# Relations between quantum error correction & gauge theory


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(本多正純)


based on a work w/o collaborators in progress

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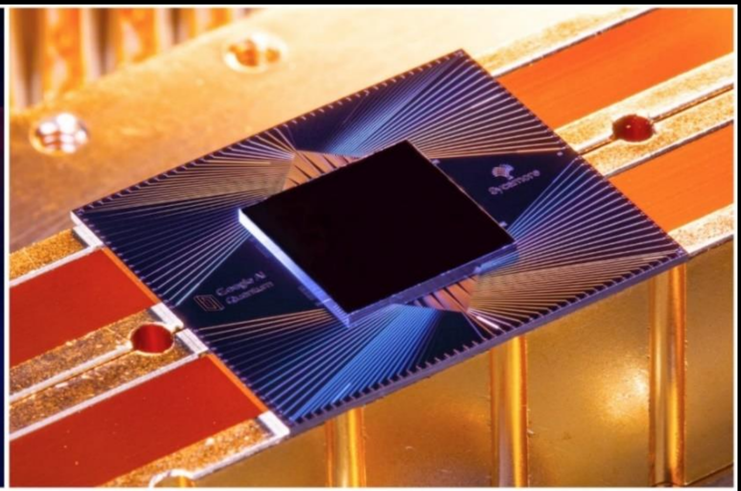
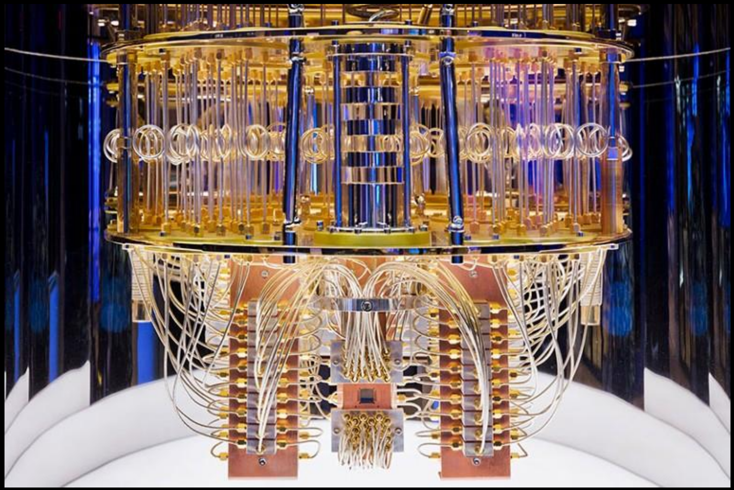
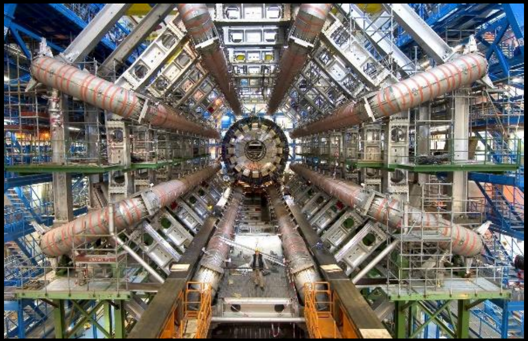
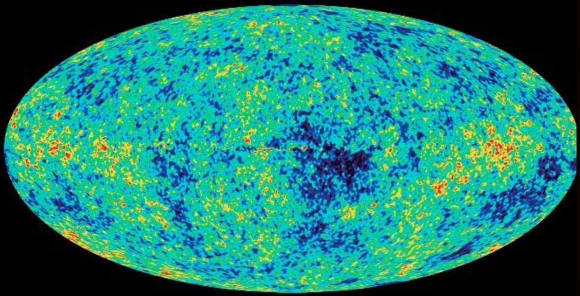
# Quantum simulation in future?









etc...

Quantum simulation is a promising approach
if $^{\exists}$much computational resource in future

Challenges:

- to get sufficient # of qubits to implement quantum error correction (QEC)
- to identify efficient ways to put gauge theory on quantum computers

This talk:

Quantum simulation is a promising approach if $^{\exists}$much computational resource in future

Challenges:

- to get sufficient # of qubits to implement quantum error correction (QEC)
- to identify efficient ways to put gauge theory on quantum computers

This talk:

relations between QEC & gauge theory

# relations between QEC & gauge theory

## Motivations

[Spirit may be similar to Gustafson-Lamm '23, Liu's talk etc...]

(some points elaborated later)

1. $^{\exists}$explicit examples

   ex.) Toric code = $\boldsymbol{Z_2}$ lattice gauge theory [Kitaev '97]

2.

3.

4.

# relations between QEC & gauge theory

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1. $^{\exists}$explicit examples

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2. Conceptual similarities:

   QEC = redundant description of logical qubits

   Gauge theory = redundant description of physical states

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3. Nature = Gauge theory & Nature = Quantum computer

   ⟹ Gauge theory may know something on QEC?

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4. ∃ proposals on relations among QEC & concepts in HEP

   ex.) Holography, Black hole information, 2d CFT

   [Almheiri-Dong-Harlow '14, Hayden-Preskill '07, Dymarsky-Shapere '20, Kawabata-Nishioka-Okuda '22, etc...]

