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Hadronic susceptibilities for b→c transitions from two-point correlation functions

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Susceptibilities

Two-point function of a $b \to c$ flavor-changing current $J_\mu = V_\mu, A_\mu$ $\left(V_\mu = \bar{c}\gamma_\mu b\,,\, A_\mu = \bar{c}\gamma_5\gamma_\mu b \right)$ splits into

Transverse (spin 1) and Longitudinal (spin 0) polarization functions $\Pi_{T,L}$:

$$\Pi_{\mu\nu}(q) = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) \Pi_T(q^2) + \frac{q_\mu q_\nu}{q^2} \Pi_L(q^2)$$

Choosing $q = (q_0, \vec{0})$, polarization functions in terms of the 2 point correlation functions $C(t)$:

$$\Pi_T(q^2) = i \int_{-\infty}^{+\infty} dt\, e^{iq\cdot t}\, C_i\,(t) \qquad \Pi_L(q^2) = i \int_{-\infty}^{+\infty} dt\, e^{iq\cdot t}\, C_0\,(t)$$

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Definition:

$$\chi_T(q^2) \equiv \frac{1}{2} \frac{\partial^2 \Pi_T}{\partial (q^2)^2} : \qquad \chi_T(q^2 = 0) = \frac{1}{12} \int_0^\infty dt\, t^4 C_i(t)$$

$$\chi_L(q^2) \equiv \frac{\partial \Pi_L}{\partial q^2} : \qquad \chi_L(q^2 = 0) = \int_0^\infty dt\, t^2 C_0(t) \xrightarrow[\text{Free from contact terms}]{\text{Ward Identity}} \frac{(m_b \mp m_c)^2}{12} \int_0^\infty dt\, t^4 C^{S,P}(t)$$

$(S, P = \bar{c}b, \bar{c}\gamma_5 b)$

Physics Motivations

- Tension with SM in lepton universality ratios $R(D^{(*)}) \equiv \frac{BR(B \to D^{(*)}\tau\nu_\tau)}{BR(B \to D^{(*)}\ell\nu_\ell)}$
- Long standing discrepancy in determination of $|V_{cb}|$ from exclusive and inclusive semileptonic *B*-decays

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Example: hadronic matrix $\langle D|V_\mu|\bar{B}\rangle$ element in $B \to D\ell\nu_\ell$ described by two form factors $f_{+,0}(q^2)$

$$\langle D|V_\mu|\bar{B}\rangle = f_+\left[(p_B + p_D)_\mu - \Delta m^2_{BD} q_\mu/q^2\right] + f_0\, \Delta m^2_{BD} q_\mu/q^2$$

$$\frac{d\Gamma(B \to D\ell\nu_\ell)}{dq^2} = \underbrace{\frac{G_F^2 \eta_{EW}^2 m_B \lambda^{1/2}}{192\pi^3}}_{\text{Known}} |V_{cb}|^2 (q^2 - m_\ell)^2 \left[c_+ \underbrace{|f_+(q^2)|^2}_{\text{From Lattice}} + c_0 \underbrace{|f_0(q^2)|^2}_{\text{From Lattice}}\right]$$

Semileptonic region:

$$m_\ell^2 \leq q^2 = (p_B - p_D)^2 \leq q^2_{max}$$

Lattice simulations more precise close to $q^2_{max} = (m_B - m_D)^2$ but no experimental data at q^2_{max}

Parametrizations to access the full semileptonic region $q^2 \to 0$ (BGL & CLN most popular)

Constraining form factors

- **Dispersion relation:** $\chi_T(q^2) \equiv \frac{1}{2}\frac{\partial^2 \Pi_T}{\partial (q^2)^2} = \frac{1}{\pi}\int_0^\infty dt \frac{\mathrm{Im}\,\Pi_T(t)}{(t-q^2)^3}$

- **Crossing symmetry** $\langle D | V_\mu | \bar{B} \rangle \to \langle 0 | V_\mu | \overline{DB} \rangle$ **:**

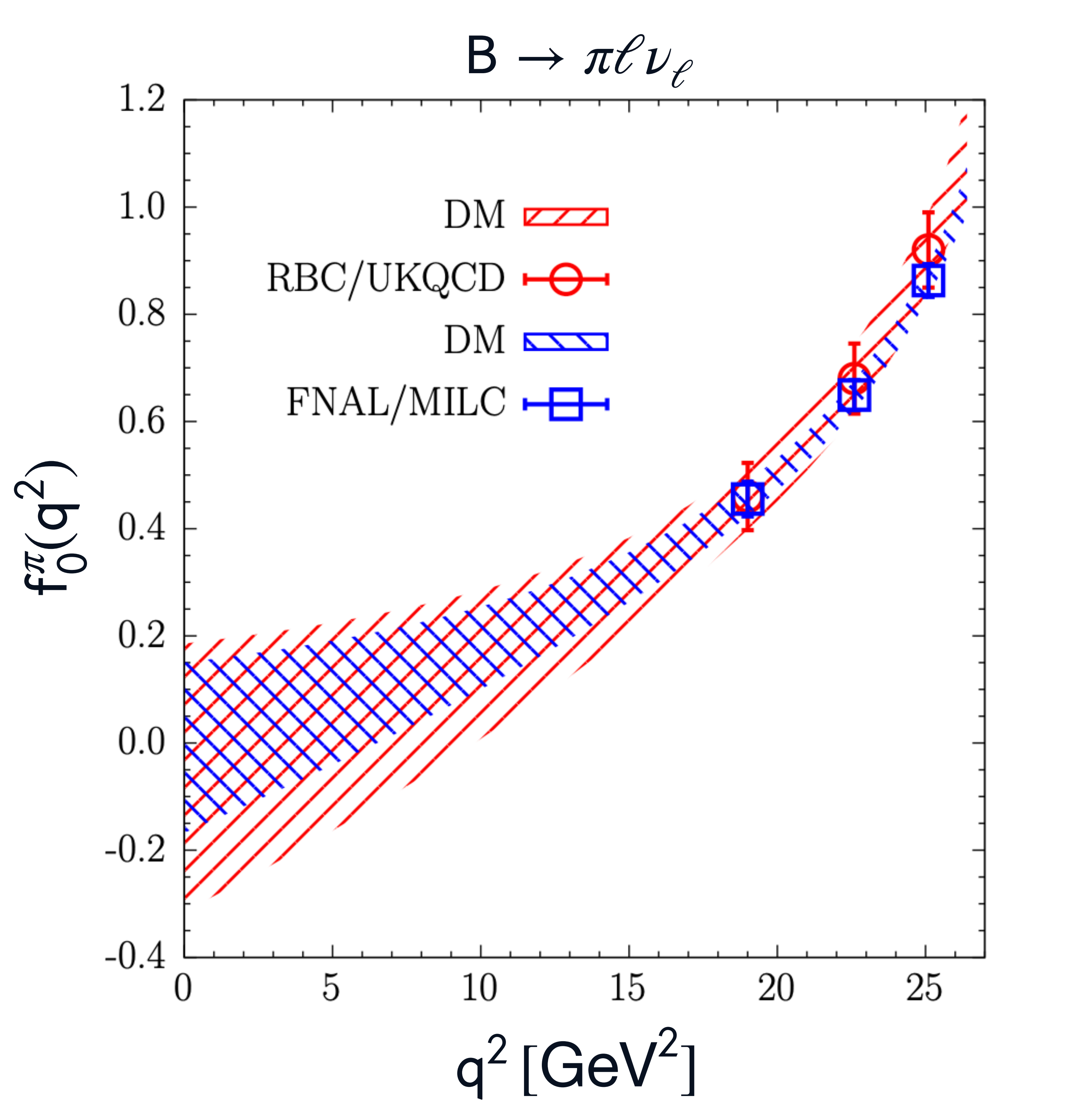
- **Unitarity:** $\mathrm{Im}\,\Pi_T(q^2) \geq \frac{1}{2}\int d^3\tilde{p}_B d^3\tilde{p}_D\, \delta^{(4)}(q - p_B - p_D)\, |\langle 0 | V | \overline{DB} \rangle|^2 + \cdots$ ✗

$$\chi_T(q^2) \geq \frac{1}{\pi}\int_{(m_B+m_D)^2}^\infty dt \frac{w(t)\,|f(t)|^2}{(t-q^2)^3}$$

- **Analyticity:** traslate this bound into semileptonic region, from lattice points at q^2_{max} extrapolate to $q^2 \to 0$ with no assumption on q^2-dependence

  See hep-ph/9509358 Lellouch, arXiv:2105.02497 Di Carlo at al.

