Sphaleron Rate from Lattice Gauge Theory

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Introduction

The "Sphaleron Rate" (imaginary linear-in-frequency part of the topological density retarded Green's function) determines the real-time relaxation rate of axial quark number for light quarks in a hot medium, and is relevant in heavy-ion collisions and electroweak baryogenesis. We recently showed how it can be determined from standard Euclidean lattice simulations of pure gauge theory via a novel saddlepoint method. We extend this work to find the sphaleron rate for (2+1)-flavor QCD with $N_\tau = 8-16$ and HISQ action at almost physical pion masses in the temperature range $0.2-3$ GeV or $1.2-18$ times the crossover temperature T_{pc}.

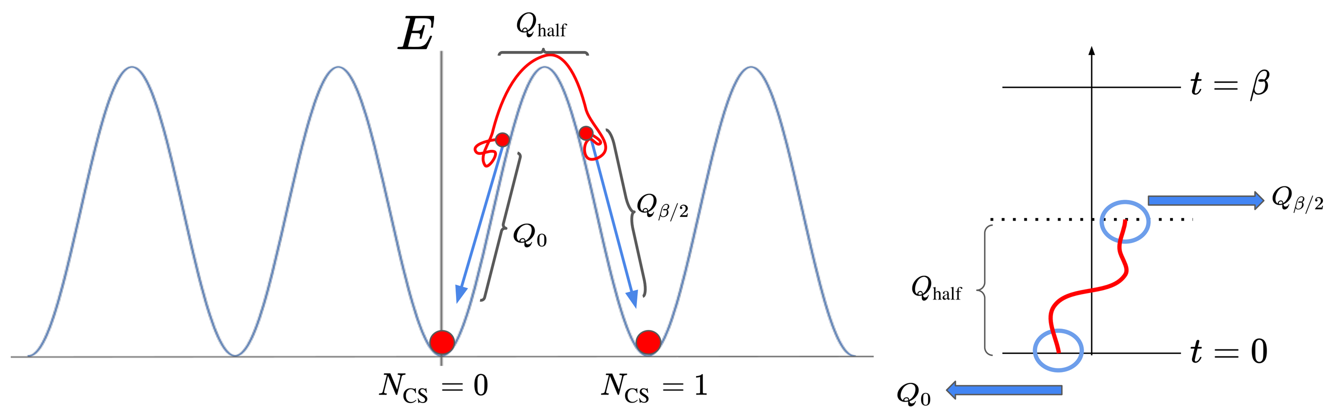
Saddlepoint method

In [1], for processes given by thermal activation over a saddlepoint we showed how to relate a set of side-by-side Euclidean fluctuations to a real-time rate. In the case of topology in gauge theories, the saddlepoint corresponds to the sphaleron solution, and we can measure its crossing of the separatrix by calculating

$$Q_S \equiv Q_0 + Q_{\rm half} - Q_{\beta/2}\,, \tag{1}$$

where Q_0 and $Q_{\beta/2}$ are defined as the topology enclosed in the $t=0$ and $t=\beta/2$ 3D slices [2] in the gradient flow direction, while $Q_{\rm half}$ is the topology enclosed in half our 4D lattice. The real-time rate is obtained with a conversion factor $2/T$ from the Euclidean rate computed on the lattice,

$$\frac{\Gamma_{{\rm sphal},s}}{T^4} = \frac{2}{T}\frac{\Gamma_{\rm Eucl}}{T^3}\,,\quad \frac{\Gamma_{\rm Eucl}}{T^3} = \frac{\langle Q_S^2\rangle}{(N_s/N_\tau)^3}\,. \tag{2}$$



$$Q_S = Q_0 + Q_{\rm half} - Q_{\beta/2}$$

Figure 1: Sketch of how we define and identify sphalerons in QCD.

