Continuum B physics II: $\mathrm{m_b}$ and $\mathrm{f_{B^{(\star)}}}$ in 2+1 flavour QCD from a combination of static and relativistic results

Alessandro Conigli

J. Frison, P. Fritzsch, A. Gérardin, G. Herdoíza, J. Heitger, S. Kuberski, C. Pena, H. Simma, R. Sommer

IFT UAM-CSIC

August 2^{nd}, 2023

Strategy for m_b [Talk by R. Sommer, 0710.2229, hep-lat/030501]

- The RGI b-quark mass is extracted from the step scaling chain

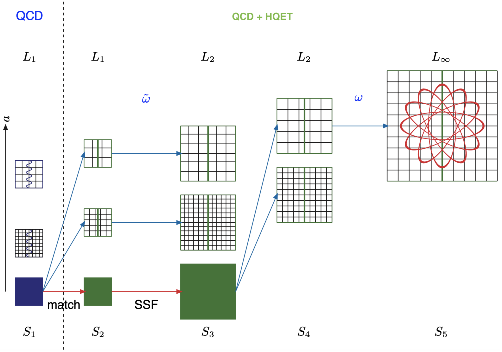
$$L_2 m_h^{\mathrm{RGI}} = \frac{L_2 m_{\mathrm{PS}} - L_2(m_{\mathrm{PS}} - m_{\mathrm{PS}}(L_2)) - L_2(m_{\mathrm{PS}}(L_2) - m_{\mathrm{PS}}(L_1))}{m_{\mathrm{PS}}(L_1)/m_h^{\mathrm{RGI}}}$$

Physical input:

$y_B = L_2 m_{\overline{B}}$

$m_h^{\mathrm{RGI}} \to m_b^{\mathrm{RGI}}$

$m_{\overline{B}} = \frac{2}{3} m_B + \frac{1}{3} m_{B_s}$



Strategy for m_b [Talk by R. Sommer, 0710.2229, hep-lat/030501]

$$L_2 m_b^{\rm RGI} = \frac{y_B - \rho_m(u_2, y_B) - \sigma_m(u_1, y_2)}{\pi_m(u_1, y_1)}, \qquad u_i = \bar{g}^2(L_i)$$

- $L_2 m_{\rm PS}(L_2) \equiv y_2 = y_B - \rho_m(u_2, y_B)$ $\qquad L_2 \to L_{\rm CLS}$ $\qquad y_B = L_2 m_{\overline{B}}$
- $L_2 m_{\rm PS}(L_1) \equiv y_1 = y_2 - \sigma_m(u_1, y_2)$ $\qquad L_1 \to L_2 = 2L_1$
- $\pi_m(u_1, y_1) = \frac{m_{\rm PS}(L_1)}{m^{\rm RGI}}$ $\qquad m_{\rm PCAC} \to m^{\rm RGI}$ in L_1 relativistic QCD
- ρ, σ finite in QCD and HQET, renormalization and matching cancel out
- Fully non-perturbative, $am_h < 1$ everywhere

Ensembles

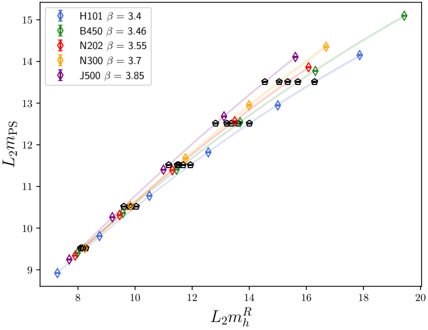
- Large volume CLS: five lattice spacings $0.04\ \mathrm{fm} \lesssim a \lesssim 0.08\ \mathrm{fm}$
  [JHEP 1107 036, JHEP1502 43, 1712.04884. 2003.13359]
  - $O(a)$-improved $N_f = 2+1$ Wilson fermions $(m_\pi = m_K \approx 420\ \mathrm{MeV})$
    $am_h^{\mathrm{RGI}} < 0.75, \qquad m_h^{\mathrm{RGI}} < 0.3 m_b^{\mathrm{RGI}}$
- Finite volume: SF b.c. projected to $Q = 0$ (critical slowing down)
  [1607.06423, 1811.02591, S. Kuberski thesis]
  - $L_2 \sim 1.0\ \mathrm{fm}$: five lattice spacings $0.03\ \mathrm{fm} \lesssim a \lesssim 0.08\ \mathrm{fm}$
    $am_h^{\mathrm{RGI}} < 0.75, \qquad m_h^{\mathrm{RGI}} < 0.3 m_b^{\mathrm{RGI}}$
  - L_1 and $2L_1$: five lattice spacings $0.02\ \mathrm{fm} \lesssim a \lesssim 0.06\ \mathrm{fm}$
    $am_h^{\mathrm{RGI}} < 0.6, \qquad m_h^{\mathrm{RGI}} < 0.5 m_b^{\mathrm{RGI}}$
  - $L_1 \sim 0.5\ \mathrm{fm}$: five lattice spacings $0.008\ \mathrm{fm} \lesssim a \lesssim 0.02\ \mathrm{fm}$
    $am_h^{\mathrm{RGI}} < 0.7, \qquad 0.9 m_b^{\mathrm{RGI}} < m_h^{\mathrm{RGI}} < 1.1 m_b^{\mathrm{RGI}}$

L_2 to L_{CLS} relativistic SSFs: interpolation in m_q, β

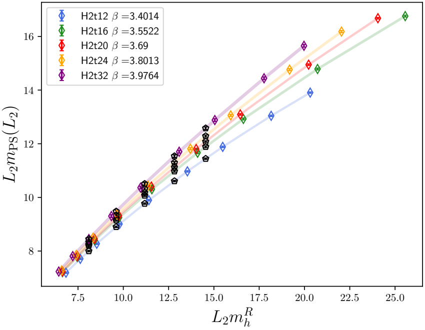
- Interpolation to target quark masses and β required to build the SSF

$$m_h^R = \frac{Z_A(\tilde{g_0}^2)}{Z_P(\tilde{g_0}^2, L_2)} m_h^{\mathrm{PCAC,I}}$$

CLS



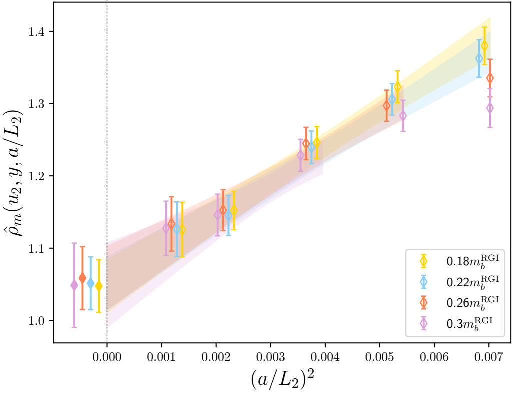
Finite Volume



$$L_2 m_{\mathrm{PS}} = p_0 + p_1(L_2 m_i^R) + p_2(L_2 m_i^R)^2, \qquad m_h^{\mathrm{RGI}} < 0.3 m_b^{\mathrm{RGI}}$$

L_2 to $L_{\rm CLS}$ relativistic SSFs: continuum limit

- Relativistic SSF: $\hat{\rho}_m(u_2, y, a/L_2) = \frac{L_2}{a}[am_{\rm PS}(am_h) - am_{\rm PS}(am_h, L_2/a)]$
  $y = L_2 m_{\rm PS}$