Figure from arXiv:2205.097421
G.Martilelli, M.Naviglio, S.Simula, L.Vittorio



Lattice setup

Ensemble	$(L^3 \cdot T)/a^4$	L [fm]	m_π[MeV]	$a\mu_{ud}$	$a\mu_s$	$a\mu_c$	$a\mu_h$	N_{cfg}
$\beta = 1.726$ $a = 0.09076(54)$ $w_0/a = 1.8352(35)$ $a\mu_s^{phys} = 0.02005(25)$ $a\mu_c^{phys} = 0.2748(27)$								
A211.53.24	$24^3 \cdot 48$	2.18	361.6(2.1)	0.0053	0.0185	0.2480	0.3390, 0.6336	615
A211.40.24	$24^3 \cdot 48$	2.18	315.7(2.0)	0.0040	0.0205	0.2900	0.3964, 0.7408	660
A211.30.32	$32^3 \cdot 64$	2.90	272.2(1.7)	0.0032			0.4635, 0.8661	615
A211.12.48	$48^3 \cdot 96$		167.1(0.8)	0.0048			0.5419, 1.0126	219
$\beta = 1.778$ $a = 0.07957(13)$ $w_0/a = 2.1299(16)$ $a\mu_s^{phys} = 0.018414(69)$ $a\mu_c^{phys} = 0.23859(78)$								
B211.25.48	$48^3 \cdot 96$	2.18	253.3(1.4)	0.0048	0.0170	0.2200	0.3007, 0.5620	300
B211.072.64	$64^3 \cdot 128$	5.09	140.1(0.2)	0.00072	0.0190	0.2572	0.3516, 0.6571	190
							0.4111, 0.7683	
							0.4807, 0.8983	
$\beta = 1.836$ $a = 0.06821(13)$ $w_0/a = 2.5045(17)$ $a\mu_s^{phys} = 0.016176(72)$ $a\mu_c^{phys} = 0.20253(74)$								
C211.060.80	$80^3 \cdot 160$	5.46	136.7(0.2)	0.00080	0.0155	0.1920	0.2625, 0.4905	165
					0.0175	0.2245	0.3069, 0.5735	
							0.3588, 0.6705	
							0.4195, 0.7840	
$\beta = 1.900$ $a = 0.05692(12)$ $w_0/a = 2.5045(17)$ $a\mu_s^{phys} = 0.013647(58)$ $a\mu_c^{phys} = 0.16710(52)$								
D211.054.96	$96^3 \cdot 192$	5.46	140.8(0.2)	0.00096	0.0130	0.1600	0.2187, 0.4087	43
					0.0140	0.1871	0.2557, 0.4779	
							0.2990, 0.5588	
							0.3496, 0.6533	

2-point correlation functions from ETMC gauge ensembles

- $N_f = 2 + 1 + 1$ flavors of Wilson-Clover twisted-mass quarks
- All dynamical quark masses close to physical values (most ensembles)
- Four values of the lattice spacing to extrapolate to the continuum limit
- Two variations (r, r) and (r, -r) of twisted mass regularization to constrain the continuum limit
- Heavy-quark masses $m_h^{(n)}$ to extrapolate to b-quark mass

b-quark mass determination

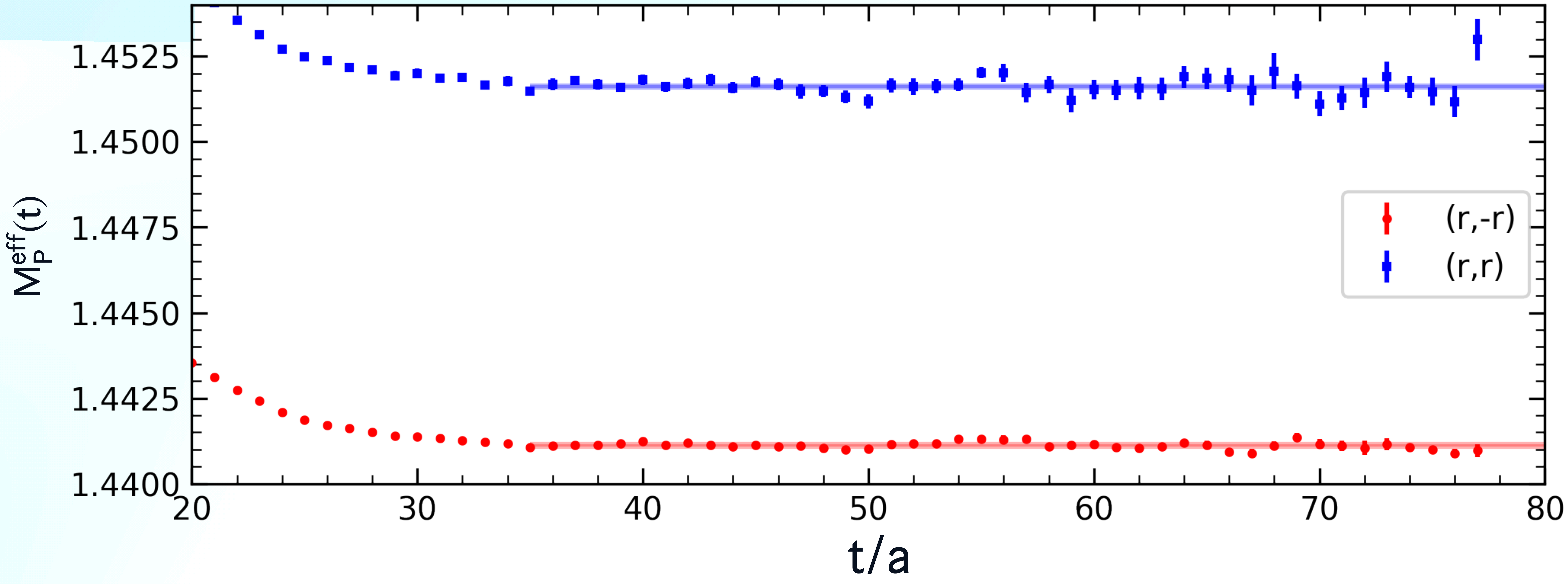
Effective mass of heavy-charmed mesons

b-quark mass determination from M_{B_c}

Extraction of meson masses $M_P(a^2, m_h^{(n)}, m_c)$ from constant fit at large time distances of effective mass $M_P^{eff}(t)$

$$M_P^{eff}(t\,; a^2, m_h^{(n)}, m_c) = \mathrm{Arccosh}\left(\frac{C^P(t) + C^P(t+2)}{2\,C^P(t+1)}\right) \xrightarrow{t \gg 0} M_P(a^2, m_h^{(n)}, m_c)$$