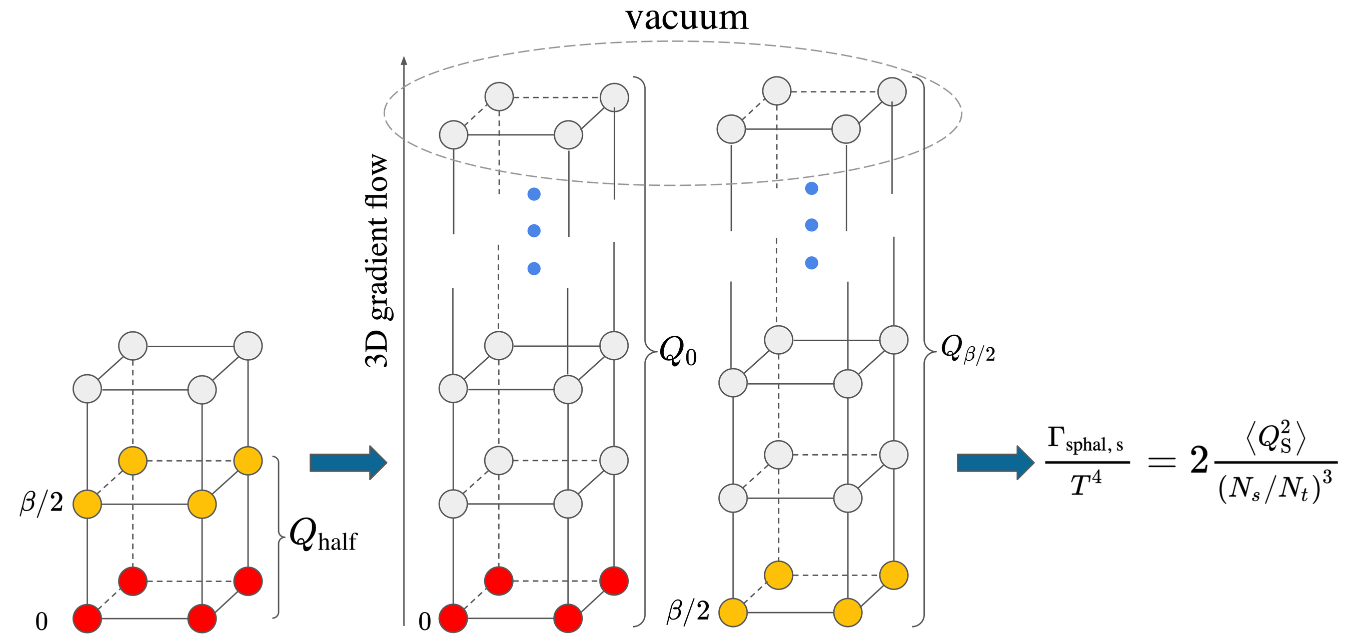


Figure 2: Sketch of the calculation performed on our 4D Euclidean lattices to extract the sphaleron rate. One dimension of the lattice is not drawn for convenience.

Observables on the lattice

The quantities Q_0, $Q_{\beta/2}$, $Q_{\rm half}$ are measured using the clover definition. To solve the 3D gradient flow equations we have implemented an Euler algorithm (we just care about ending in vacuum, not the precise configurations we pass through). Two more important techniques are used:

1. We also apply a certain amount of 4D gradient flow in order to reduce the UV noise, and obtain clean observables. This is done through a Runge-Kutta 3rd order algorithm as implemented in openQCD [3]. But the application of this flow changes the answer and we need to correct for it. The form of this correction was derived in [1] and takes the form of

$$\frac{\Gamma^{\tau_F}_{{\rm sphal},s}}{\Gamma_{{\rm sphal},s}} = \frac{2}{\pi}\sqrt{2\sum_{n=1,\ n\ {\rm odd}}^{\infty}\frac{e^{-\tau_F n^2}}{n^2}}\,, \tag{3}$$

where we have defined $\tau_F \equiv 8\pi^2\tau_F/(a^2 N_\tau^2)$. This way, we will add enough flow until topological number Q_S is close enough to an integer, while the individual contributions in Eq. **(1)** do not have to be.