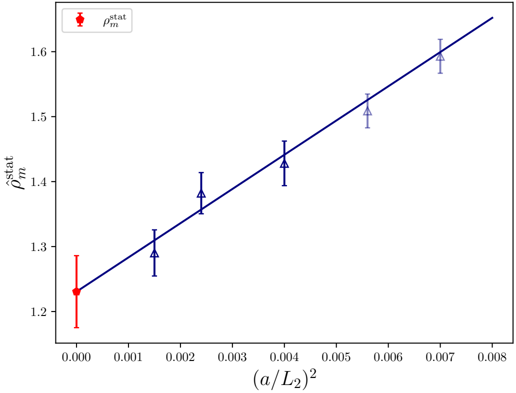
Points with $am_h^{\rm RGI} > 0.8$ excluded

$m_h^{\rm RGI} < 0.3m_b^{\rm RGI}$

fit function: $\hat{\rho}_m = p_0 + p_1(a/L_2)^2$, $\lim_{a\to 0} \hat{\rho}_m(u_2, y, a/L_2) = \rho_m(u_2, y)$

L_2 to $L_{\rm CLS}$ static SSF

- Static SSF: $\hat{\rho}_m^{\rm stat}(u_2, a/L_2) = \frac{L_2}{a}\left[aE^{\rm stat} - aE^{\rm stat}(L_2)\right]$



Improvement coefficient $c_A^{\rm stat}$ not known for Lüscher-Weisz action

$\Downarrow$

1-loop $c_A^{\rm stat}$ from Wilson action with 200% error

fit function: $\hat{\rho}_m^{\rm stat} = p_0 + p_1 (a/L_2)^2$, $\lim_{a\to 0} \hat{\rho}_m^{\rm stat}(u_2, a/L_2) = \rho_m^{\rm stat}(u_2)$

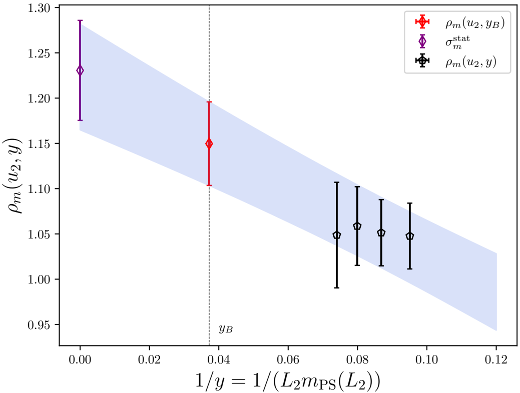
L_2 to $L_{\rm CLS}$: combining static and relativistic SSFs

Static SSF

A static SSF $\sigma_m^{\rm stat}(u_2)$

Relativistic SSF

A set of $\rho_m(u_2, y)$



$y_B = L_2 m_{\overline{B}}$

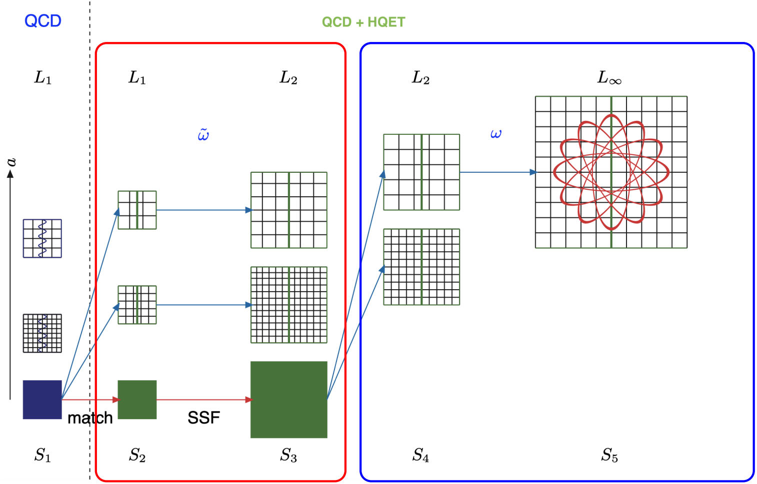
$m_{\overline{B}} = 5308.5(2)$ MeV

$m_h^{\rm RGI} < 0.3 m_b^{\rm RGI}$

fit function: $\rho_m = s_m + o_m y^{-1}$, $\rho_m(u_2, y_B) = 1.152(45)$

General Strategy [1312.1566]

$$L_2 m_b^{\rm RGI} = \frac{y_B - \rho_m(u_2, y_B) - \sigma_m(u_1, y_2)}{\pi_m(u_1, y_1)}$$

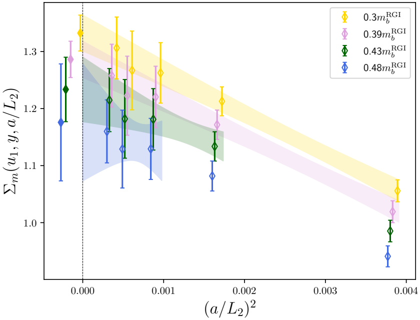


L_1 to L_2 SSFs

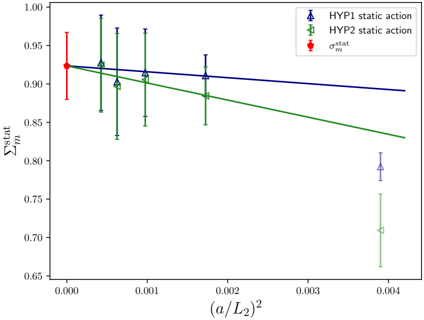
- Continuum limit of relativistic and static step scaling functions

$\Sigma_m(u_1, y, a/L_2)$

$m_h^{\mathrm{RGI}} < 0.5 m_b^{\mathrm{RGI}}$



$\Sigma_m^{\mathrm{stat}}(u_1, a/L_2)$



fit function: $\Sigma_m^{(\mathrm{stat})} = p_0 + p_1 (a/L_2)^2$,

c_A^{stat} from Wilson action with 200 % error

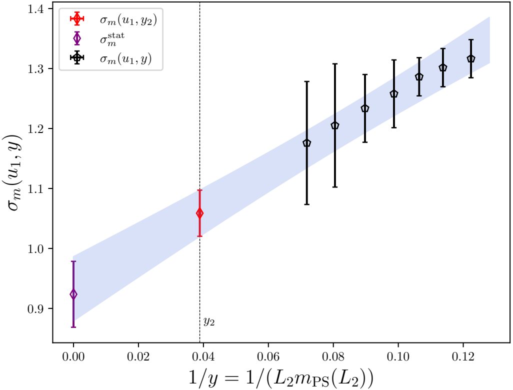
L_1 to L_2: combining static and relativistic SSFs