Effective mass at $m_h^{(n)} = 2.2\,m_c$

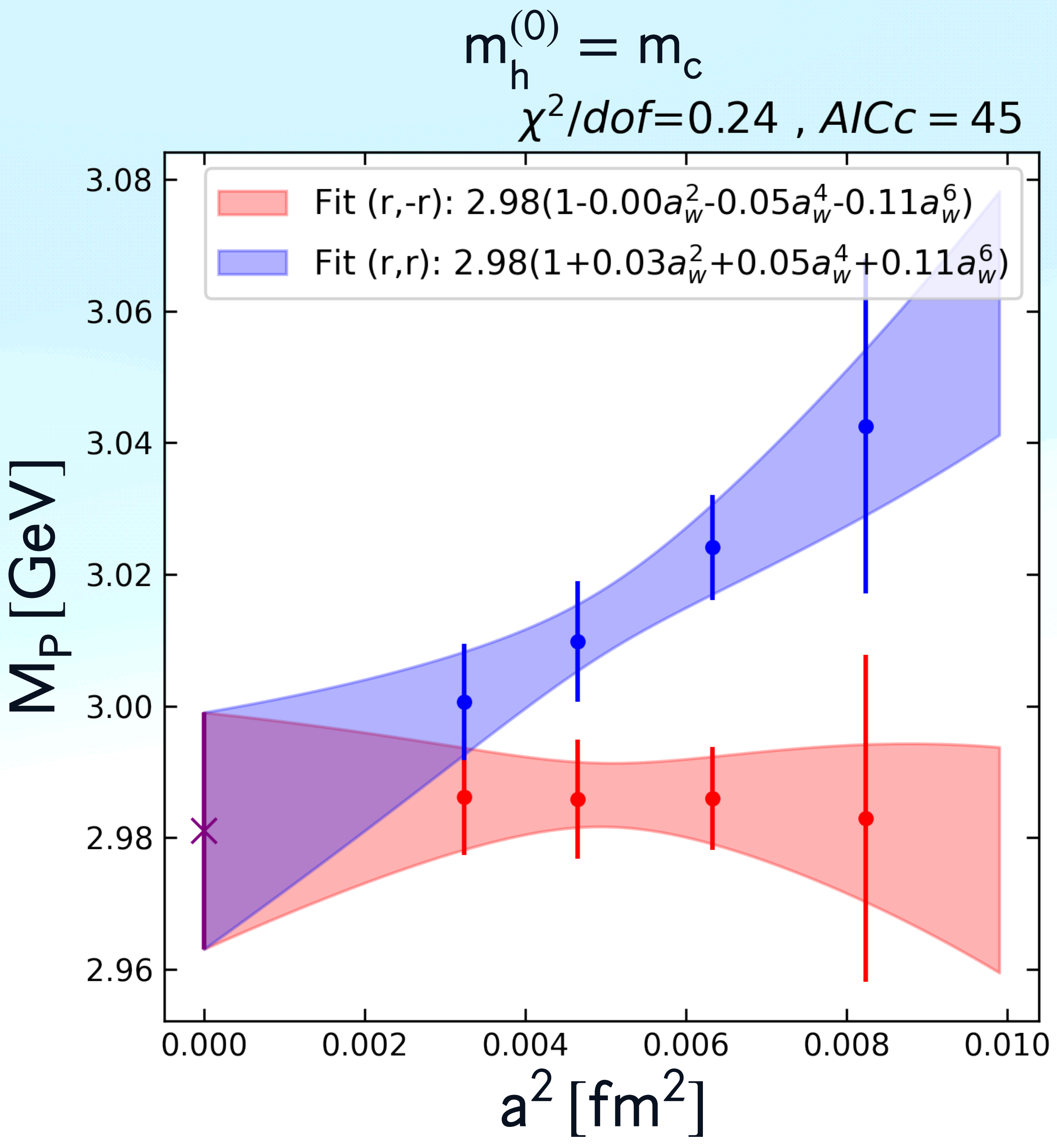


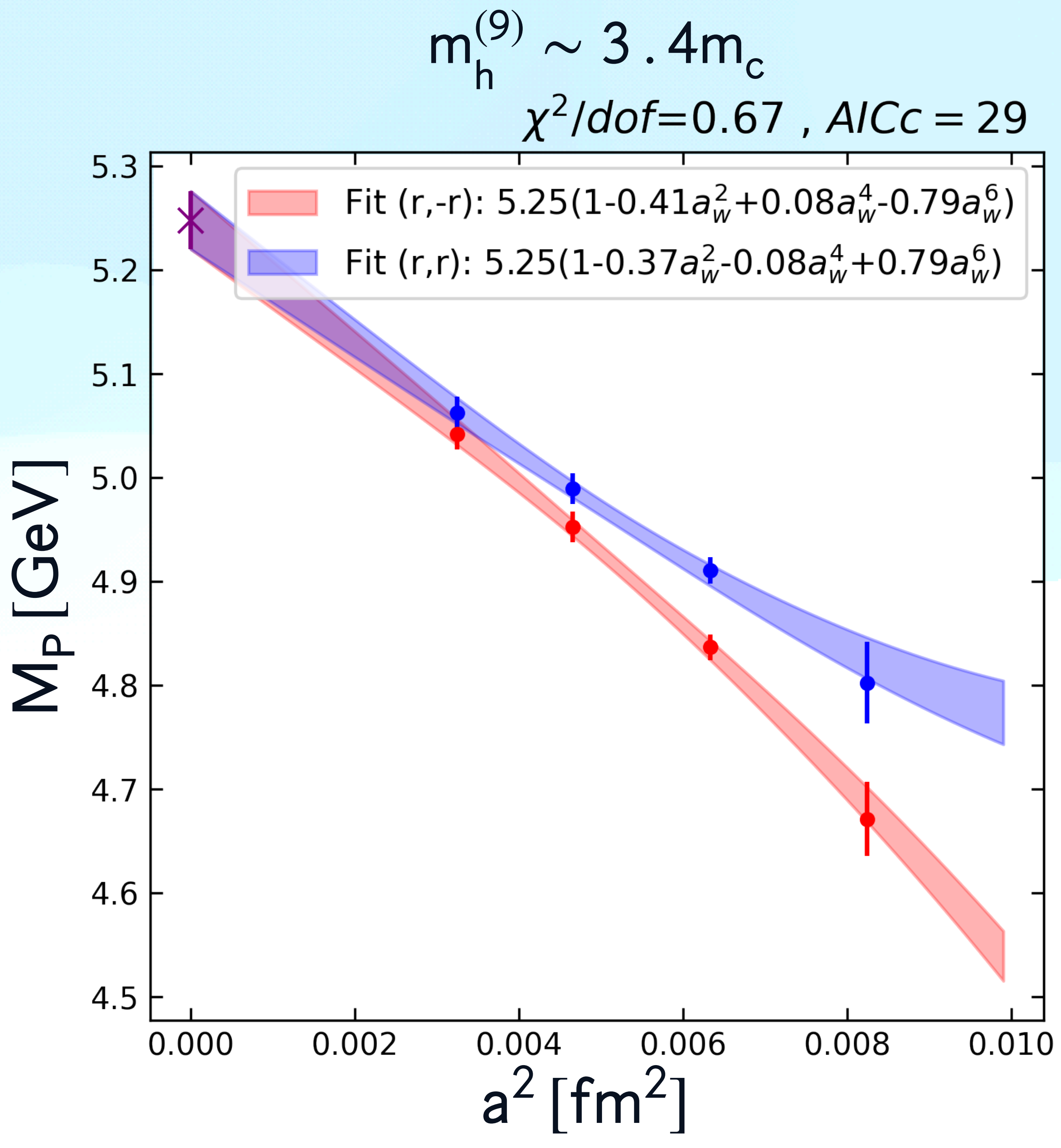
Continuum limit of $M_P(a^2; m_h^{(n)}, m_c)$

- Combined fit of (r, r) and (r, -r) regularizations imposing same continuum limit
- Highly correlated data $\rho \sim 1$: we study mean and difference of the two data sets
- Polynomial fit ansatz up to $\mathcal{O}(a^6)$:

$$M_P^{(r,r)}(a^2; m_h^{(n)}, m_c) = M_P(m_h^{(n)}, m_c)\left(1 + P_1^{(r,r)}a^2 + P_2^{(r,r)}a^4 + P_3^{(r,r)}a^6\right)$$

$$M_P^{(r,-r)}(a^2; m_h^{(n)}, m_c) = M_P(m_h^{(n)}, m_c)\left(1 + P_1^{(r,-r)}a^2 - P_2^{(r,r)}a^4 - P_3^{(r,r)}a^6\right)$$





$M_P(m_c, m_c) \equiv M_{\eta_c} = 2981(18)$ MeV

b-quark mass

Simple phenomenological ansatz determination: mass of heavy meson goes to infinity when its heavy quark constituent mass $m_h \to \infty$:

$$M_P^{-1}(m_h^{(n)}, m_c) = \frac{m_c}{m_h} B_1 \left(1 + B_2 \frac{m_c}{m_h} + B_3 \frac{m_c^2}{m_h^2} + B_4 \frac{m_c^3}{m_h^3} \right)$$