2. As we evolve the 3D slices, after some amount of flow has been applied, we use blocking in order to reduce the total amount of links. Effectively what we are doing is discarding UV contributions that are not important for us, but we reduce the total amount of links and increase each step-size of flow.

It is important to add here that in order for the calculation to work we need to be in a regime without instantons, as otherwise our picture of small fluctuations around a saddlepoint is not correct.

Determination of α_s from gradient flow

Using the results from [4], we determine the MS coupling from the energy as a function of gradient flow,

$$\tau_F^2\langle E(\tau_F)\rangle = \frac{3\alpha_s}{4\pi}\left(1 + k_1\alpha_s + k_2\alpha_s^2\right) \tag{4}$$

with $k_1 \approx 1.098$ and $k_2 \approx -0.982$ for SU(3) theory without fermions, where we use the improved definition [5] of $\langle E(\tau_F)\rangle \equiv \langle F_{\mu\nu}(\tau_F)F^{\mu\nu}(\tau_F)\rangle$, using Zeuthen flow [6], and the scale is given by $\mu = 1/\sqrt{8\tau_F}$. We choose to calculate it at the scale of

$$\mu = \pi T \rightarrow \tau_F/a^2 = \frac{1}{8(\pi N_\tau)^2}, \tag{5}$$

which is enough to reduce the lattice artifacts even on coarse lattices, without overflowing. We can then run the coupling to any other scale within reach of a perturbative expansion using the β−function at five loops [7].