Static SSF

A static SSF $\sigma_m^{\mathrm{stat}}(u_1)$

Relativistic SSF

A set of $\sigma_m(u_1, y)$



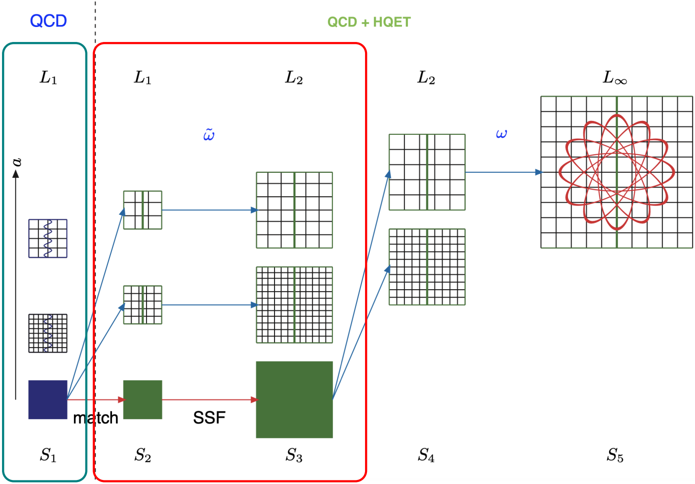
$$y_2 = y_B - \rho_m(u_2, y_B)$$

$$m_h^{\mathrm{RGI}} < 0.5 m_b^{\mathrm{RGI}}$$

fit function: $\sigma_m = s_m + o_m y^{-1}$, $\sigma_m(u_1, y_2) = 1.058(49)$

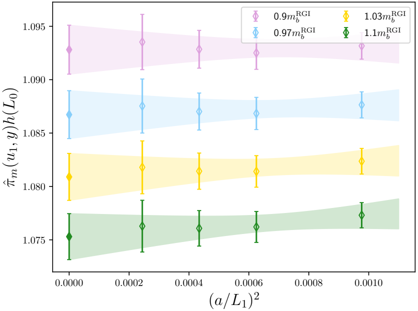
General Strategy [1312.1566]

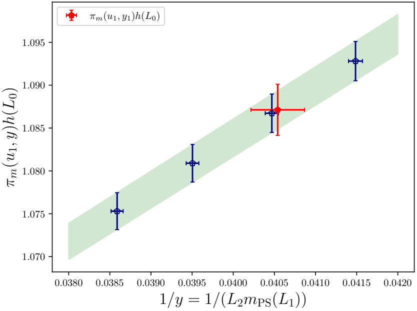
$$L_2 m_b^{\mathrm{RGI}} = \frac{y_B - \rho_m(u_2, y_B) - \sigma_m(u_1, y_2)}{\pi_m(u_1, y_1)}$$



$$\hat{\pi}_m(u_1, y, a/L_1) = \frac{am_{\rm PS}(m_h, a/L_1)}{am^{\rm RGI}(m_h)} \qquad m^{\rm RGI} = h(L_0)\frac{Z_A}{Z_P(L_0)} m_h^{\rm PCAC,I}$$

$$0.9m_b^{\rm RGI} < m_h^{\rm RGI} < 1.1m_b^{\rm RGI}$$





fit function: $\hat{\pi}_m = p_0 + p_1(a/L_1)^2,$

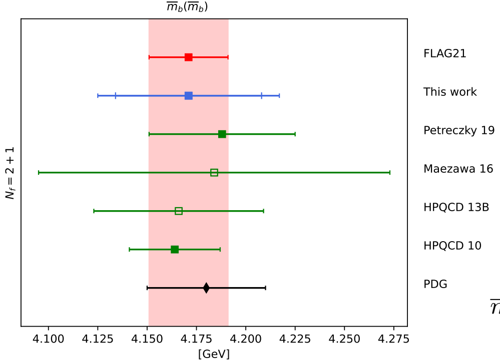
fit function: $\pi_m = c_0 + c_1 y^{-1}$

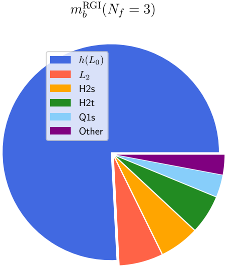
$$y_1 = y_2 - \sigma_m(u_1, y_2), \qquad \pi_m(u_1, y_1)h(L_0) = 1.0871(30)$$

Full step scaling for m_b [preliminary ($m_\pi = m_K \approx 420$ MeV)]

- Fully non-perturbative, $am_h < 1$ everywhere

$$m_b^{\rm RGI}(N_f = 3) = \frac{y_B - \rho_m(u_2, y_B) - \sigma_m(u_1, y_2)}{\pi_m(u_1, x_1)} L_2^{-1} = 6.605(61) \text{ GeV } [0.9\%]$$





$$\overline{m}_b(\overline{m}_b, N_f = 4) = 4.171(40)(27)_\Lambda \text{ GeV}$$

$$\Lambda^{(3)}_{\overline{\rm MS}} = 338(12) \text{ MeV}$$

Step scaling for $f_{\overline{B}^{(\star)}}$

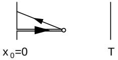
$f_{\overline{B}^{(\star)}}$ decay constants: step scaling chain

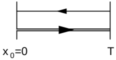
- Step scaling chain for $\hat{f}_{B^{(\star)}} = f_{B^{(\star)}}\sqrt{m_{B^{(\star)}}}$:

$$\ln\left(\frac{L_2^{3/2}\hat{f}_{B^{(\star)}}}{2}\right) = \Phi_{A_0(\vec{V})}(u_1, y_1) + \sigma_{A_0(\vec{V})}(u_1, y_2) + \rho_{A_0(\vec{V})}(u_2, y_B)$$

- $\rho_{A_0(\vec{V})}(u_2, y) = \ln\left(\frac{L_2^{3/2}\hat{f}_{B^{(\star)}}}{2}\right) - \Phi_{A_0(\vec{V})}(u_2, y)$ $\quad L_2 \to L_{\mathrm{CLS}}$

- $\sigma_{A_0(\vec{V})}(u_1, y) = \Phi_{A_0(\vec{V})}(u_2, y) - \Phi_{A_0(\vec{V})}(u_1, y)$ $\quad L_1 \to L_2$

- $\Phi_{A_0(\vec{V})}(u_1, y) =$ (numerator) 

over (denominator) 

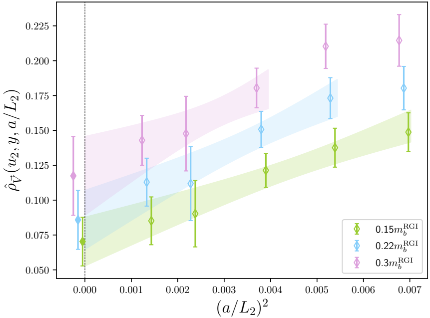
$^{1/2}$ $\quad$ relativistic b-quark in L_1

L_2 to $L_{\rm CLS}$ SSFs

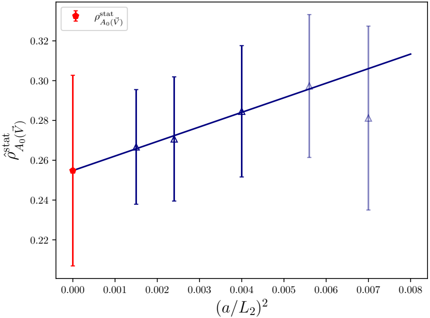
- Continuum limit of relativistic and static step scaling functions

$$\hat{\rho}_{\vec{V}}(u_2, y, a/L_2)$$

$$m_h^{\rm RGI} < 0.3 m_b^{\rm RGI}$$



$$\hat{\rho}^{\rm stat}_{A_0(\vec{V})}(u_2, a/L_2) = \log\left(\frac{\Phi_f^{\rm stat}}{\Phi_f^{\rm stat}(a/L_2)}\right)$$



fit function: $\hat{\rho}^{(\rm stat)}_{A_0(\vec{V})} = p_0 + p_1 (a/L)^2,$

$c_A^{\rm stat}$ from Wilson action with 200 % error

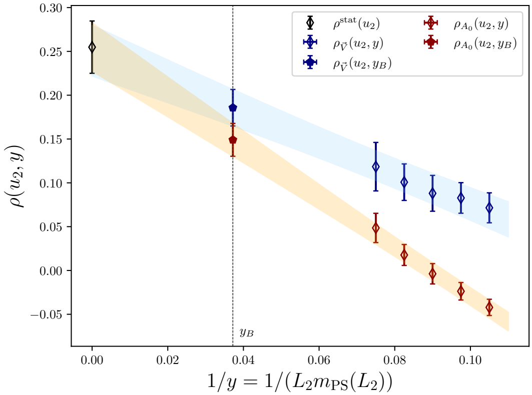
L_2 to L_{CLS}: combining static and relativistic SSFs