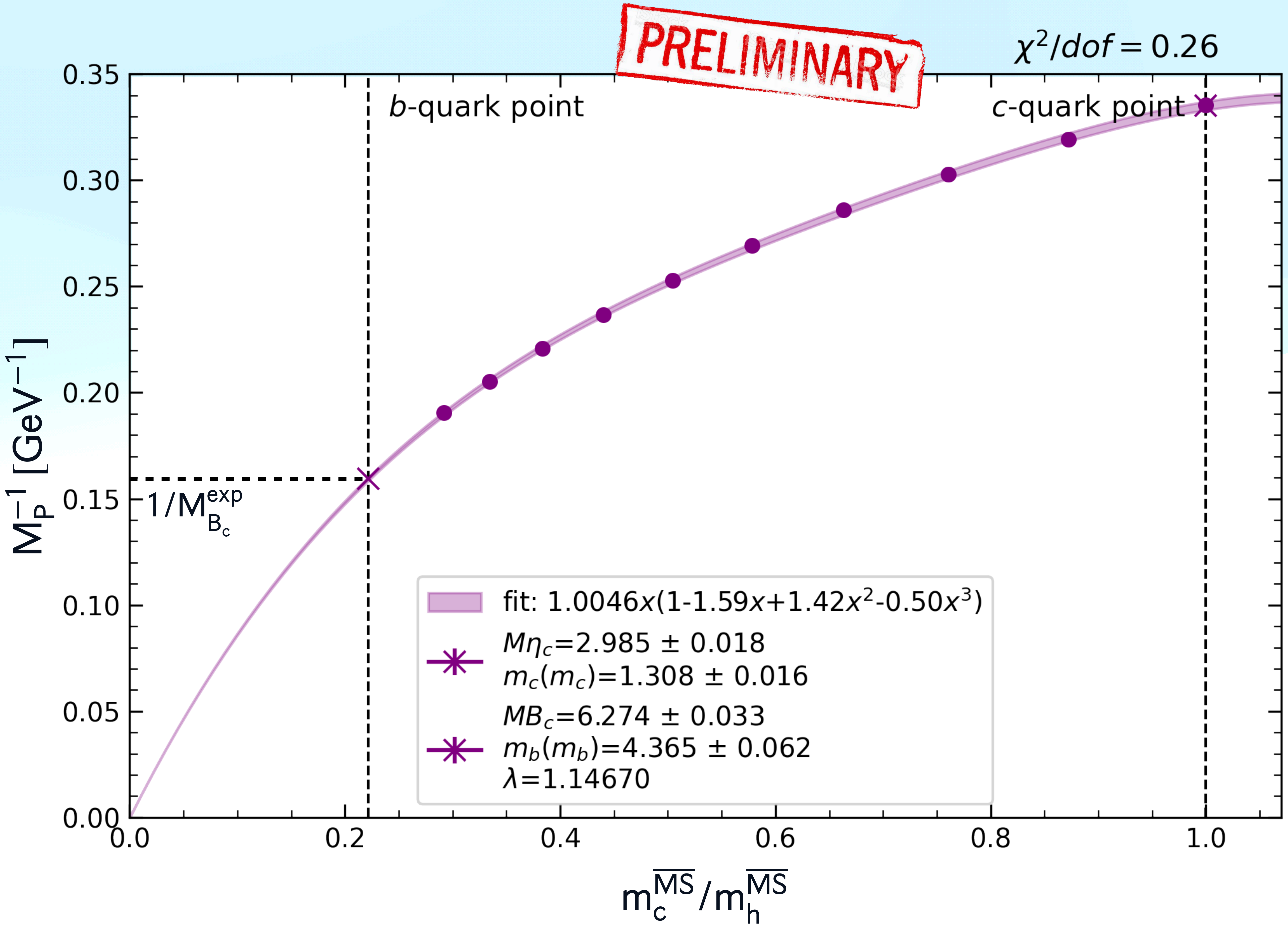
Fix m_b/m_c imposing

$$M_P(m_b, m_c) = M_{B_c}^{exp}$$

$$\Rightarrow m_b^{\overline{MS}}/m_c^{\overline{MS}} = 4.5075$$

Multiplying by $m_c^{\overline{MS}}$(3GeV)=1.039(17) :
arXiv: 2104.13408 ETM '21

$$m_b^{\overline{MS}}(3\text{GeV}) = 4.683(77)\,\text{GeV}$$
$$m_b(m_b) = 4.365(62)\,\text{GeV}$$



Susceptibilities

Update w.r.t. ETM '21

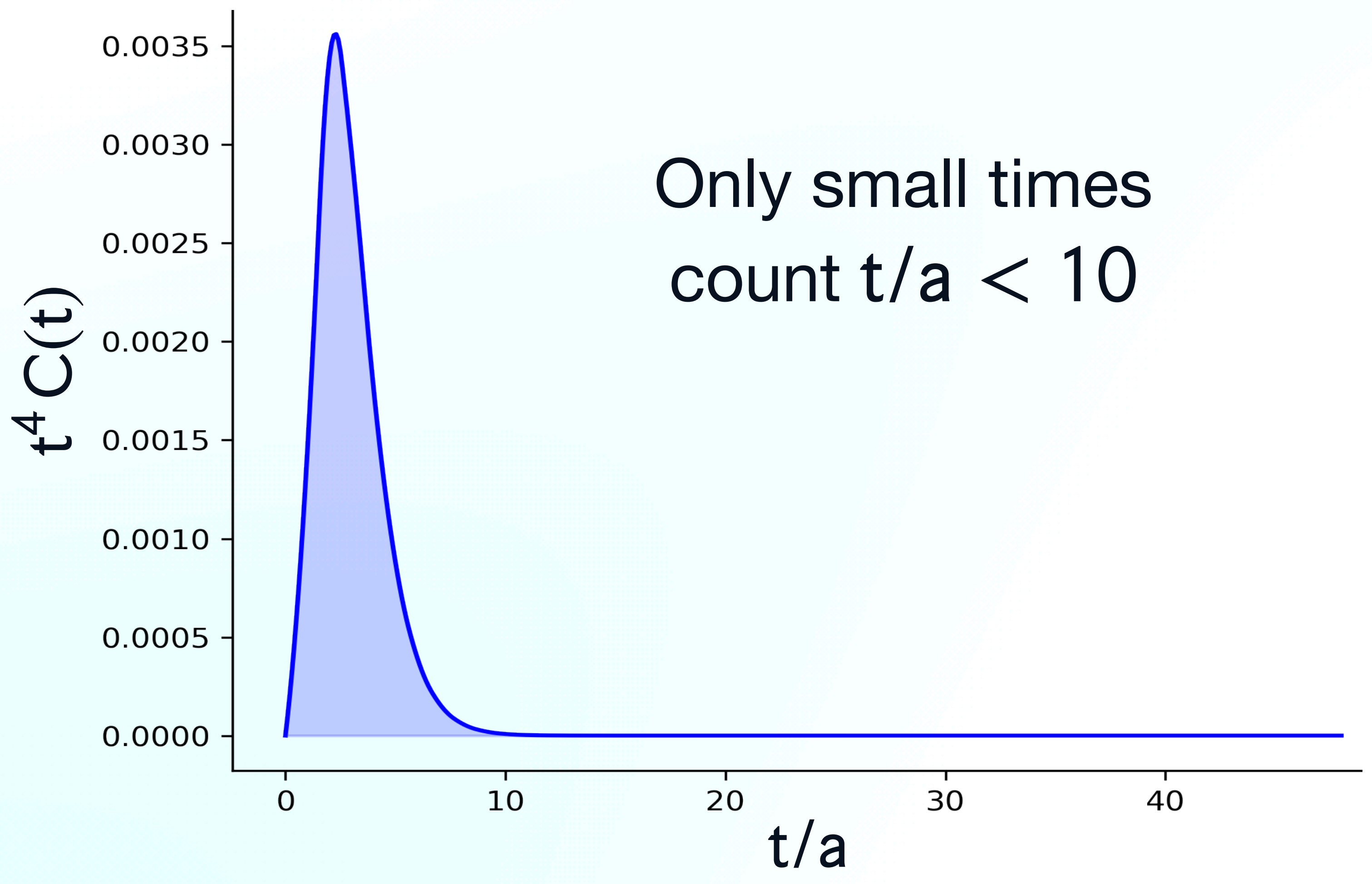
arXiv:2105.07851
G.Martilelli, S.Simula, L.Vittorio