# What I'm doing...

[MH, work in progress]

to make dictionary for classes of codes/gauge theories:

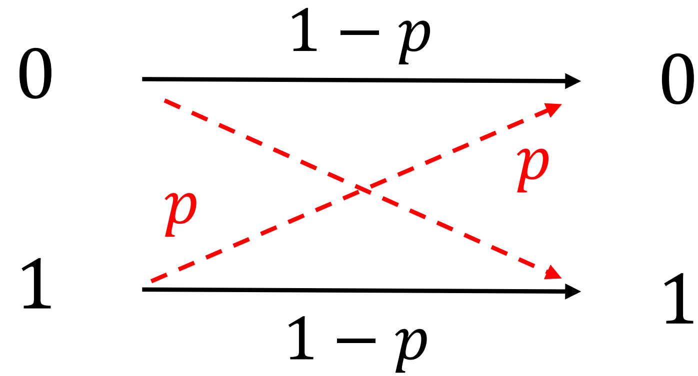
| QEC | Gauge theory |
| --- | --- |
| errors | unphysical process |
| logical qubits | physical states |
| "no error conditions" (stabilizer) | Gauss law & min(energy) |
| logical operator | gauge invariant operator |
| ancilla for recovery | additional matters |
| ⋮ | ⋮ |

# Contents

# Errors in classical computers

Computer interacts w/ environment → error/noise

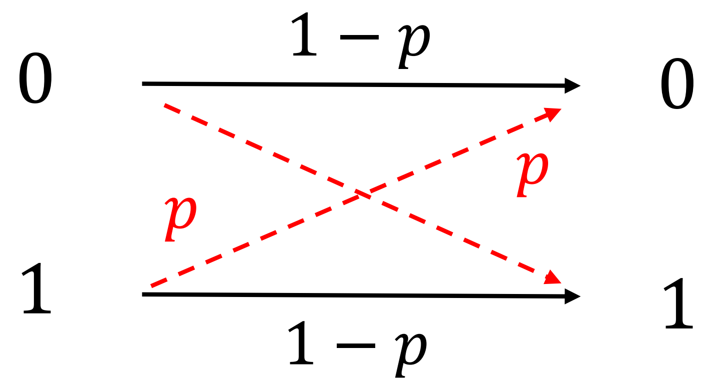
one bit →




Suppose we send a bit but have “error” in probability $p$

# Errors in classical computers

Computer interacts w/ environment ⇒ error/noise


Suppose we send a bit but have “error” in probability $p$

A simple way to correct errors:

① Duplicate the bit (encoding): $0 \to 000, \quad 1 \to 111$

② Error detection & correction by “majority voting”:

$$001 \to 000, \quad 011 \to 111, \quad \text{etc…}$$

⇒ $P_{\text{failed}} = 3p^2(1-p) + p^3$ (improved if $p < 1/2$)

# Errors in quantum computers

Computer interacts w/ environment ⇒ error/noise

▪ Unknown unitary operators are multiplied:

(in addition to decoherence & measurement errors)

$$|\psi\rangle \xrightarrow{\text{error!}} U|\psi\rangle$$

not only bit flip!

# Errors in quantum computers

Computer interacts w/ environment ➡ error/noise

- Unknown unitary operators are multiplied:

(in addition to decoherence & measurement errors)

$$|\psi\rangle \xrightarrow{\text{error!}} U|\psi\rangle$$

*not only bit flip!*

- have to detect errors & act "inverse of errors" to recover w/o destroying states

- need more qubits as in the classical case

# Ex.) 3-qubit bit flip code

## Bit flip error

$$|\psi\rangle \rightarrow \quad X|\psi\rangle \qquad \text{w/ probability } p$$

## Encoding

## Error detection

# Ex.) 3-qubit bit flip code

Bit flip error

$$|\psi\rangle \to X|\psi\rangle \quad \text{w/ probability } p$$

Encoding

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle \longrightarrow |\psi_E\rangle = c_0|000\rangle + c_1|111\rangle$$

Error detection

# Ex.) 3-qubit bit flip code

## Bit flip error

$$|\psi\rangle \to \quad X|\psi\rangle \qquad \text{w/ probability } p$$

## Encoding

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle \longrightarrow |\psi_E\rangle = c_0|000\rangle + c_1|111\rangle$$

