as QCD in $1+1$ dims

Backup

Generating Monte Carlo configurations

Heat bath algorithm

Start with an initial field configuration $\{\phi_{x,\tau}\}$

1. focus on $\phi_{x,\tau}$ at some site (x,τ)
2. update $\phi_{x,\tau}$ while fixing the rest (heat bath)
3. repeat 1 and 2 for all sites

Repeating the sweep many times, the field configuration $\{\phi_{x,\tau}\}$ starts to distribute with $P(\{\phi_{x,\tau}\}) \propto \exp(-S_E(\{\phi_{x,\tau}\}))$.

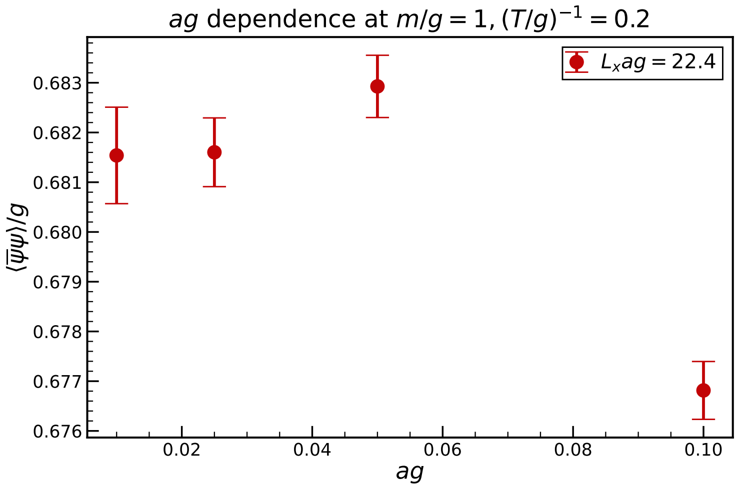
$$P(\phi_{x,\tau}) \propto \exp\left\{-2I(ag)\left(\phi_{x,\tau} - \frac{\overline{\phi}_{x,\tau}}{I(ag)}\right)^2\right\}$$

$$\times \exp\left\{\frac{e^{\gamma}}{2\pi^{3/2}}(m/g)(ag)^2\mathcal{O}(1/ag)\cos\left(2\sqrt{\pi}\phi_{x,\tau} - \theta\right)\right\},$$

$\overline{\phi}_{x,\tau} := (\phi_{x,\tau+1} + \phi_{x,\tau-1} + \phi_{x+1,\tau} + \phi_{x-1,\tau})/4, I(ag) := 1 + (ag)^2/4\pi.$

Generate a Gaussian random number, apply the rejection sampling

ag dependence at high temperature and large fermion mass



The continuum limit is approximately achieved at $ag \lesssim 0.025$.

Chiral symmetry at $m = 0$ in Schwinger model

At $m = 0$, the action has the chiral symmetry

$$U(1)_\mathrm{V} \times U(1)_\mathrm{A} \times SU(N_f)_\mathrm{V} \times SU(N_f)_\mathrm{A}.$$

$U(1)_\mathrm{A}$ is explicitly broken by the chiral anomaly.

Spontaneous continuous symmetry breaking is prohibited in $1 + 1$ dims model (except the Higgs mechanism). Coleman, '73

- $N_f \geq 2$
  $\left\langle \overline{\psi}\psi \right\rangle \neq 0 \Longrightarrow$ spontaneous $SU(N_f)_\mathrm{A}$ symmetry breaking, which contradicts Coleman's theorem.
- $N_f = 1$
  We don't have $SU(N_f)_\mathrm{A}$ symmetry from the beginning.
  $\Longrightarrow$
  $\left\langle \overline{\psi}\psi \right\rangle \neq$ does not contradict Coleman's theorem.