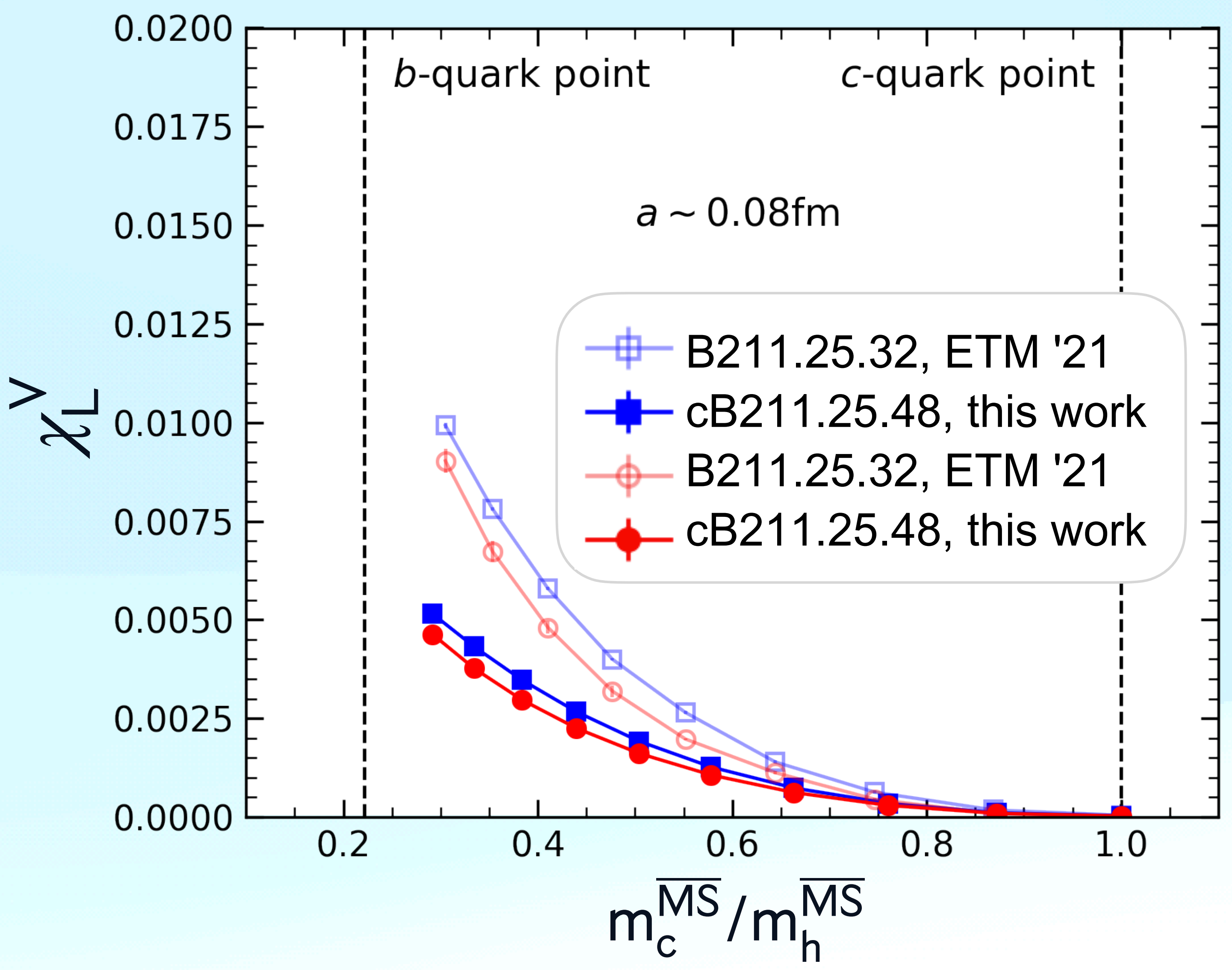
Clover term and improved precision

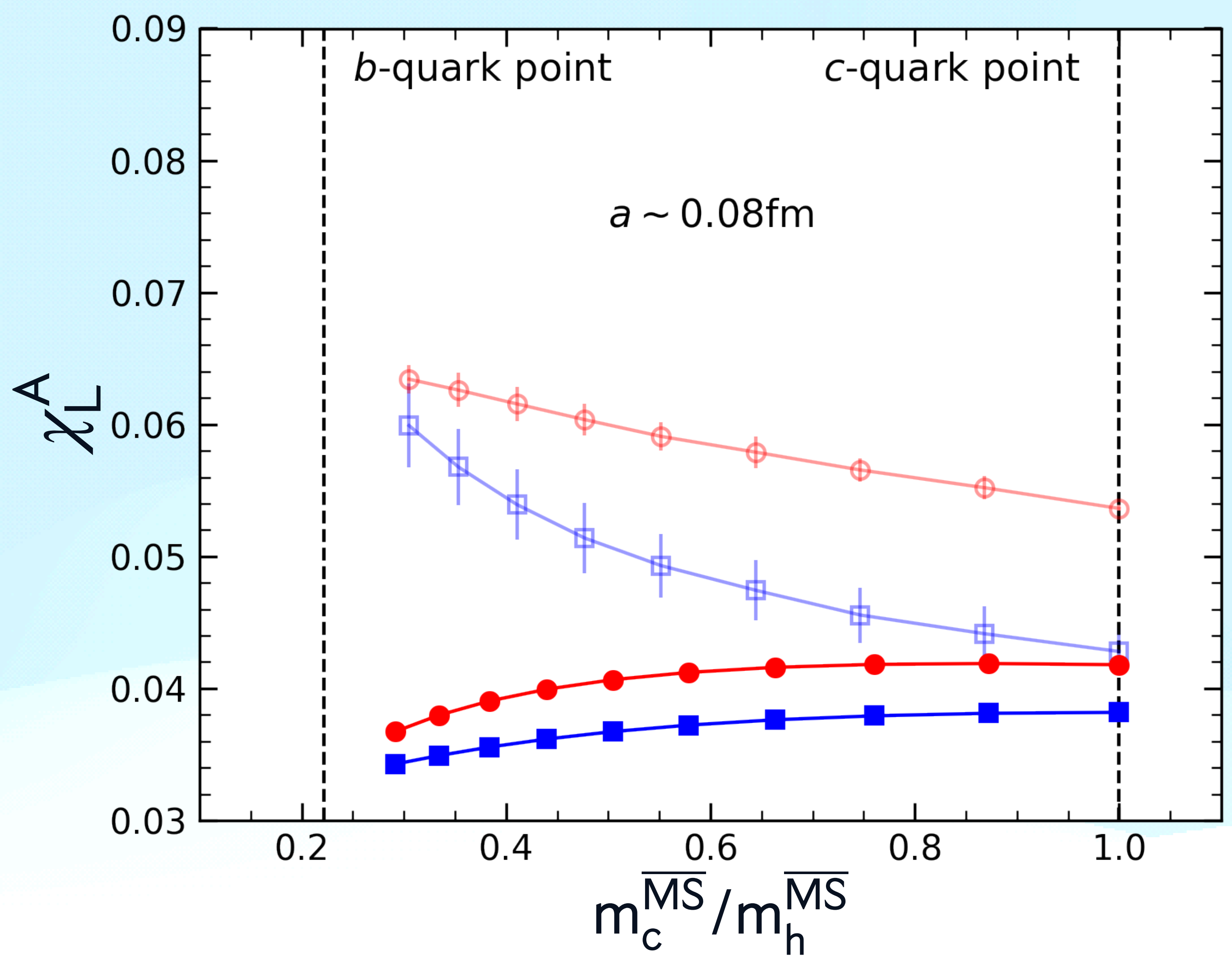
Computed quantities for each ensemble:

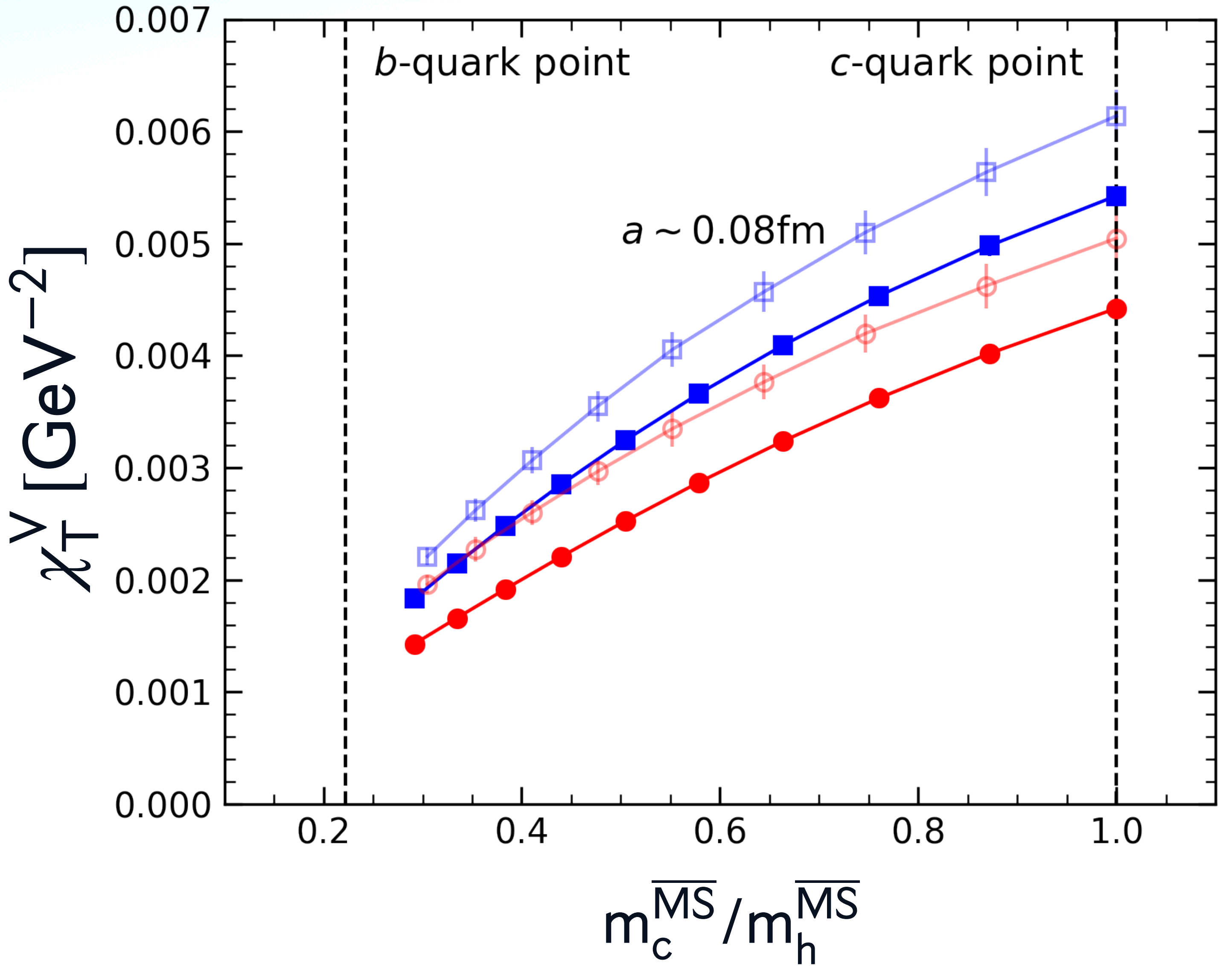
$$\chi_L^{V,A}(a^2, m_h^{(n)}, m_c) = \frac{WI(m_h^{(n)} \mp m_c)^2}{12} \int_0^\infty dt\, t^4\, C^{S,P}(t)$$