Static SSF

A static SSF $\rho_f^{\text{stat}}(u_2)$

Relativistic SSFs

A set of $\rho_{A_0(\vec{V})}(u_2, y)$



$$y_B = L_2 m_{\overline{B}}$$

$$L_2 \approx 1.0 \text{ fm}$$

$$m_h^{\text{RGI}} < 0.3 m_b^{\text{RGI}}$$

$$\rho(u_2, y) = s_f + o_f y^{-1}, \qquad \rho_{\vec{V}}(u_2, y_B) = 0.186(20), \qquad \rho_{A_0}(u_2, y_B) = 0.149(18)$$

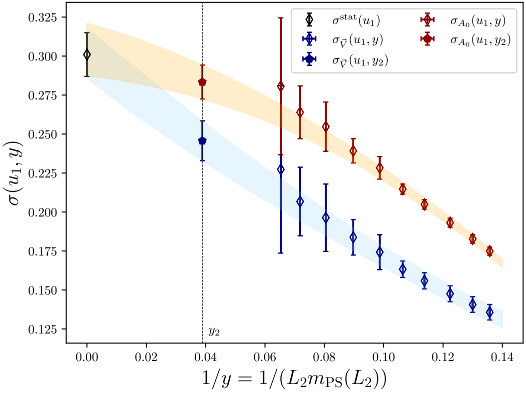
L_1 to L_2: combining static and relativistic SSFs

Static SSF

A static SSF $\sigma_f^{\rm stat}(u_1)$

Relativistic SSFs

A set of $\sigma_{A_0(\vec{V})}(u_1, y)$



$$y_2 = y_B - \rho_m(u_2, y_B)$$
$$L_1 \approx 0.5 \text{ fm}$$

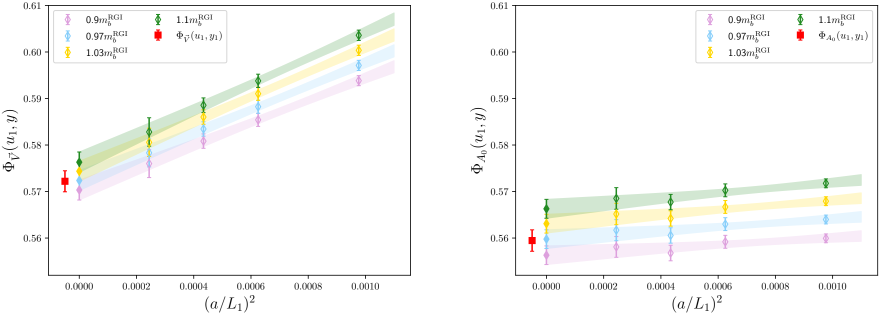
$$m_h^{\rm RGI} < 0.5 m_b^{\rm RGI}$$

$$\sigma_{A_0(\vec{V})} = s_f + o_f y^{-1} + c_f y^{-2}, \qquad \sigma_{\vec{V}}(u_1, y_2) = 0.246(13), \qquad \sigma_{A_0}(u_1, y_2) = 0.283(11)$$

- Cutoff effects under control

$$0.9m_b^{\rm RGI} < m_h^{\rm RGI} < 1.1m_b^{\rm RGI}$$

$\Phi_{\vec{V}}(L_1)$ $\Phi_{A_0}(L_1)$

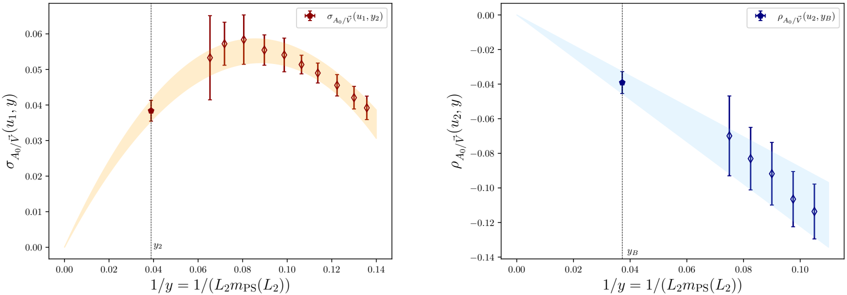


$$\Phi_{A_0(\vec{V})} = p_0 + p_1(a/L)^2$$

$y_1 = y_2 - \sigma_m(u_1, y_2)$, $\Phi_{\vec{V}}(y_1, L_1) = 0.5722(22)$, $\Phi_{A_0}(y_1, L_1) = 0.5594(23)$

A glimpse of $f_{\overline{B}}/f_{\overline{B}^\star}$ [preliminary]

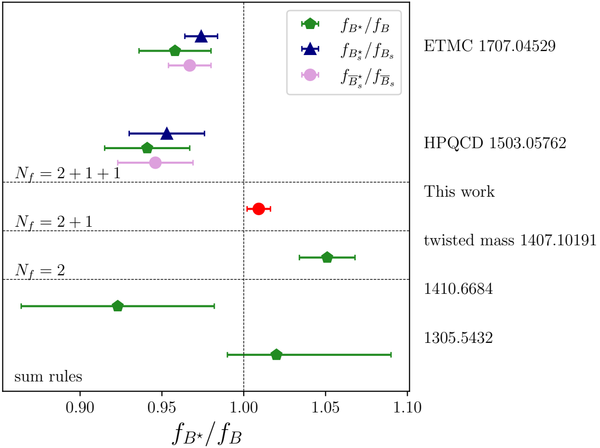
$$\ln\left(\hat{f}_{\overline{B}}/\hat{f}_{\overline{B}^\star}\right) = \Phi_{A_0/\vec{V}}(u_1, y_1) + \sigma_{A_0/\vec{V}}(u_1, y_2) + \rho_{A_0/\vec{V}}(u_2, y_B)$$



$$f_{\overline{B}^\star}/f_{\overline{B}} = 1.009(7)\ [0.7\ \%]$$

From the independent determinations: $f_{\overline{B}^\star}/f_{\overline{B}} = 1.008(10)$

Results comparison for $f_{B^\star}/f_B$ [preliminary ($m_\pi = m_K \approx 420$ MeV)]



$f_{\overline{B}^\star} = 207.0(5.4)$ MeV [2.6 %], $\quad f_{\overline{B}} = 205.4(4.7)$ MeV [2.2 %]

$N_f = 2+1$ FLAG21 from f_B and f_{B_s}: $\quad f_{\overline{B}} = 204.1(3.1)$ MeV [1.5 %]

Conclusions and outlook

Summary

- Precision B-physics from fully non-perturbative calculations by combining static and relativistic results in the continuum
- Cutoff effects well under control with $am_h < 1$ everywhere

