Bootstrapping for Quantum Mechanical Systems

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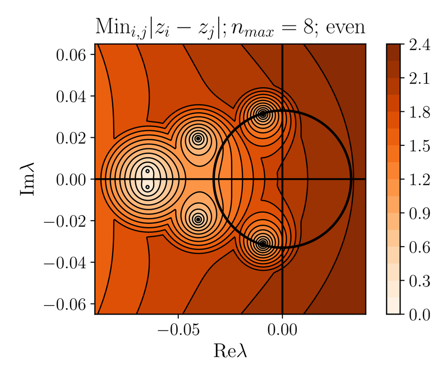
Motivation

- Universal quantum computers allow for real time evolution in quantum field theory. One of the simplest examples is $\lambda\phi^4$ [1,2,3].
- In [4], a digitization inspired by Gaussian quadrature for the path integral where ϕ is restricted to the zeros of the n_{max} order Hermite polynomial is used. With the following modification and resulting commutation relations, the Dyson Interaction picture is preserved:
  - $[a, a^\dagger] = \mathbf{1} - n_{max}|n_{max}-1\rangle\langle n_{max}-1|$
  - $[a^\dagger a, a] = -a$
  - $[a^\dagger a, a^\dagger] = a^\dagger$
- It is shown in [4] that since digitization provides a field cutoff, the perturbation series in λ will converge with a radius determined from complex singularities, for sufficiently small $|\lambda|$ [5]. See below for case when $n_{max} = 8$ (from [4]).



- In 1+1 dimensions these singularities pinch the real axis for $\lambda = \lambda_c$, which indicates second order phase transition.
- For the undigitized anharmonic oscillator, the moments $\langle E|x^{2l}|E\rangle$ can be recursively calculated [6,7].

Bootstrap Foundation

- We build up a moment recursion starting with three identities and constraints:

$$\langle [H, O]\rangle = 0 \tag{1}$$

$$\langle HO\rangle = E\langle O\rangle \tag{2}$$

$$\langle O^\dagger O\rangle \geq 0 \tag{3}$$

- By using these as a starting point, combining with $O = x^m, x^m p, x^{m-1}$ yielding an equation that can be used for moment recursion:

$$\begin{aligned} 0 &= 2mE\langle x^{m-1}\rangle \\ &+ \frac{1}{4}m(m-1)(m-2)\langle x^{m-3}\rangle \\ &- \langle x^m V'(x)\rangle - 2m\langle x^{m-1}V(x)\rangle \end{aligned} \tag{4}$$

- Now we have a moment recursion that relates x and E values, captures the dynamics of the Hamiltonian, and returns the virial theorem for m=1.
- A Hankel matrix can be constructed from the positivity of the norm:

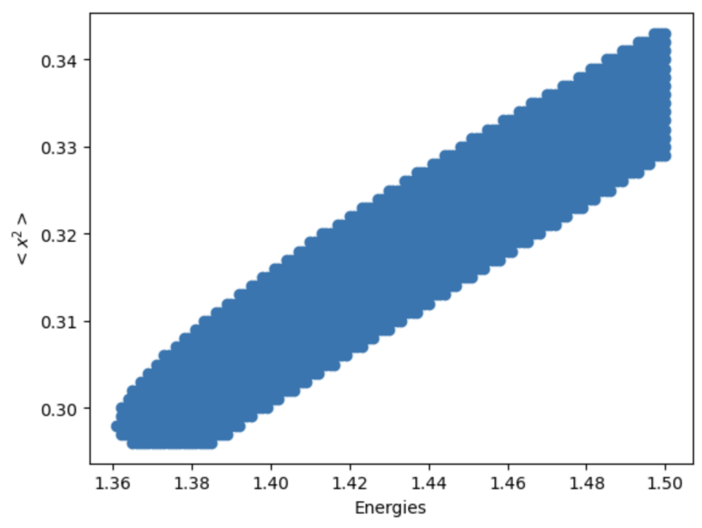
$$0 \leq \langle O^\dagger O\rangle = \sum_{ij} c_i^* \langle x^{i+j}\rangle c_j \equiv \sum_{ij} c_i^* M_{ij} c_j \tag{5}$$