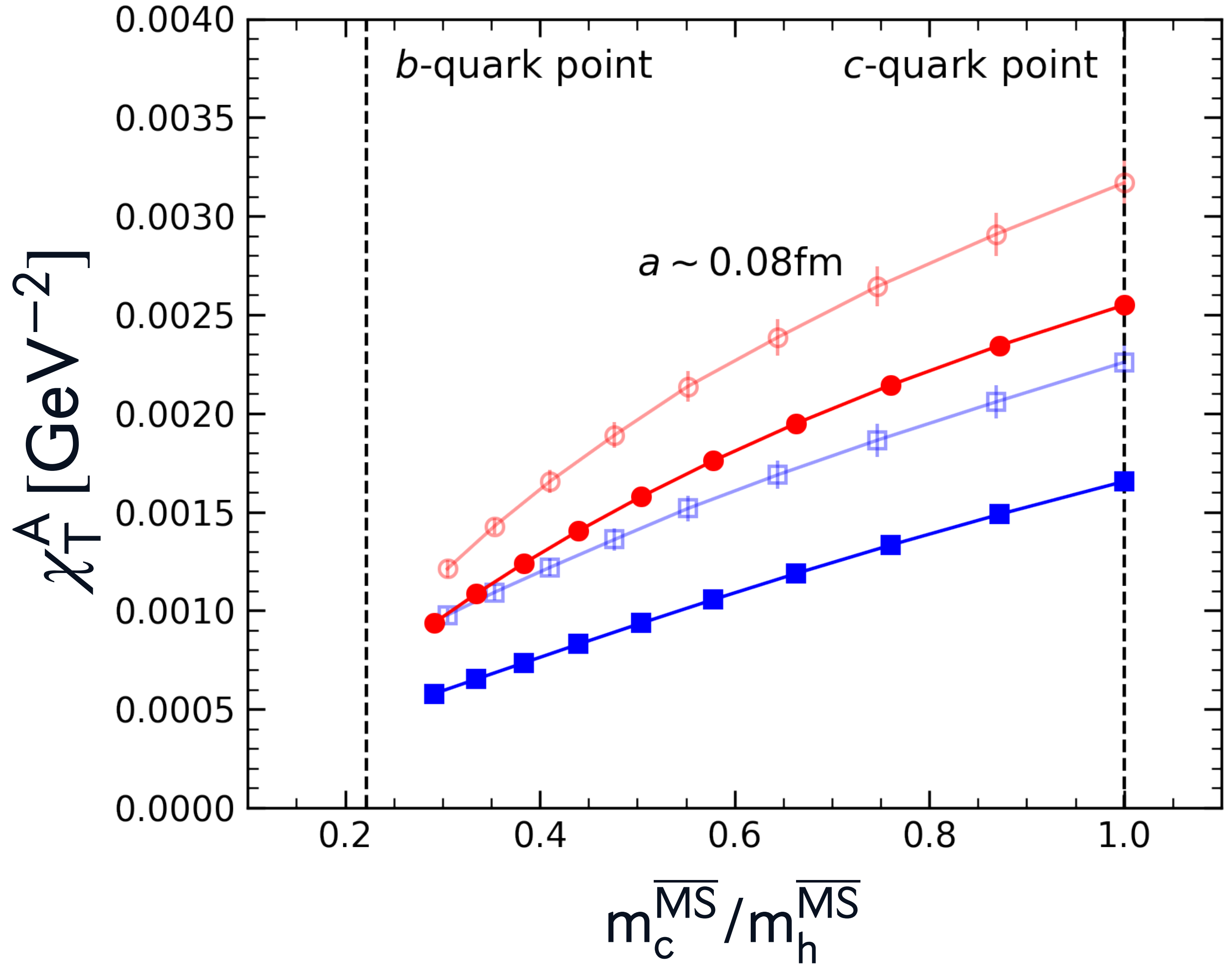
$$\chi_T^{V,A}(a^2, m_h^{(n)}, m_c) = \frac{1}{12} \int_0^\infty dt\, t^4\, C_i^{V,A}(t)$$











Perturbative Subtraction

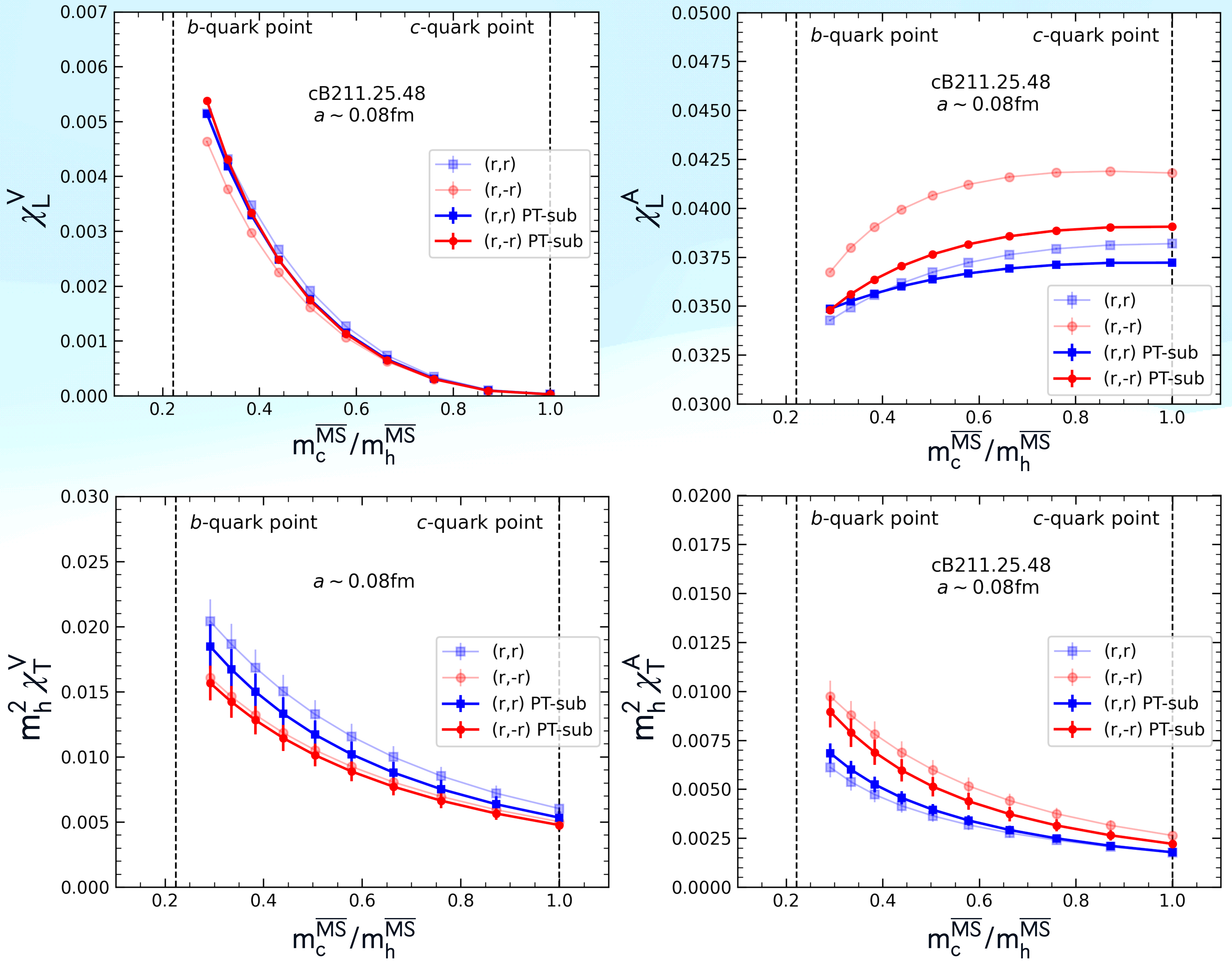
Following arXiv: 2105.02497

$$\chi(a^2) \rightarrow \chi(a^2) - \underbrace{\left(\chi^{FT}(a^2) - \chi^{PT}_{LO}\right)}_{\text{Cut-off effect of Free Theory}}$$

χ^{PT}_{LO} : the perturbative value is known from hep-ph/9705252 Grinstein et al.

χ^{FT}: susceptibilities from correlation functions in Free Theory(FT): $\alpha_s = 0$, computed for every ensemble setup

$$\chi^{FT}(a^2) = \chi^{PT}_{LO} + \text{ cut-off effects}$$



Continuum limit of $\chi_L^{V,A}$

- Polynomial fit ansatz up to $\mathcal{O}(a^6)$:

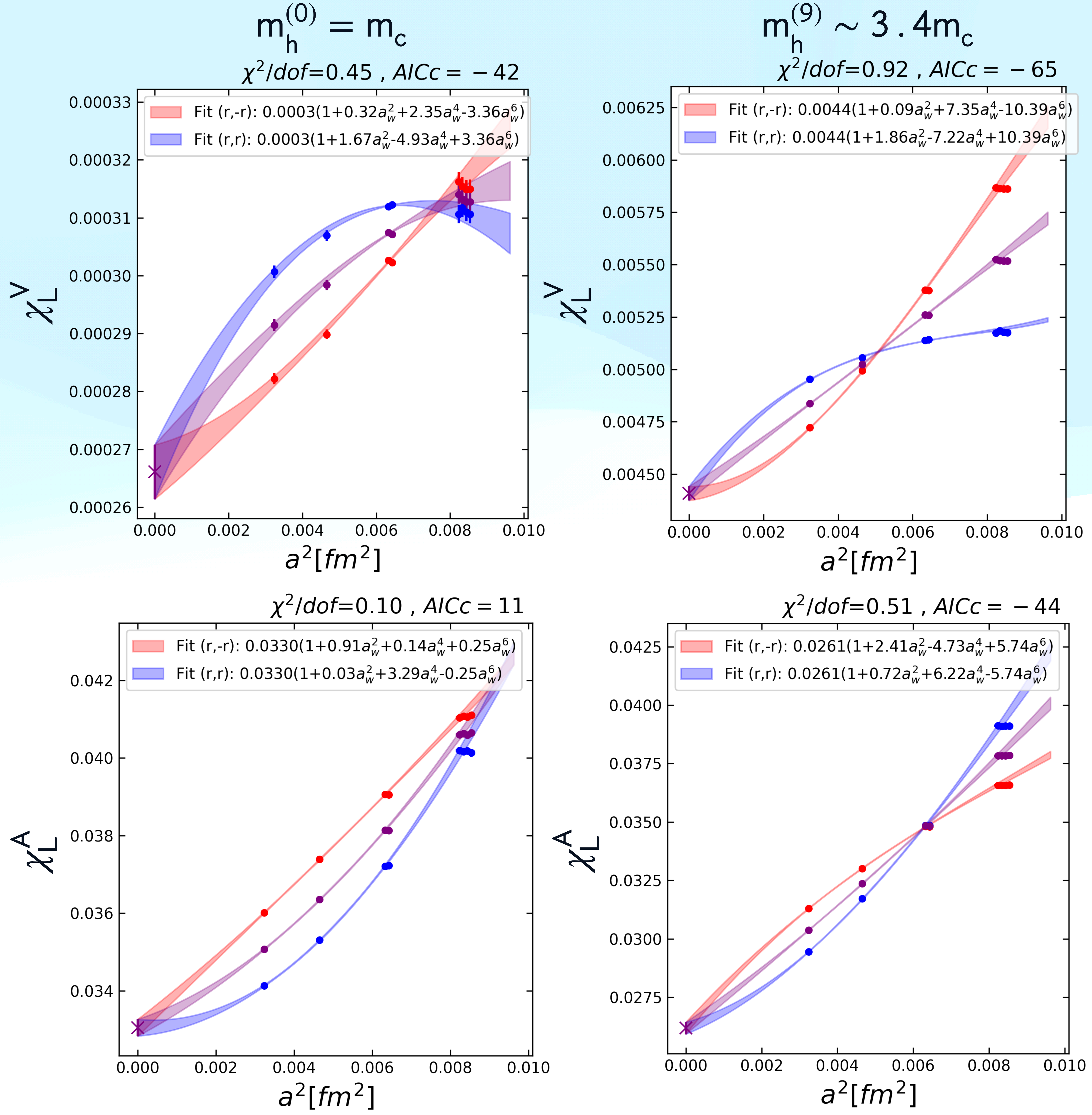
$$\chi_L^{(r,r)}(a^2; m_h^{(n)}, m_c) = \chi_L(m_h^{(n)}, m_c)\left(1 + P_1^{(r,r)}a^2 + P_2^{(r,r)}a^4 + P_3^{(r,r)}a^6\right)$$

$$\chi_L^{(r,-r)}(a^2; m_h^{(n)}, m_c) = \chi_L(m_h^{(n)}, m_c)\left(1 + P_1^{(r,-r)}a^2 + P_2^{(r,-r)}a^4 - P_3^{(r,r)}a^6\right)$$

- Combined fit of (r, r) and (r, -r) regularizations imposing same continuum limit

- Highly correlated data $\rho \sim 1$: we study mean and difference of the two data sets

- Mean shows reduced $\mathcal{O}(a^{4,6})$ effects



Continuum limit of $m_h^2 \chi_T^{V,A}$

- Polynomial fit ansatz up to $\mathcal{O}(a^6)$:

$$m_h^2\, \chi_T^{(r,r)}(a^2; m_h^{(n)}, m_c) = m_h^2 \chi_T(m_h^{(n)}, m_c)\left(1 + P_1^{(r,r)} a^2 + P_2^{(r,r)} a^4 + P_3^{(r,r)} a^6\right)$$

$$m_h^2\, \chi_T^{(r,-r)}(a^2; m_h^{(n)}, m_c) = m_h^2 \chi_T(m_h^{(n)}, m_c)\left(1 + P_1^{(r,-r)} a^2 + P_2^{(r,-r)} a^4 - P_3^{(r,r)} a^6\right)$$

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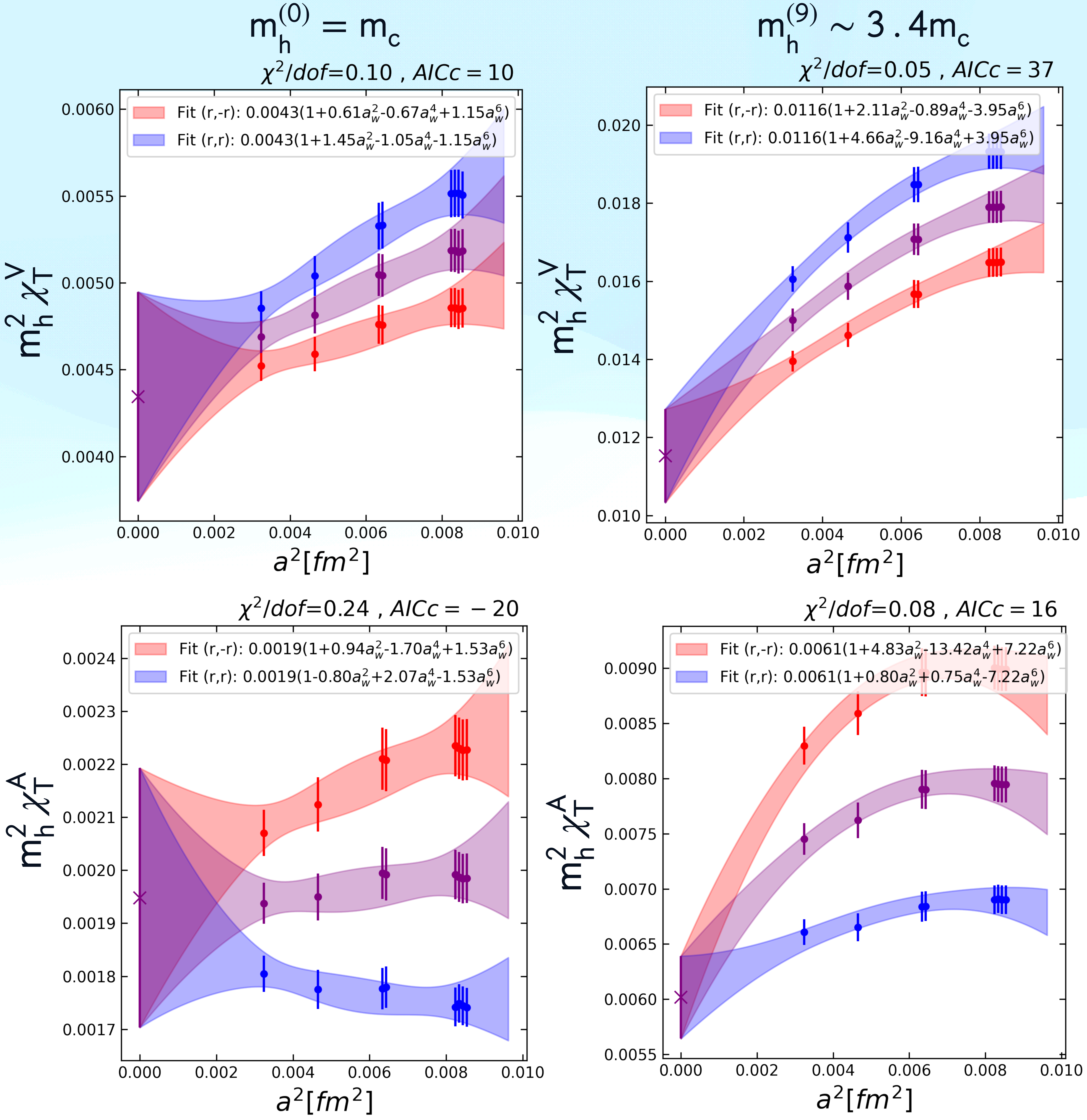