Improvements

- One loop computation for c_A^{stat} in progress
- Assessing the systematics: cuts in y and a
- Consider additional small volumes definitions (different θ angles)

Future

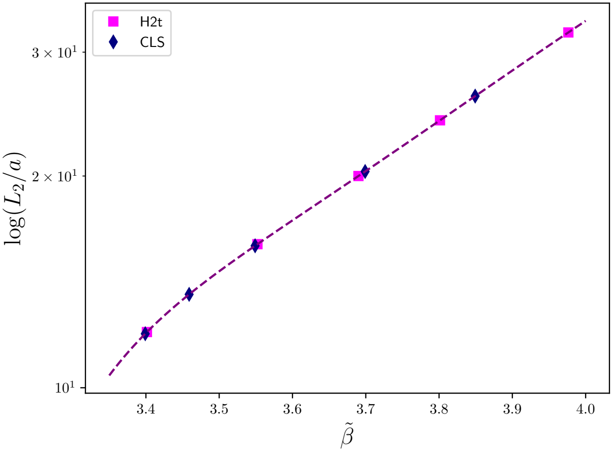
- Ongoing extension across more ensembles at physical point
- Extension to semi-leptonic decays using the results for $f_{B^\star}$

Thank You!

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 813942

Related talks:
R. Sommer: *"A strategy for B-physics observables in the continuum limit"*

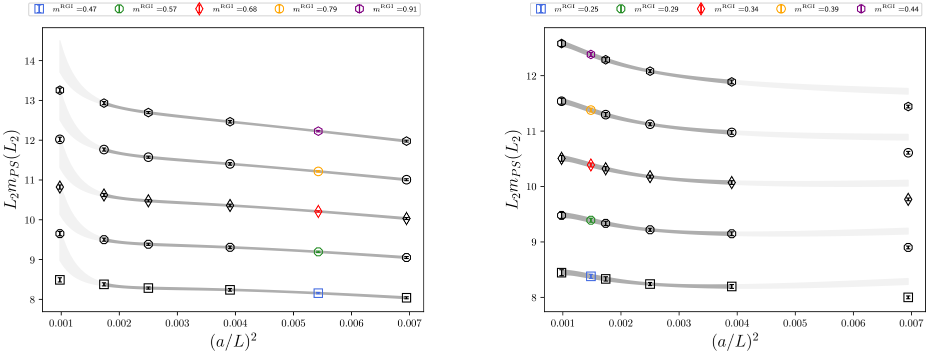
L_2 interpolation at target $\tilde{\beta}$



$$f(\tilde{\beta}, p) = p_0 + p_1\tilde{\beta} + p_2/\tilde{\beta} + p_3/\tilde{\beta}^2$$

L_2 to $L_{\rm CLS}$ relativistic SSFs: interpolation in m_q, β

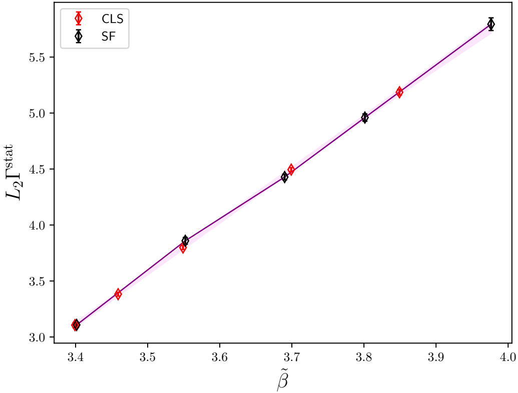
- Interpolation to target quark masses and β required to build the SSF



$$L_2 m_{\rm PS}(a/L) = p_0 + p_1\left(\frac{a^2}{L^2}\right) + p_2\left(\frac{a^2}{L^2}\right)^{\frac{3}{2}} + p_3\left(\frac{a^2}{L^2}\right)^2, \qquad 0.2m_b < m_h < 0.5m_b$$

L_2 to L_{CLS} static SSF

- Static SSF: $\hat{\rho}_m^{\mathrm{stat}}(u_2, a/L_2) = \frac{L_2}{a}[aE^{\mathrm{stat}} - aE^{\mathrm{stat}}(L_2)]$

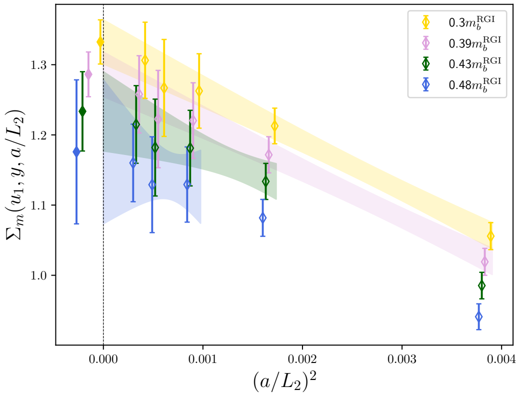


$$f(\tilde{\beta}, p) = p_0\tilde{\beta} + \frac{p_1}{\tilde{\beta}} + \frac{p_2}{\tilde{\beta}^2}$$

L_1 to L_2 relativistic SSFs: continuum limit

- Relativistic SSF:

$$\Sigma_m(u_1, y, a/L_2) = \frac{L_2}{a}\left[am_{\rm PS}(am_h, L_2/a) - am_{\rm PS}(am_h, L_1/a)\right]$$



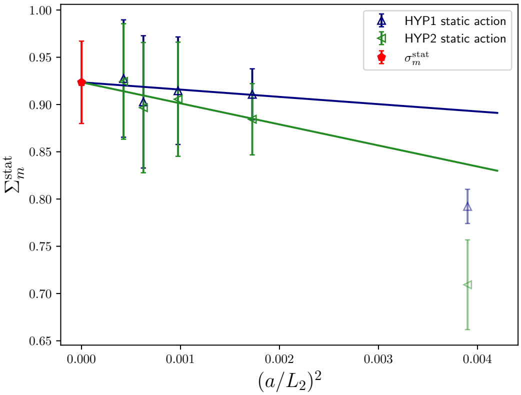
Points with $am^{\rm RGI} > 0.8$ excluded

$$0.3am_b^{\rm RGI} < m_h^{\rm RGI} < 0.5am_b^{\rm RGI}$$

fit function: $\Sigma_m = p_0 + p_1(a/L)^2,$ $\lim_{a\to 0} \Sigma_m(u_1, y, a/L) = \sigma_m(u_1, y)$

L_1 to L_2 static SSF

- Finite-a static SSF: $\Sigma_m^{\rm stat}(u_1, a/L_2) = \frac{L_2}{a}\left[aE^{\rm stat}(L_2) - aE^{\rm stat}(L_1)\right]$



Improvement coefficient $c_A$stat not known for Lüscher-Weisz action

$\Downarrow$

$c_A^{\rm stat}$ from Wilson action with 200% error

fit function: $\Sigma_m^{\rm stat} = p_0 + p_1(a/L)^2$, $\lim_{a\to 0} \Sigma_m^{\rm stat}(u_1, a/L_2) = \sigma_m^{\rm stat}(u_1)$

Step scaling for $f_{\overline{B}}$ and $f_{\overline{B}^\star}$

$f_{B^{(\star)}}$ decay constants: relativistic definitions

- Observables in large CLS volume

$$\hat{f}_{B^{(\star)}} = \sqrt{m_{B^{(\star)}}} f_{B^{(\star)}}, \qquad \Phi_{\vec{V}} = \ln\left(\frac{L_2^{3/2}}{2}\hat{f}_{B^\star}\right), \qquad \Phi_{A_0} = \ln\left(\frac{L_2^{3/2}}{2}\hat{f}_B\right)$$