Pure-gauge results

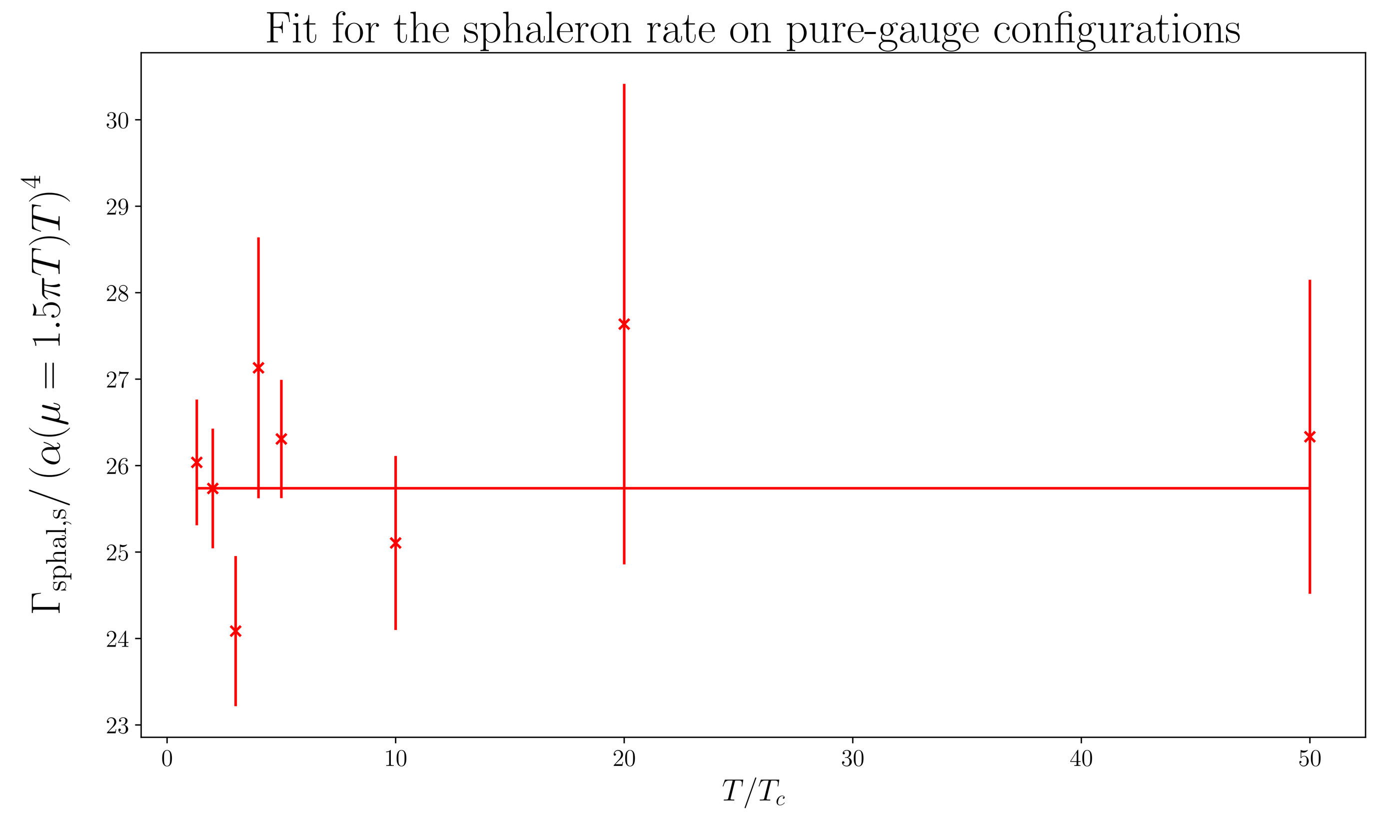


Figure 3: Sphaleron rate determined on pure-gauge configurations with $N_\tau = 10$, as a function of T/T_c, where T_c is the critical temperature of the pure-gauge theory, $T_c \sim 287$ MeV.

The pure-gauge results have been generated using the HMC algorithm with the Wilson action, as implemented in [3]. Figure 3 shows the updated data from [1]. The dependence on volume, lattice spacing, and gradient flow depth were thoroughly investigated in the paper. Assuming that the sphaleron rate scales with $(\alpha_s T)^4$, we have performed a fit where the coupling α_s has been found from gradient flow and matching to a known observable (see previous section). Fitting to the data yields

$$\Gamma_{{\rm sphal},s} \approx A\,(\alpha_s(\mu = B\pi T)T)^4 \rightarrow \Gamma_{{\rm sphal},s} \approx (25.7 \pm 1.3)\,(\alpha_s(\mu = 1.5\pi T)T)^4 \tag{6}$$

Full QCD results

We have reused ensembles generated with the RHMC algorithm for 2+1 flavors of highly improved staggered quarks (HISQ) in the sea, with a physical strange and almost physical light quarks at $m_l/m_s = 1/20$, as provided by the HotQCD and TUMQCD collaborations. For those data, we propose a fit in the form of

$$\Gamma_{{\rm sphal},s} = \left(A + \frac{C}{N_\tau^2}\right)(\alpha_s(\mu = B\pi T)T)^4 \rightarrow \Gamma_{{\rm sphal},s} = \left(14.6 \pm 0.6 + \frac{23.7 \pm 2.5}{N_\tau^2}\right)(\alpha_s(\mu = 1.6\pi T)T)^4 \tag{7}$$