## Error detection

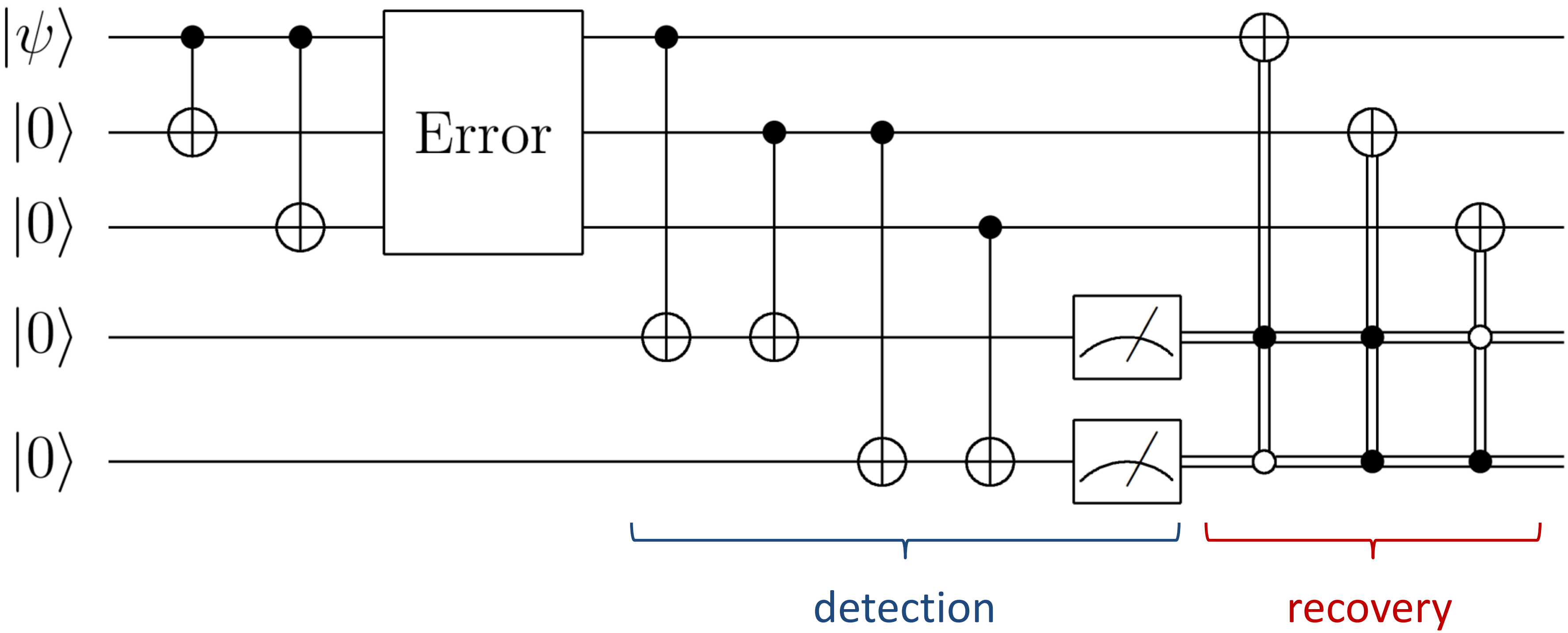
If error occurs once, we can detect the error by knowing

$$Z_1Z_2 \;\&\; Z_2Z_3$$

"No error" condition:

$$(Z_1Z_2)|\psi_E\rangle = |\psi_E\rangle\,, \qquad (Z_2Z_3)|\psi_E\rangle = |\psi_E\rangle$$

# Error recovery in 3-qubit bit flip code




As in the classical case, it fails if $^{\exists}$multiple "errors":

$$P_{\text{failed}} = 3p^2(1-p) + p^3 \qquad \text{(improved if } p < 1/2\text{)}$$

# Quantum Error Correction

## 1.Encoding

$$|\psi\rangle \in \mathcal{H} \longrightarrow |\psi_E\rangle \in \mathcal{H}_E \qquad (\mathcal{H} \subset \mathcal{H}_E)$$

## 2. Error detection

Take set of operators $\{O_1, \cdots\}$ s.t.

$$O_i|\psi_E\rangle = |\psi_E\rangle, \quad O_i(\text{error})|\psi_E\rangle \neq (\text{error})|\psi_E\rangle$$

Then find eigenvalues of $O_i$'s using ancillary qubits

## 3. Error recovery

Act "inverse of error" based on the eigenvalues

# Contents

# Conceptual similarity?

## Quantum error correction:

description of logical qubits by more qubits

Ex.) 3-qubit bit flip code

$$c_0|0\rangle + c_1|1\rangle \longrightarrow c_0|000\rangle + c_1|111\rangle$$

## Gauge theory:

# Conceptual similarity?

## Quantum error correction:

description of logical qubits by more qubits

Ex.) 3-qubit bit flip code

$$c_0|0\rangle + c_1|1\rangle \longrightarrow c_0|000\rangle + c_1|111\rangle$$

## Gauge theory:

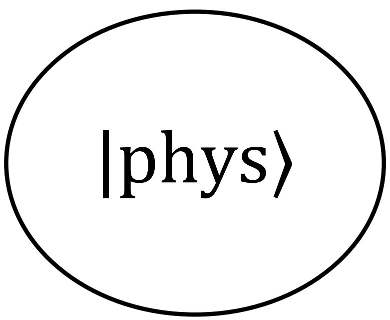
description of physical states by larger state space

Ex.) $U(1)$ gauge theory + matters

$$\nabla \cdot \hat{\boldsymbol{E}}(x)|\mathrm{phys}\rangle = \hat{\rho}(x)|\mathrm{phys}\rangle \qquad \text{"Gauss law"}$$