- Observables in finite volume

$$\Phi_{\vec{V}}(L) = \ln\left(\frac{k_{\vec{V}}(T/2)}{\sqrt{K_1(T)}}\right), \qquad \Phi_{A_0}(L) = \ln\left(-\frac{f_A(T/2)}{\sqrt{f_1(T)}}\right)$$

$$\Phi_{\vec{V}}(L), \Phi_{A_0}(L) \xrightarrow[T\to\infty]{} \ln\left(\frac{L^{3/2}}{\sqrt{2}}\hat{f}_{B^{(\star)}}\right)$$

$f_{B^{(\star)}}$ decay constants: static definitions

- Large volume static observables

$$\Phi_f^{\mathrm{stat}} = \frac{p^{\mathrm{stat}} L_2^{3/2}}{\sqrt{2}}$$

- Finite volume static observables

$$\Phi_f^{\mathrm{stat}}(a/L) = \frac{-Y_{\mathrm{RGI}}^{\mathrm{stat}}(L)}{(L/a)^{3/2}}$$

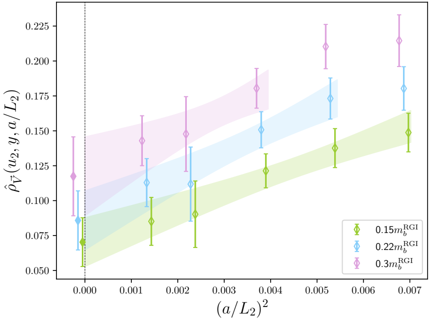
- Static step scaling functions

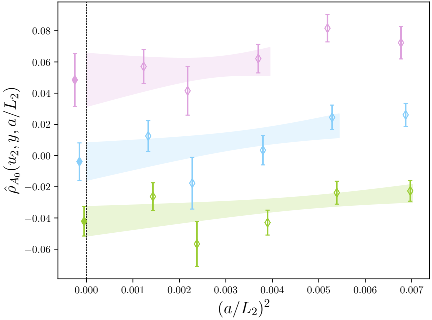
$$\rho_f^{\mathrm{stat}}(u_2) = \ln\left(\frac{\Phi_f^{\mathrm{stat}}}{\Phi_f^{\mathrm{stat}}(a/L_2)}\right), \qquad \sigma_f^{\mathrm{stat}}(u_1) = \ln\left(\frac{\Phi_f^{\mathrm{stat}}(a/L_2)}{\Phi_f^{\mathrm{stat}}(a/L_1)}\right)$$

L_2 to $L_{\rm CLS}$ relativistic SSFs

- Continuum limit of relativistic step scaling functions

$\hat{\rho}_{\vec{V}}(u_2, y, a/L_2)$ $\hat{\rho}_{A_0}(u_2, y, a/L_2)$

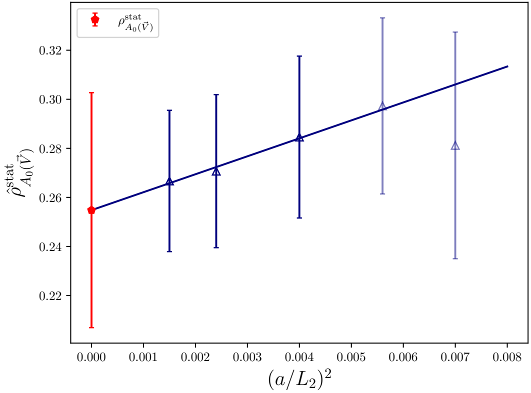




fit function: $\hat{\rho}_{A_0(\vec{V})} = p_0 + p_1 (a/L)^2$, $\lim_{a\to 0} \hat{\rho}_{A_0(\vec{V})}(u_2, y, a/L_2) = \rho_{A_0,(\vec{V})}(u_2, y)$

L_2 to L_{CLS} static SSF

- Finite a static SSF: $\hat{\rho}_f^{\mathrm{stat}}(u_2, a/L_2) = \log\left(\frac{\Phi_f^{\mathrm{stat}}}{\Phi_f^{\mathrm{stat}}(a/L_2)}\right)$



Improvement coefficient c_A^{stat} not known for Lüscher-Weisz action

$\Downarrow$

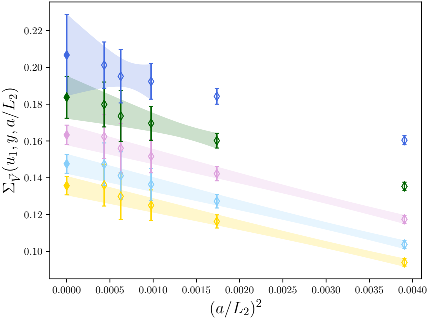
c_A^{stat} from Wilson action with 200% error

fit function: $\hat{\rho}_f^{\mathrm{stat}} = p_0 + p_1(a/L)^2,$ $\lim_{a\to 0} \hat{\rho}_f^{\mathrm{stat}}(u_2, a/L_2) = \rho_f^{\mathrm{stat}}(u_2)$

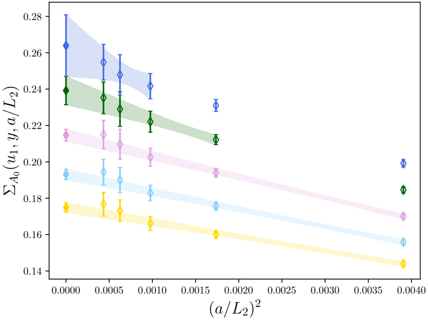
L_1 to L_2 relativistic SSFs

- Continuum limit of relativistic step scaling functions

$\Sigma_{\vec{V}}(u_1, y, a/L_2)$



$\Sigma_{A_0}(u_1, y, a/L_2)$

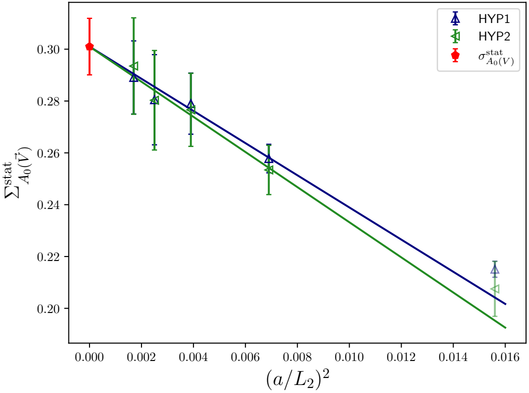


fit function: $\Sigma_{A_0(\vec{V})} = p_0 + p_1 (a/L)^2,$

$$\lim_{a \to 0} \Sigma_{A_0(\vec{V})}(u_1, y, a/L_2) = \sigma_{A_0(\vec{V})}(u_1, y)$$

L_1 to L_2 static SSF: continuum limit

- Finite-a static SSF: $\Sigma_f^{\rm stat}(u_1) = \log\left(\frac{\Phi_f^{\rm stat}(a/L_2)}{\Phi_f^{\rm stat}(a/L_1)}\right)$