Wick's theorem

$$\phi(x) =: \int_{-\infty}^{\infty} \frac{dk}{2\pi}\left[\frac{1}{2\omega(k,\mu)}\right]^{1/2}\left(a(k,\mu)e^{-ikx} + a^\dagger(k,\mu)e^{ikx}\right)$$
$$=: \phi^+ + \phi^-$$

where $\omega = \sqrt{k^2+\mu^2}$, and μ is an arbitrary real number.
By trivial manipulations, we find

$$\phi^2 = \mathcal{N}_\mu\phi^2 + \left[\phi^+, \phi^-\right]$$
$$= \mathcal{N}_\mu\phi^2 + \Delta(x=0,\mu),$$

where $\Delta(x;\mu)$ is the Feynman propagator.
Similarly, we find

$$\phi^3 = \mathcal{N}_\mu\phi^3 + \binom{3}{2}\Delta(0;\mu)\mathcal{N}_\mu\phi,$$

$$\phi^4 = \mathcal{N}_\mu\phi^4 + \binom{4}{2}\Delta(0;\mu)\mathcal{N}_\mu\phi^2 + \frac{\binom{4}{2}}{2}\Delta(0;\mu)^2,$$

$$\vdots$$

$e^{i\beta\phi} = \exp\big(-\beta^2\Delta(0;\mu)/2\big)\mathcal{N}_\mu e^{i\beta\phi}$ from Wick's theorem

$$
\begin{aligned}
e^{i\beta\phi} &= \left[1 + i\beta\phi + \frac{(i\beta)^2}{2!}\phi^2 + \frac{(i\beta)^3}{3!}\phi^3 + \frac{(i\beta)^4}{4!}\phi^4 + \cdots\right] \\
&= 1 + i\beta\mathcal{N}_\mu\phi + \frac{(i\beta)^2}{2!}\big[\mathcal{N}_\mu\phi^2 + \Delta(0;\mu)\big] \\
&\quad + \frac{(i\beta)^3}{3!}\big[\mathcal{N}_\mu\phi^3 + {}_3C_2\Delta(0;\mu)\mathcal{N}_\mu\phi\big] \\
&\quad + \frac{(i\beta)^4}{4!}\left[\mathcal{N}_\mu\phi^4 + {}_4C_2\Delta(0;\mu)\mathcal{N}_\mu\phi^2 + \frac{{}_4C_2}{2}\Delta(0;\mu)^2\right] + \cdots \\
&= \mathcal{N}_\mu e^{i\beta\phi} \\
&\quad + \frac{(i\beta)^2}{2!}\Delta(0;\mu)\left[1 + \frac{i\beta}{3!}2!\,{}_3C_2\mathcal{N}_\mu\phi + \frac{(i\beta)^2}{4!}2!\,{}_4C_2\mathcal{N}_\mu\phi^2 + \cdots\right] \\
&\quad + \frac{(i\beta)^4}{4!}\frac{{}_4C_2}{2}\Delta(0;\mu)^2[1 + \cdots] + \cdots \\
&= \exp\big(-\beta^2\Delta(x;\mu)/2\big)\mathcal{N}_\mu e^{i\beta\phi}
\end{aligned}
$$

Abelian bosonization of QCD in 1 + 1 dims?

For gluon field and electric field in the matrix representation

$$\left(F_{\mu\nu}\right)^{ij} := \sum_{a=1}^{N_c^2-1} \lambda_a^{ij} F_{\mu\nu}^a, \quad \left(A_\mu\right)^{ij} = \sum_{a=1}^{N_c^2-1} \lambda_a^{ij} A_\mu^a$$

Baluni, '80 proposed a special gauge fixing

$$\left(F_{\mu\nu}\right)^{ij} = 0 \quad \text{for} \quad i \neq j, \quad \left(A_\mu\right)^{ii} = 0 \quad \text{for} \quad i = 1, \dots, N_c - 1,$$

and obtained

$$\begin{aligned}
\mathcal{H} &= \mathcal{H}_0 + \mathcal{H}_1, \\
\mathcal{H}_0 &= \frac{1}{2} \sum_{i=1}^{N_c} \Big[\pi_i^2 + (\partial_x \phi_i)^2 + 2m\Lambda\Big(1 - \cos\big(2\sqrt{\pi}\phi_i\big)\Big)\Big], \\
\mathcal{H}_1 &= \frac{g^2}{8\pi N_c} \sum_{i,k} (\phi_i - \phi_k)^2, \\
&\quad + \left(\sqrt{\pi}\Lambda^2\right) \sum_{i,j} \left\{ 1 - \int_0^1 d\gamma \cos 2\sqrt{\pi}\gamma(\phi_i - \phi_k) \right\}.
\end{aligned}$$