# Gauge theory on QC w/ error correction

When we don’t solve Gauss law before simulation…



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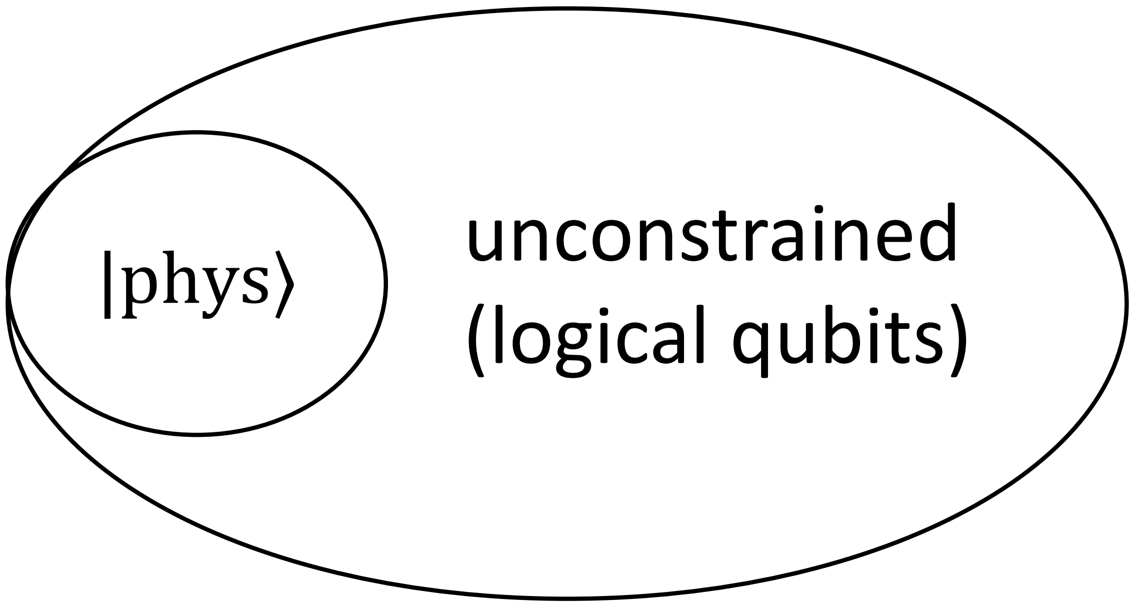

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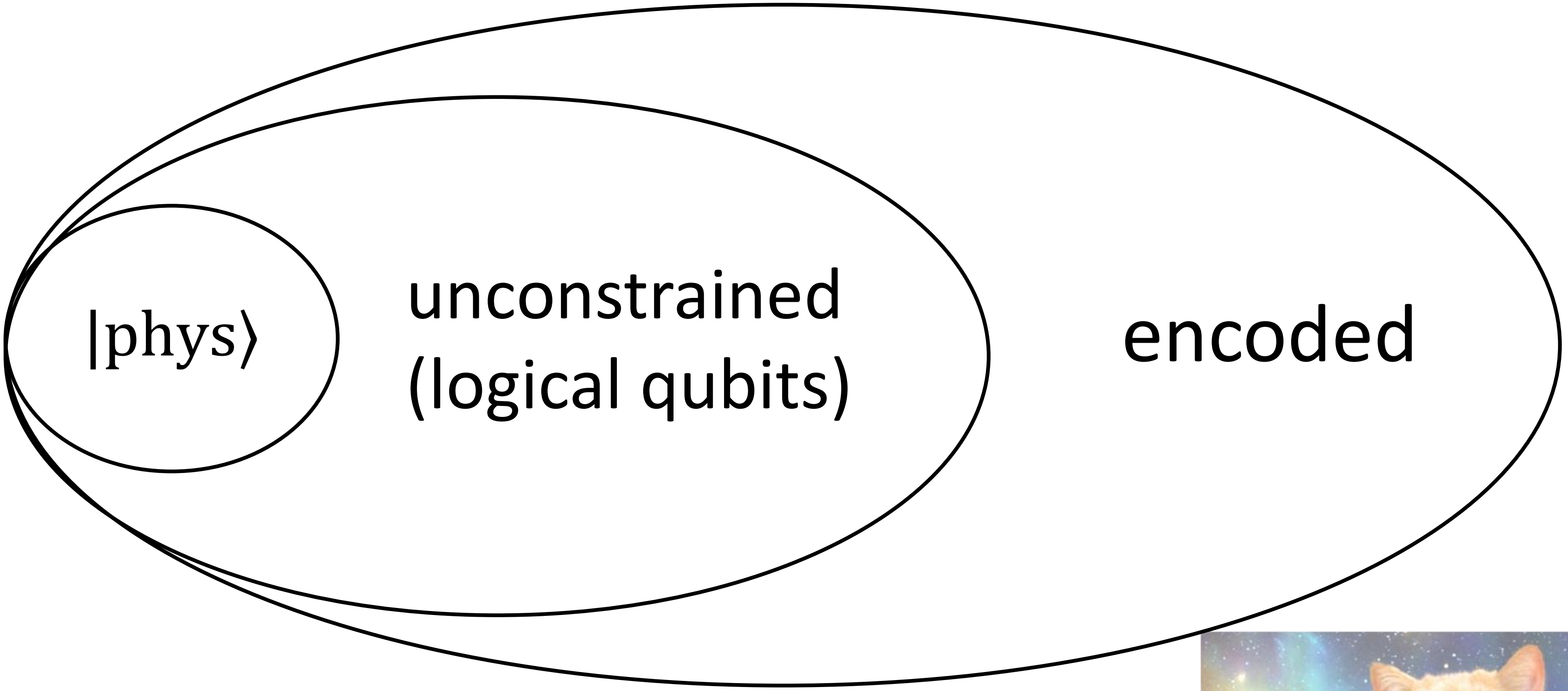