Susceptibilities at *b*-quark point

Known static limit:

$$\lim_{m_h\to\infty} \chi_L^{V,A}(m_h) = \frac{1}{8\pi^2}$$

$$\lim_{m_h\to\infty} m_h^2\chi_T^{V,A}(m_h) = \frac{3}{32\pi^2}$$

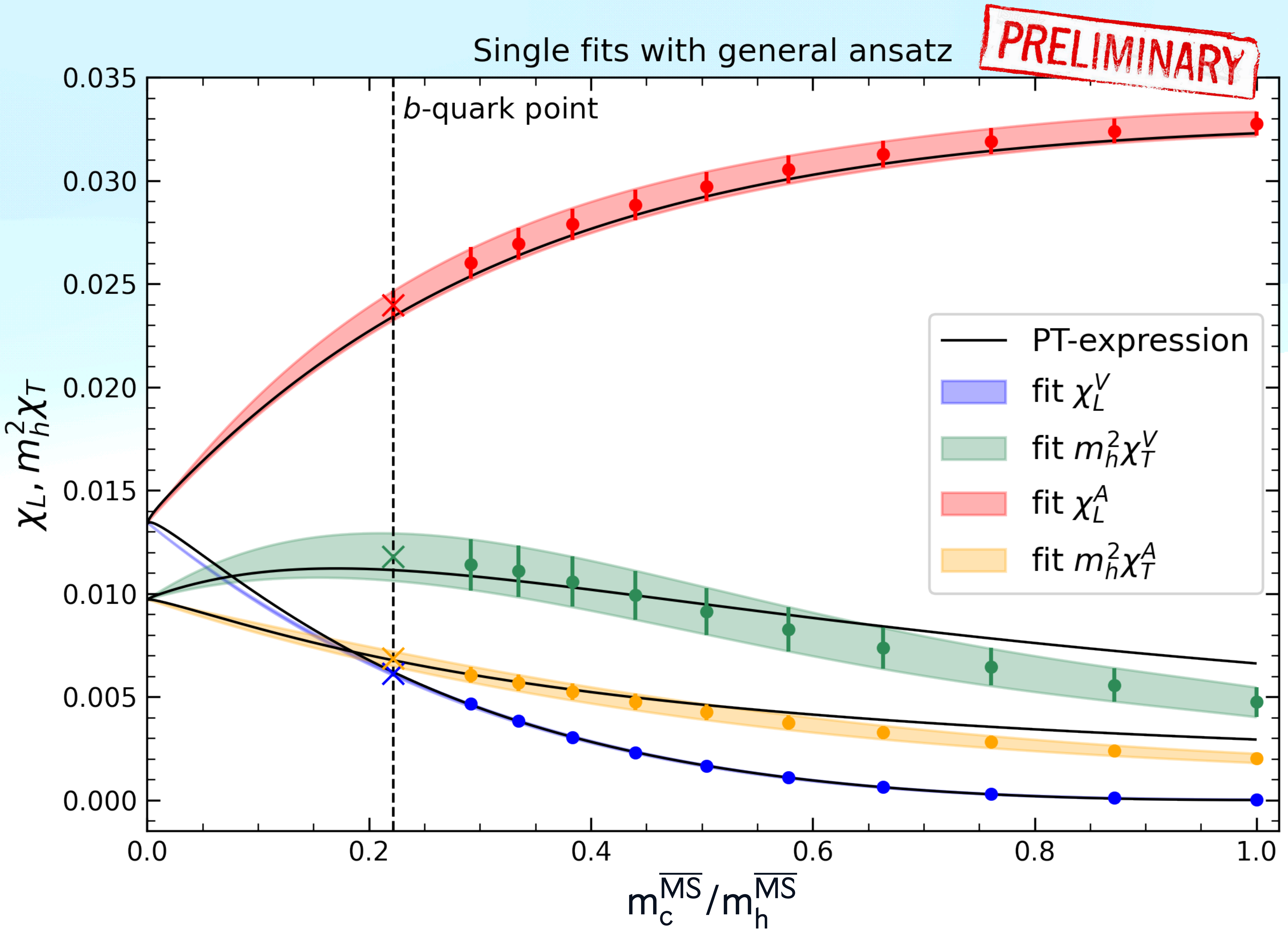
Fit ansatz imposing static limit:

$$\chi_L(m_h^{(n)}, m_c) = \frac{1}{8\pi^2}\left(1 + B_1\frac{m_c}{m_h} + B_2\frac{m_c^2}{m_h^2} + B_3\frac{m_c^3}{m_h^3}\right)$$

$$m_h^2\chi_T(m_h^{(n)}, m_c) = \frac{3}{8\pi^2}\left(1 + B_1\frac{m_c}{m_h} + B_2\frac{m_c^2}{m_h^2} + B_3\frac{m_c^3}{m_h^3}\right)$$

PT expressions known at NNLO

hep-ph/9705252 J.Grigo at al.



Susceptibilities at *b*-quark point

Our Results:

$\chi_L^V(m_b) = 6.243(45) \times 10^{-3}$

$\chi_L^A(m_b) = 2.362(63) \times 10^{-2}$

$\chi_T^V(m_b) = 5.37(48) \times 10^{-4}\,[\mathrm{GeV}^{-2}]$

$\chi_T^A(m_b) = 3.12(18) \times 10^{-4}\,[\mathrm{GeV}^{-2}]$

Fit ansatz imposing static limit:

$$\chi_L(m_h^{(n)}, m_c) = \frac{1}{8\pi^2}\left(1 + B_1\frac{m_c}{m_h} + B_2\frac{m_c^2}{m_h^2} + B_3\frac{m_c^3}{m_h^3}\right)$$

$$m_h^2\chi_T(m_h^{(n)}, m_c) = \frac{3}{8\pi^2}\left(1 + B_1\frac{m_c}{m_h} + B_2\frac{m_c^2}{m_h^2} + B_3\frac{m_c^3}{m_h^3}\right)$$

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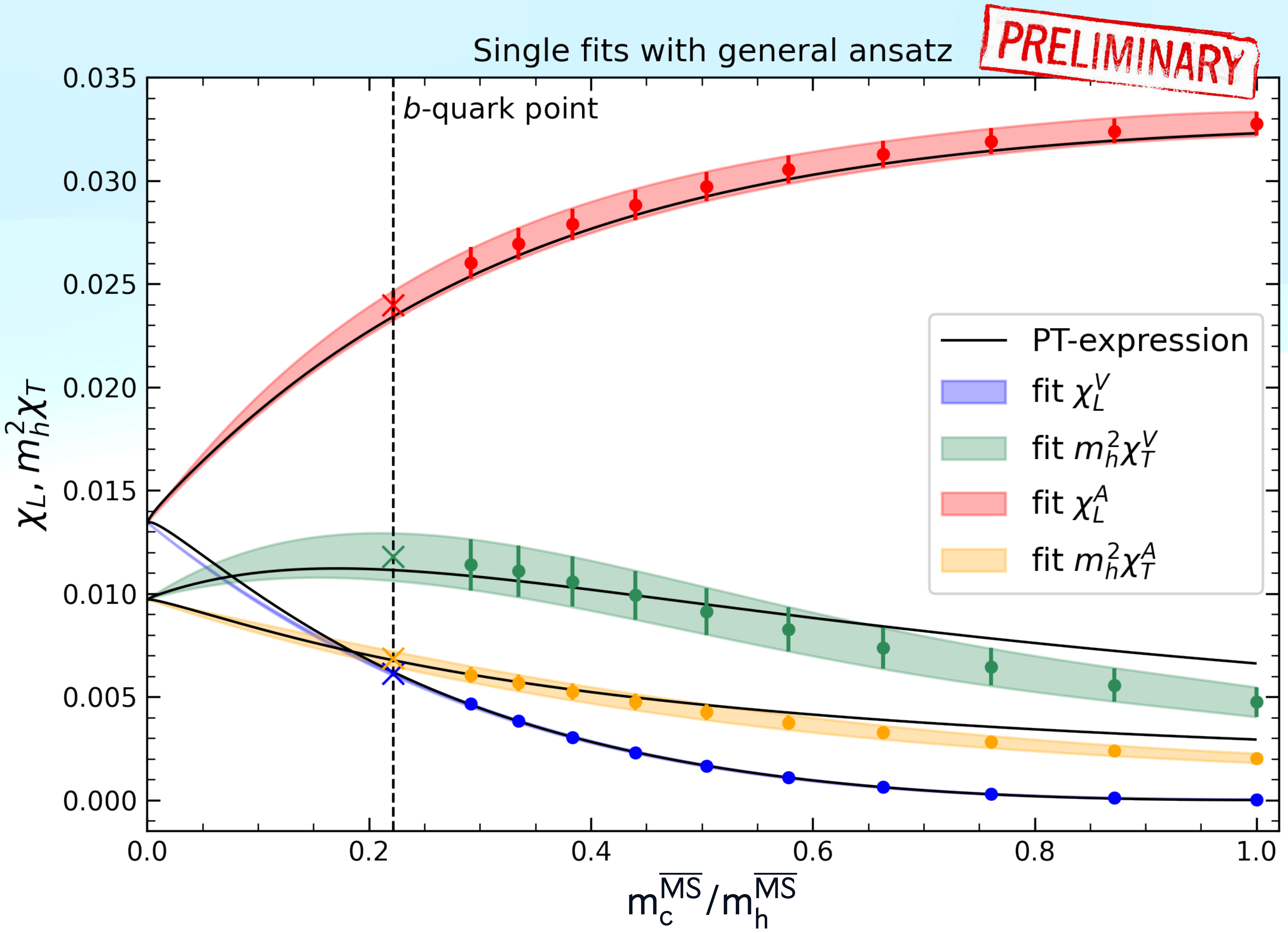

Conclusions

- Susceptibilities are important ingredients to constrain the from factors of $b \to c$ transitions, which are important to clarify the tension in $R_{D^{(*)}}$ and in the determination of $|V_{cb}|$

- We have presented a non perturbative calculation of the Susceptibilities from $N_f = 2 + 1 + 1$ ETM ensembles at physical point and at four lattice spacings.

- Results are well compatible with the perturbative expressions, encouraging results, but comparison with previous analysis only possible after results are final.

To do list:

- Large discretisation effects, to improve robustness of the continuum limit, on going simulations of a 5^{th} lattice spacing

- Perform a global fit of all the susceptibilities with a fit ansatz based on Operator Product Expansion to extract leading condensate term