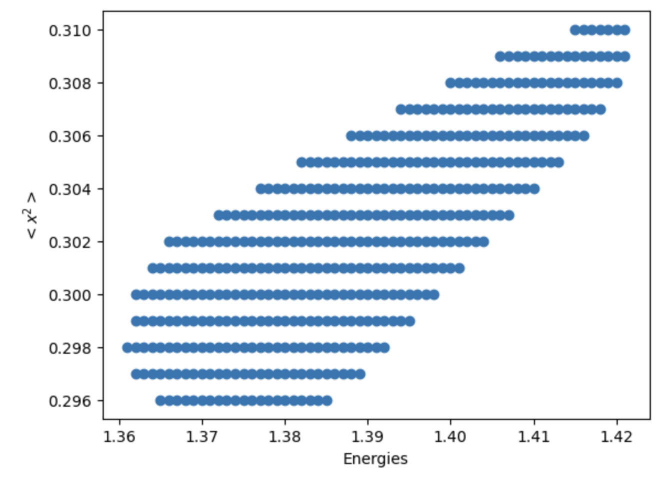
Bootstrap Algorithm

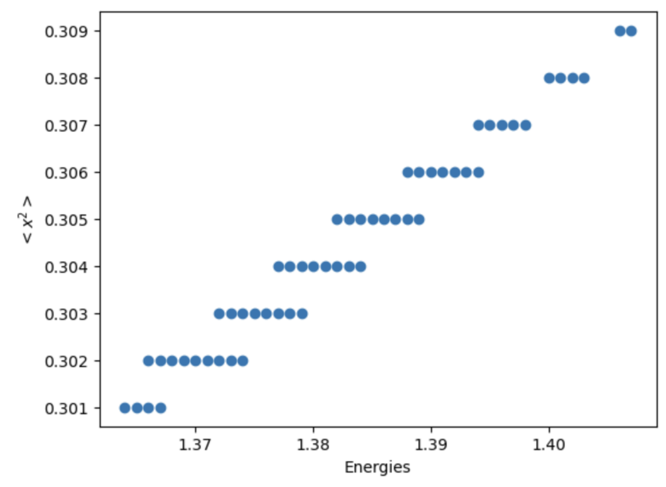
- Start with a Hamiltonian with a given potential and find a recursive statement for this potential with equation (4).
- We take a search space S and a set of trial points A within S.
  1. For each point in A, generate 2K-2 points of the moment sequence for each.
  2. For these terms, construct a KxK Hankel Matrix where $M_{ij} = \langle x^{i+j}\rangle$ for each point in A.
  3. Check positive definiteness of this matrix. If positive definite accept this point, if not throw it out.
  4. Obtain a set of values at depth K: B_K. Note: $B_K \subseteq A \subset S$ and, working through different values of K: $B_{K+1} \subseteq B_K$

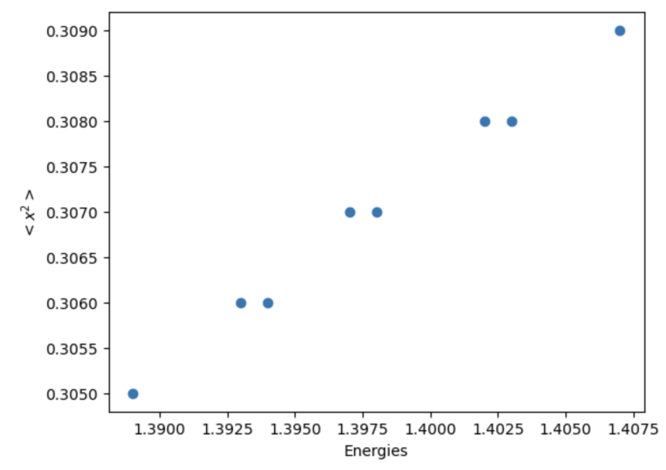
Infinite Anharmonic Oscillator

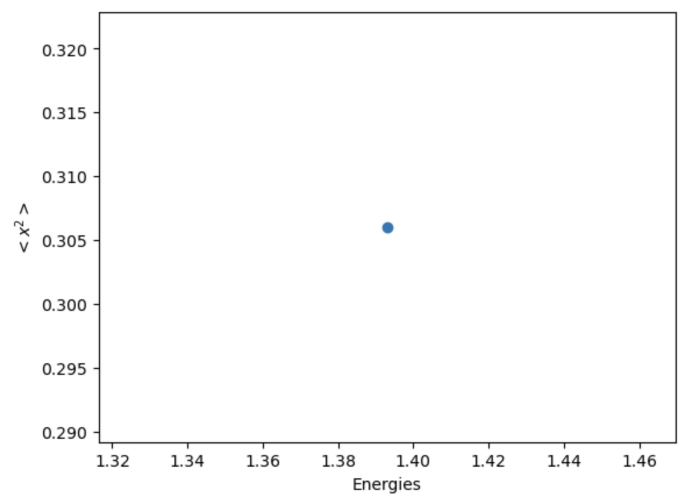
- How does the quantum mechanical bootstrap of [6,7] behave when solving well known systems? Below, the process is repeated for the anharmonic oscillator, reproducing the results from [6].
- $H = p^2 + gx^2 + hx^4$ with $g = h = m = \omega = \hbar = 1$











- From top down, depths at K= 7,8,9,10,12

Questions to be Addressed

- How does this bootstrap apply to digitized cases, and what, if any changes are to be made?
- What kind of information can be extracted from the digitized cases?