redundancy$^2$ !!

# Gauge theory on QC w/ error correction (cont'd)

Could we avoid the redundancy$^2$ ??

Possible hints:

- Nature = quantum computer
- Nature = gauge theory

⇨ quantum computer = gauge theory ??

Gauge theory knows something on error correction?

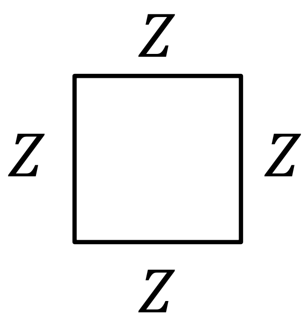
(I don't have a clear answer at this moment
but I'm trying to make connections precise)

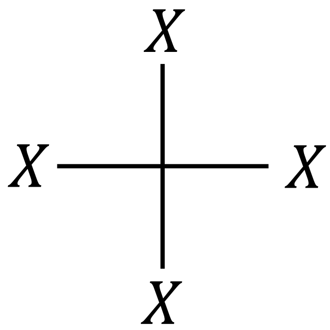
# Ex.) Toric code

[Kitaev '97]

Consider 2d periodic square lattice and put qubits on edges

$$H = -J \sum_{\text{face}} \prod_{e \in \partial(\text{face})} Z_e - J \sum_{\text{vertex}} \prod_{e | \partial e = \text{vertex}} X_e$$



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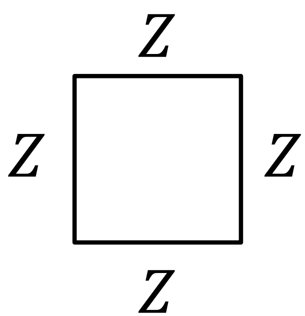


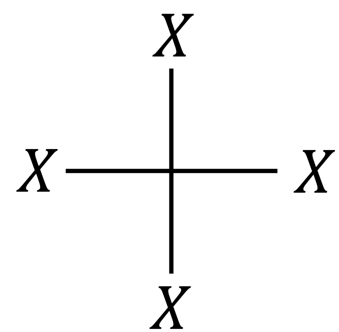


“No error” condition = minimum energy condition:

$$\prod_{e \in \partial(\text{face})} Z_e |\psi_E\rangle = |\psi_E\rangle, \quad \prod_{e | \partial e = \text{vertex}} X_e |\psi_E\rangle = |\psi_E\rangle$$

⇨ logical op. = products of $X, Z$ along nontrivial cycles

# Ex.) Toric code (cont'd)

$\boldsymbol{Z_2}$ gauge theory on 2d square lattice: $(U \sim e^{iA}, \Pi \sim e^{iE} \in \boldsymbol{Z_2})$

$$H = g^2 \sum_e \Pi_e - J \sum_{\text{face}} \prod_{e \in \partial(\text{face})} U_e$$

$(\Pi_e U_{e'} \Pi_e^\dagger = -\delta_{ee'} U_e)$

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Gauss law: $(\Pi_e U_{e'} \Pi_e^\dagger = -\delta_{ee'} U_e)$

$$\prod_{e | \partial e = \text{vertex}} \Pi_e \, |\text{phys}\rangle = |\text{phys}\rangle$$

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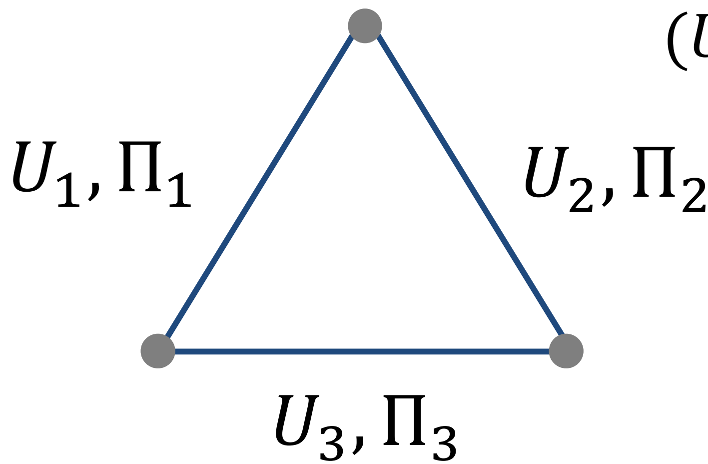
Ground state for $g = 0$:

$$\prod_{e|\partial e = \text{vertex}} U_e \,|\text{ground}\rangle = |\text{ground}\rangle$$