Improvement coefficient $c_A^{\rm stat}$ not known for Lüscher-Weisz action

$\Downarrow$

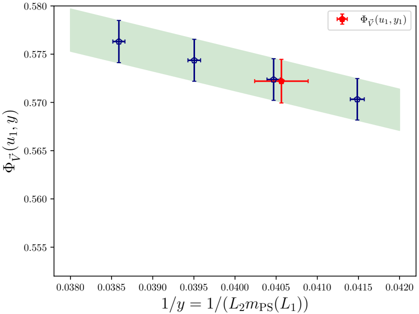
$c_A^{\rm stat}$ from Wilson action with 200% error

$$\Sigma_f^{\rm stat} = p_0 + p_1(a/L)^2, \qquad \lim_{a\to 0}\Sigma_f^{\rm stat}(u_1) = \sigma_f^{\rm stat}(u_1)$$

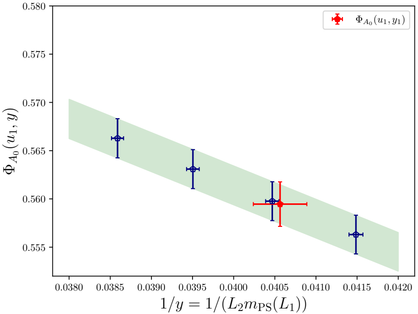
Relativistic QCD in L_1 $[0.0078\ \mathrm{fm} \lesssim a \lesssim 0.021\ \mathrm{fm}]$

- Cutoff effects well under control

$\Phi_{\vec{V}}(L_1)$



$\Phi_{A_0}(L_1)$



$$f(y,p) = p_0 + p_1 y^{-1}$$

$$y_1 = y_2 - \sigma_m(u_1, y_2), \qquad \Phi_{\vec{V}}(y_1, L_1) = 0.5722(22), \qquad \Phi_{A_0}(y_1, L_1) = 0.5594(23)$$

Results for $f_{\overline{B}^{(\star)}}$ [preliminary ($m_\pi = m_K = 420$ MeV)]

- Full step scaling chain

$$\ln\left(\hat{f}_{\overline{B}^\star}\frac{L_2^{3/2}}{2}\right) = \Phi_{\vec{V}}(u_1, y_1) + \sigma_{\vec{V}}(u_1, y_2) + \rho_{\vec{V}}(u_2, y_B) = 1.004(24)$$

$$\ln\left(\hat{f}_{\overline{B}}\frac{L_2^{3/2}}{2}\right) = \Phi_{A_0}(u_1, y_1) + \sigma_{A_0}(u_1, y_2) + \rho_{A_0}(u_2, y_B) = 0.992(21)$$

- Using the flavour-averaged mass combinations

$$m_{\overline{B}^\star} = 5354.60(61)\ \text{MeV}, \qquad m_{\overline{B}} = 5308.5(2)\ \text{MeV}$$

$$f_{\overline{B}^\star} = 207.0(5.4)\ \text{MeV}\ [2.6\ \%], \qquad f_{\overline{B}} = 205.4(4.7)\ \text{MeV}\ [2.2\ \%]$$

FLAG21 $f_{\overline{B}} = 204.1(3.1)$ MeV [1.5 %]

Step scaling for $f_{B^\star}$

$f_{B^\star}$ decay constant

- Observable in large volume

$$f_{\mathrm{V}}^{\mathrm{bare}}(m_h) = \sqrt{\frac{2}{L^3 m_{\mathrm{V}}}} \langle 0 | \hat{V}_k | \mathrm{V} \rangle \qquad \langle 0 | \hat{V}_k | \mathrm{V} \rangle = \sqrt{f_{V_k V_k}(x_0, y_0)} e^{m_{\mathrm{V}}/2(x_0 - y_0)}$$

$$f_{\mathrm{V}}^{R} = Z_V \left[1 + a b_v m_{q,h}\right] f_{\mathrm{V}}^{\mathrm{bare}}$$

$$\Phi_{\vec{V}} = \ln\left(\frac{L_2^{3/2}}{2} \hat{f}_{B^\star}\right), \qquad \hat{f}_{B^\star} = \sqrt{m_{B^\star}} f_V$$

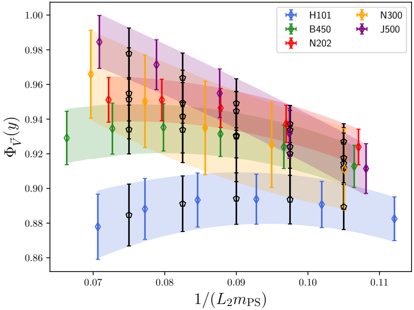
- Observables in finite volume

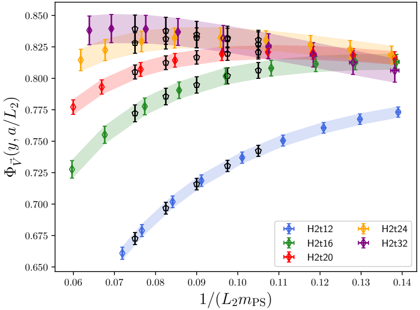
$$\Phi_{\vec{V}}(L) = \log\left(\frac{k_{\vec{V}}(T/2)}{\sqrt{K_1(T)}}\right) \xrightarrow[T \to \infty]{} \ln\left(\frac{L^{3/2}}{\sqrt{2}} \hat{f}_{B^\star}\right)$$

L_2 to L_{CLS} relativistic SSF

- Interpolation to target quark masses and β required to build the SSF

$$\hat{\rho}_{\vec{V}}(u_2, y, a/L) = \Phi_{\vec{V}}(y) - \Phi_{\vec{V}}(y, L_2)$$

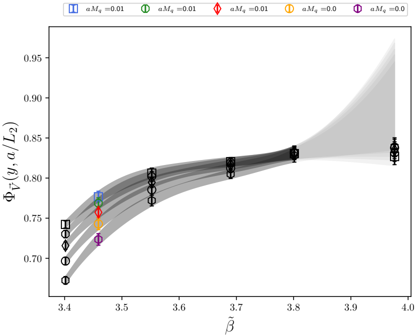


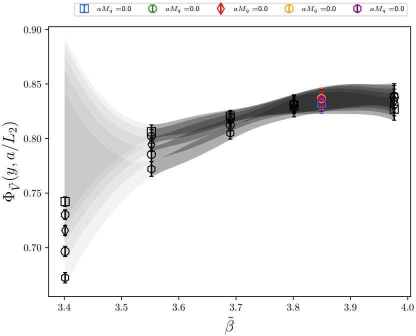


L_2 to $L_{\rm CLS}$ relativistic SSF

- Interpolation to target quark masses and β required to build the SSF

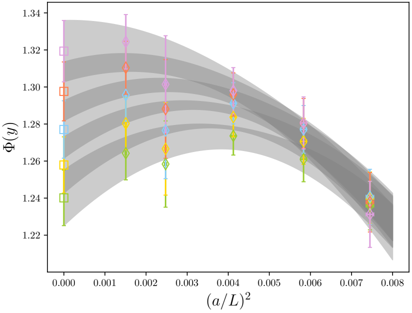
$$\hat{\rho}_{\vec{V}}(u_2, y, a/L) = \Phi_{\vec{V}}(y) - \Phi_{\vec{V}}(y, L_2)$$