Digitized Oscillators

- Now, by taking $H_{nmax}(\hat{x}) = 0$ we can re-express the $x^{n_{max}}$ moment in terms of lower powers of x, via the Hermite polynomial. For example the $n_{max} = 4$ case:

$$\hat{x}^4 = 3\hat{x}^2 - \frac{3}{4} \tag{6}$$

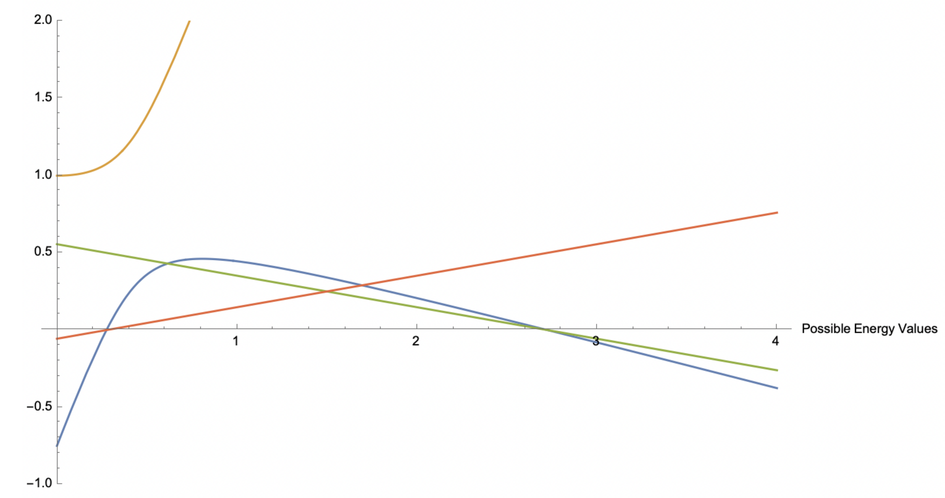
- Now, moments larger than n_{max} are superfluous. This limits us to a $\frac{n_{max}}{2} x \frac{n_{max}}{2}$ Hankel Matrix.
- New terms are introduced from the modified relations, for instance in the $n_{max} = 4$ case: $\langle E|n_{max}-1\rangle$

Digitized Oscillators

- Knowing $\langle E|\hat{x}^0|E\rangle, \langle E|\hat{x}^2|E\rangle, ..., \langle E|\hat{x}^{n_{max}-2}|E\rangle$ is equivalent to the knowledge of $|\langle x_j|E\rangle|^2 \geq 0$ for the $\frac{n_{max}}{2}$ positive zeros of the Hermite polynomial of degree n_{max}
- We predict that the positivity constraint imposed by the eigenvalues of the $\frac{n_{max}}{2} x \frac{n_{max}}{2}$ Hankel matrix are equivalent to the constraints on energy also imposed by the positivity of $|\langle x_j|E\rangle|^2 \geq 0$. We will attempt to show this for a digitized harmonic and anharmonic oscillator with $n_{max} = 4$.

Digitized Results

- The digitized harmonic oscillator:



- The orange and blue are from the eigenvalue process, while the green and red are from the Hermite polynomial process. The inner product is set to zero, but if included would just provide a shift of the graph. Energy falls within the range $\frac{3-\sqrt{6}}{2}$ to $\frac{3+\sqrt{6}}{2}$. Here we have:

$$\langle x^2\rangle = E - 2|\langle E|3\rangle|^2 \tag{7}$$

- When looking at the digitized anharmonic oscillator we get:

$$\begin{aligned} \langle x^2\rangle &= \frac{1}{6\lambda+1}(E - 2|\langle E|3\rangle|^2 \\ &+ 3\lambda\sqrt{6}\langle E|1\rangle\langle 3|E\rangle \\ &+ 9\lambda|\langle E|2\rangle|^2 + 9\lambda|\langle E|3\rangle|^2) \end{aligned} \tag{8}$$

- It is easy to see how this will return similar results to the digitized harmonic case. When $\lambda = 0$ it returns this case. The only differences for differing λ being a rescaling and coordinate shift of the same graph.
- For $\lambda = \frac{1}{2}$ and $\lambda = 1$ they would respectively have energy fall in ranges of $6 - 2\sqrt{6}$ to $6 + 2\sqrt{6}$ for the former and $\frac{21-7\sqrt{6}}{2}$ to $\frac{21+7\sqrt{6}}{2}$ for the latter.

References

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