In identification ($U$-basis)~(computational basis),
this is the same condition as the toric code

# Ex.) $\boldsymbol{Z_2}$ lattice gauge theory on 3 sites

$(U \sim e^{iA}, \Pi \sim e^{iE} \in \boldsymbol{Z_2})$




## Hamiltonian:

$(\Pi_m U_n \Pi_m^\dagger = -\delta_{mn} U_n)$

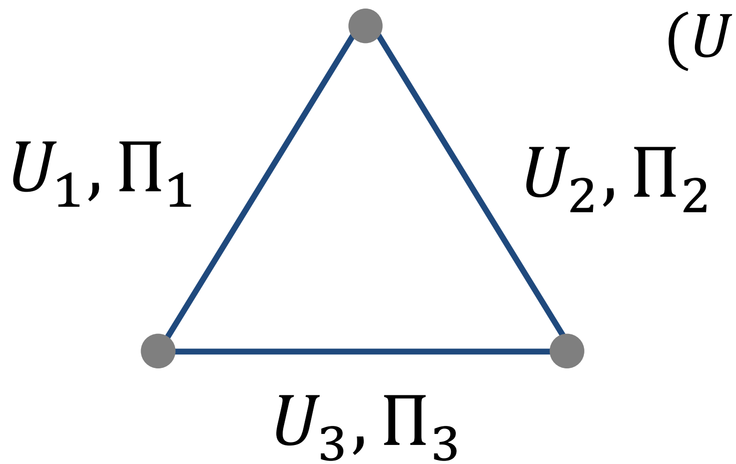
$$H = -J \sum_{n=1}^{3} \left( \Pi_n + \Pi_n^\dagger \right)$$

## Gauss law:

$$\Pi_n \Pi_{n-1}^\dagger |\mathrm{phys}\rangle = |\mathrm{phys}\rangle$$

# Ex.) $\boldsymbol{Z_2}$ lattice gauge theory on 3 sites (cont'd)

$(U \sim e^{iA}, \Pi \sim e^{iE} \in \boldsymbol{Z_2})$




$$\Pi_n \Pi_{n-1}^\dagger |\text{phys}\rangle = |\text{phys}\rangle$$

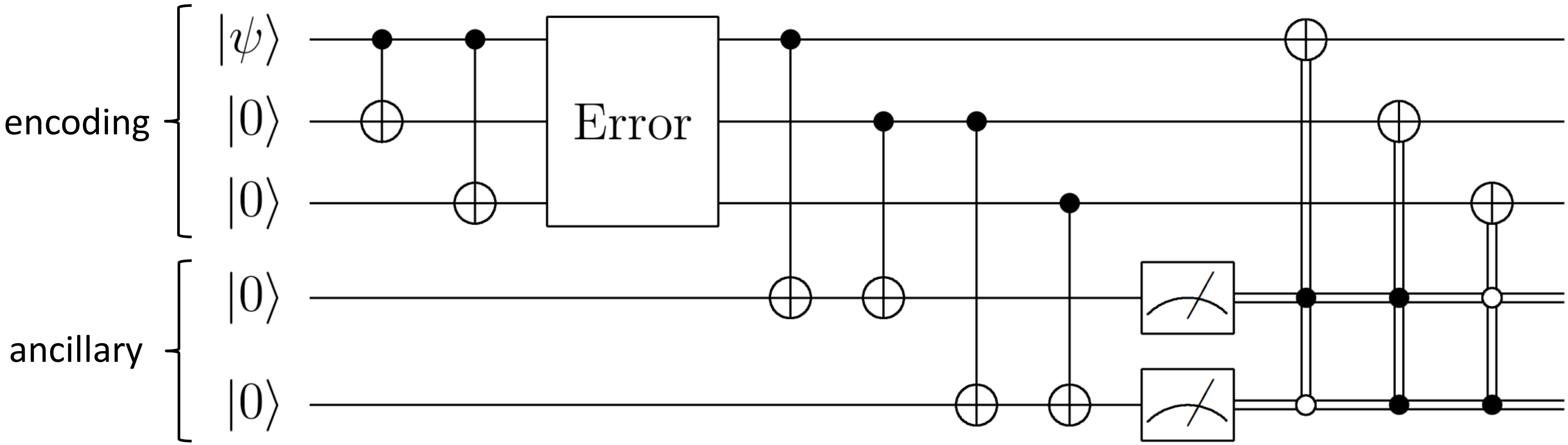
Taking (computational basis) ~ (eigenstate of $\Pi_n$)

⇩

$$Z_1 Z_2 |\text{phys}\rangle = |\text{phys}\rangle\,, \quad Z_2 Z_3 |\text{phys}\rangle = |\text{phys}\rangle$$

"no error" condition in 3-qubit bit flip code!