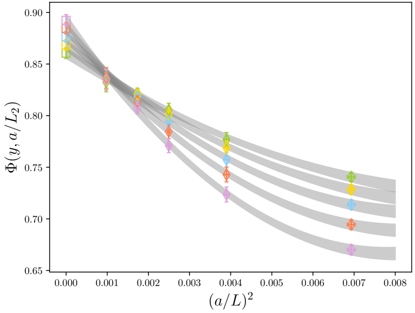




L_2 to $L_{\rm CLS}$: continuum limit of Φ

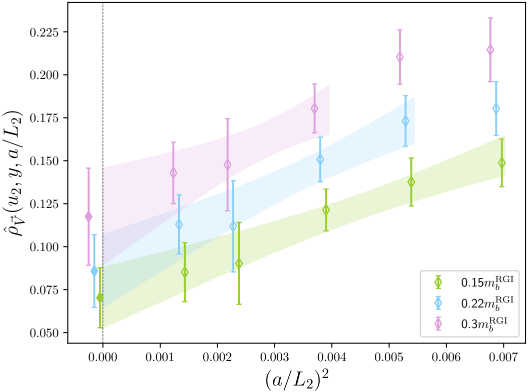
- Continuum limit of $\Phi(y)$ (left) and $\Phi(y, a/L_2)$ (right) following interpolation in m_h and $\tilde{\beta}$





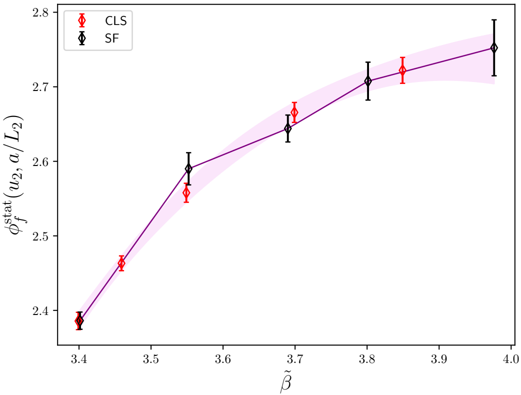
L_2 to L_{CLS}: continuum limit of SSF

- Continuum limit of $\hat{\rho}_{\vec{V}}(u_2, x, a/L)$



L_2 to L_{CLS} static SSF

- Finite-a static SSF: $\Sigma_f^{\mathrm{stat}}(u_2, a/L_2) = \log\left(\frac{\phi_f^{\mathrm{stat}}(u_2)}{\phi_f^{\mathrm{stat}}(u_2, a/L_2)}\right)$



$$f(\tilde{\beta}, p) = p_0\tilde{\beta} + \frac{p_1}{\tilde{\beta}} + \frac{p_2}{\tilde{\beta}^2}$$

Step scaling for f_B

f_B decay constant

- Observable in large volume

$$\langle 0|\, \hat{A}_0 \,|\mathrm{PS}\rangle = \sqrt{f_{A_0A_0}(x_0,y_0)}e^{m_{\mathrm{PS}}/2(x_0-y_0)} \qquad f_{\mathrm{PS}}^{\mathrm{bare}}(m_h) = \sqrt{\frac{2}{L^3 m_{\mathrm{PS}}}}\,\langle 0|\, \hat{A}_0 \,|\mathrm{PS}\rangle$$

$$\hat{f}_{\mathrm{PS}}(m_h) = Z_A\left[1 + ab_a m_{q,h}\right] f_{\mathrm{PS}}^{\mathrm{bare}}$$

$$\Phi_{A_0}(y) = \ln\left(\frac{L_2^{3/2}}{2}\hat{f}_B\right), \qquad \hat{f}_B = \sqrt{m_B} f_{\mathrm{PS}}$$

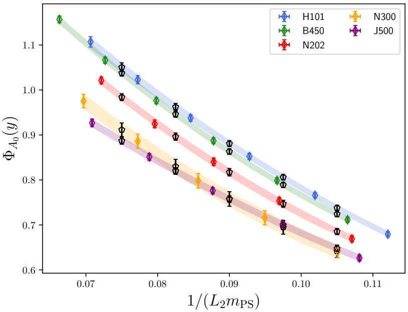
- Observables in finite volume

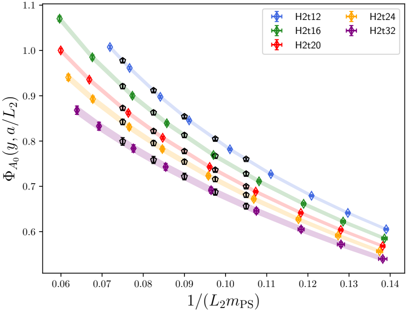
$$\Phi_{A_0}(y,L) = \log\left(-\frac{f_A(T/2)}{\sqrt{f_1(T)}}\right) \xrightarrow[T\to\infty]{} \ln\left(\frac{L^{3/2}}{\sqrt{2}}\hat{f}_B\right)$$

L_2 to L_{CLS} relativistic SSF

- Interpolation to target quark masses and β required to build the SSF

$$\hat{\rho}_{A_0}(u_2, y, a/L_2) = \Phi_{A_0}(y) - \Phi_{A_0}(y, a/L_2)$$

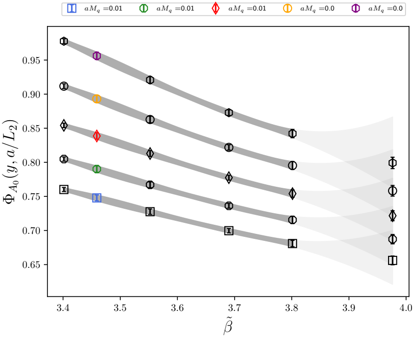


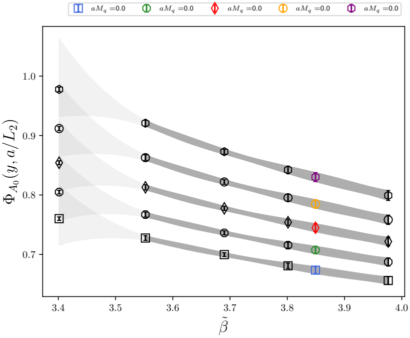


L_2 to L_{CLS} relativistic SSF

- Interpolation to target quark masses and β required to build the SSF

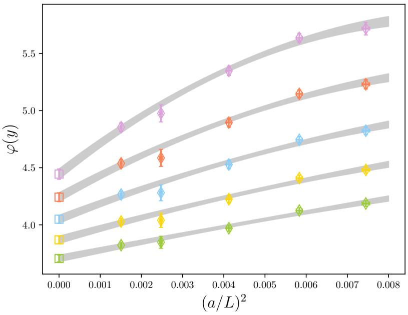
$$\hat{\rho}_{A_0}(u_2, y, a/L_2) = \Phi_{A_0}(y) - \Phi_{A_0}(y, a/L_2)$$

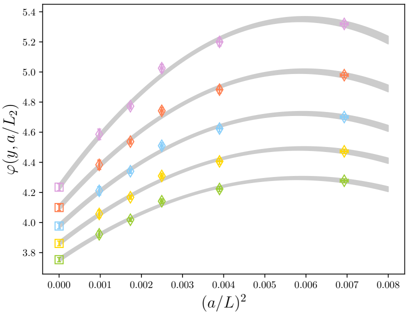




L_2 to $L_{\rm CLS}$: continuum limit of Φ_{A_0}

- Continuum limit of $\Phi_{A_0}(y)$ (left) and $\Phi_{A_0}(y, a/L_2)$ (right) following interpolation in m_h and $\tilde{\beta}$





L_2 to L_{CLS}: continuum limit of SSF

- Continuum limit of $\hat{\rho}_{A_0}(u_2, x, a/L_2)$