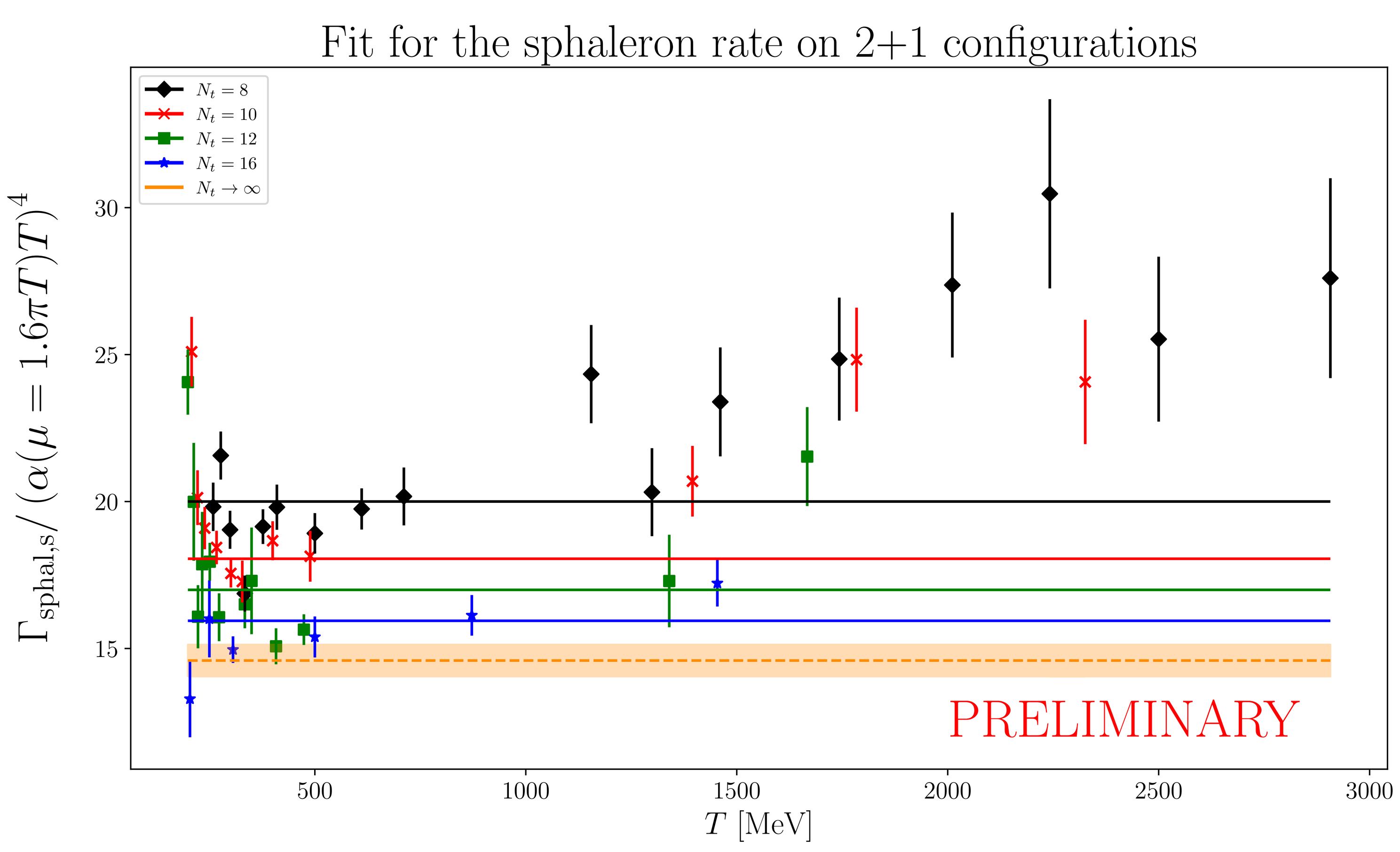


Figure 4: Sphaleron rate determined on 2+1 configurations with different N_τ, fitted to the proposed formula.

Outlook

Many of the lower temperature configurations in the full QCD case with small N_τ ($\sim 8, 10$) suffer from big fluctuations in Q_S that need large amounts of 4D gradient flow to tame. This leads to overflowing the lattice and being unable to reliably correct for this flow. Some ideas to solve this issue would be to apply inverse-blocking techniques, or using improved definitions of the field-strength tensor on the lattice, in order to be able to use lower amounts of flow. We are currently investigating these approaches.

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