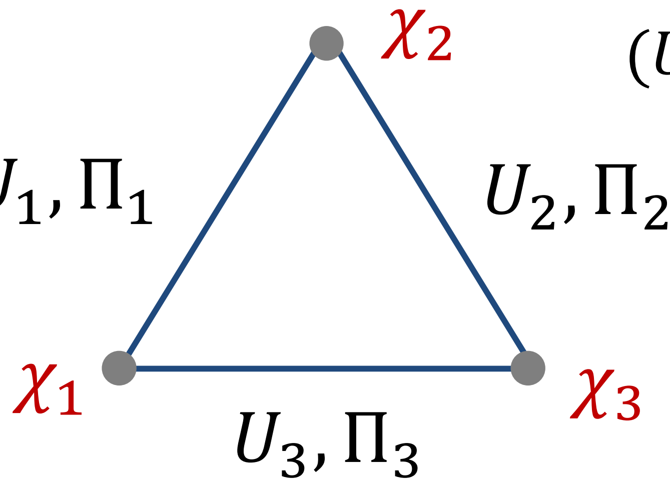
# Error detection & recovery




Is there analogue of this in gauge theory?

—— Ancilla may be matter on sites (next slide)

# $\boldsymbol{Z}_2$ lattice gauge theory w/ a complex fermion




$(U \sim e^{iA}, \Pi \sim e^{iE} \in \boldsymbol{Z}_2)$

## Hamiltonian:

$$H = -J \sum_{n=1}^{3} \left(\Pi_n + \Pi_n^\dagger\right) + w \sum_{n=1}^{3} (\chi_{n+1}^\dagger U_n \chi_n - \chi_n^\dagger U_n^\dagger \chi_{n+1})$$

## Commutation relation:

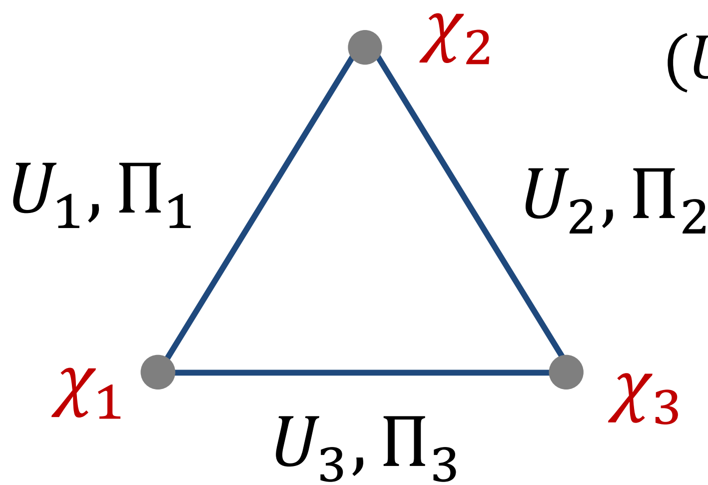
$$\Pi_m U_n \Pi_m^\dagger = -\delta_{mn} U_n \qquad \left\{\chi_m, \chi_n^\dagger\right\} = \delta_{mn}$$

## Gauss law:

$$\Pi_n \Pi_{n-1}^\dagger |\text{phys}\rangle = e^{i\pi \chi_n^\dagger \chi_n} |\text{phys}\rangle$$

# $\boldsymbol{Z_2}$ lattice gauge theory w/ a complex fermion (cont'd)

$(U \sim e^{iA}, \Pi \sim e^{iE} \in \boldsymbol{Z_2})$




$$\Pi_n \Pi_{n-1}^\dagger |\text{phys}\rangle = e^{i\pi \chi_n^\dagger \chi_n} |\text{phys}\rangle$$

Taking (computational basis) $\sim$ (eigenstate of $\Pi_n$)

$\Downarrow$

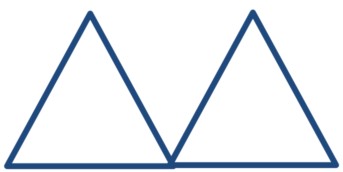
$$Z_1 Z_2 |\text{phys}\rangle = e^{i\pi \chi_n^\dagger \chi_n} |\text{phys}\rangle, \quad Z_2 Z_3 |\text{phys}\rangle = e^{i\pi \chi_n^\dagger \chi_n} |\text{phys}\rangle$$

Measuring Fermion charge = Syndrome measurement?

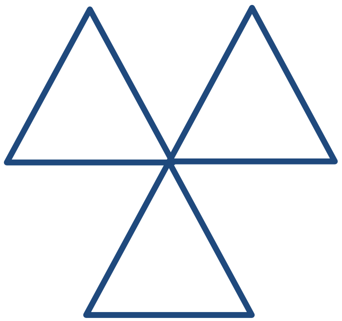
# Some generalizations

[MH, work in progress]

- $\boldsymbol{Z_2}$ theory on 1d periodic lattice w/ $(2n+1)$ sites = $[2n+1,1,2n+1]$ code



$= [6,2,3]$,



$= [9,3,3], \cdots$

- Phase flip code is done by changing basis
- Shor code seems to need products of plaquettes
- $\boldsymbol{Z_2} \to \boldsymbol{Z_N}$ makes qubit qudit w/ $d = N$
- 5-qubit perfect code is a special case of variant of toric code [Bonilla Ataides etal. '20]

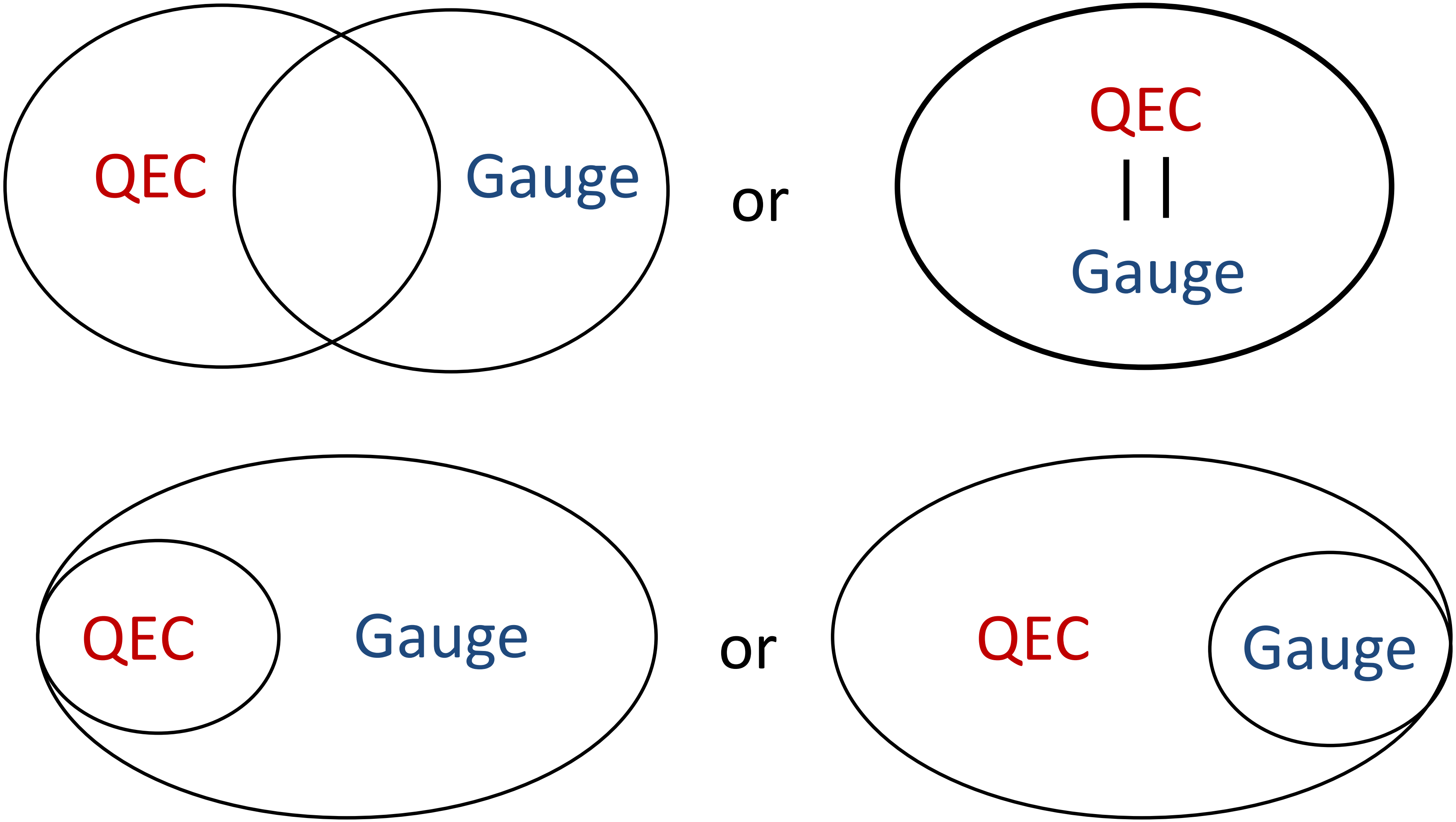
# Summary

# Summary

[MH, work in progress]

*"QEC/Gauge correspondence":*

| QEC | Gauge theory |
| --- | --- |
| errors | unphysical process |
| logical qubits | physical states |
| "no error conditions" (stabilizer) | Gauss law & min(energy) |
| logical operator | gauge invariant operator |
| ancilla for recovery | additional matters |
| ⋮ | ⋮ |




???

Thanks!

# Appendix

# Code distance

## Hamming distance:

$$d(x, y) \equiv \text{(\# of different components between } x \text{ \& } y\text{)}$$

## Hamming weight:

$$\mathrm{wt}(x) \equiv d(x, 0)$$

In particular,

$$\mathrm{d(x, y)} = \mathrm{wt}(x + y)$$

## Distance of a code $C$:

$C$: "$[n, k, d]$ code"

$$\mathrm{d(C)} \equiv \min_{y_1, y_2 \in C,\, y_1 \neq y_2} d(y_1, y_2) = \min_{y \in C, y \neq 0} \mathrm{wt}(y)$$

If $d(C) = 2t + 1$, then we can correct errors on up to $t$-bits