Lattice investigation of the general 2HDM with SU(2) gauge fields

Guilherme Catumba
Atsuki Hiraguchi; George W.-S. Hou; Karl Jansen; Ying-Jer Kao; C.-J. David Lin; Alberto Ramos; Mugdha Sarkar

IFIC – Valencia
NYCU

August 3, 2023

SU(2) Two Higgs Doublet model – Motivation

SU(2) gauge theory with 2 fundamental Higgs in 4D

- Single Higgs – simplest way to generate EWSB
- Minimal SM extension – possible new features
  - New source of CP violation
  - Phase structure – Baryogenesis
- Mimics the SM at low energies

SU(2) gauge theory with 2 fundamental Higgs in 4D

- Single Higgs – simplest way to generate EWSB
- Minimal SM extension – possible new features
  - New source of CP violation
  - Phase structure – Baryogenesis
- Mimics the SM at low energies

Fundamental Representation and Gauge Fields

$$\Phi_i(x) = \begin{pmatrix} \phi_i^+(x) \\ \phi_i^0(x) \end{pmatrix} \quad i = 1, 2,$$

$$\mathbb{A}_\mu = -igA_\mu^a \sigma_a/2,$$

$$D_\mu = \partial_\mu + \mathbb{A}_\mu,$$

$$G_{\mu\nu} = \partial_\mu \mathbb{A}_\nu - \partial_\nu \mathbb{A}_\mu + [\mathbb{A}_\mu, \mathbb{A}_\nu]$$

SU(2) Two Higgs Doublet model – Scalar Potential

$$\mathcal{L}_{\text{2HDM}} = (D_\mu \Phi_1)^\dagger (D_\mu \Phi_1) + (D_\mu \Phi_2)^\dagger (D_\mu \Phi_2) + V_{\text{2HDM}} - \frac{1}{2g^2} \operatorname{Tr}[G_{\mu\nu} G_{\mu\nu}]$$

$$\begin{aligned} V_{\text{2HDM}} = \; & \mu_{11}^2 (\Phi_1^\dagger \Phi_1) + \mu_{22}^2 (\Phi_2^\dagger \Phi_2) + \mu_{12}^2 \operatorname{Re}(\Phi_1^\dagger \Phi_2) \\ & + \eta_1 (\Phi_1^\dagger \Phi_1)^2 + \eta_2 (\Phi_2^\dagger \Phi_2) + \eta_3 (\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) + \eta_4 (\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1) \\ & + \eta_5 \operatorname{Re}(\Phi_1^\dagger \Phi_2)^2 + \operatorname{Re}(\Phi_1^\dagger \Phi_2) \left[\eta_6 (\Phi_1^\dagger \Phi_1) + \eta_7 (\Phi_2^\dagger \Phi_2) \right] \end{aligned}$$

SU(2) Two Higgs Doublet model – Scalar Potential

$$\mathcal{L}_{\text{2HDM}} = (D_\mu \Phi_1)^\dagger (D_\mu \Phi_1) + (D_\mu \Phi_2)^\dagger (D_\mu \Phi_2) + V_{\text{2HDM}} - \frac{1}{2g^2} \text{Tr}[G_{\mu\nu} G_{\mu\nu}]$$

$$V_{\text{2HDM}} = \mu_{11}^2 (\Phi_1^\dagger \Phi_1) + \mu_{22}^2 (\Phi_2^\dagger \Phi_2) + \mu_{12}^2 \, \text{Re}(\Phi_1^\dagger \Phi_2) + \eta_1 (\Phi_1^\dagger \Phi_1)^2 + \eta_2 (\Phi_2^\dagger \Phi_2) + \eta_3 (\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) + \eta_4 (\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1) + \eta_5 \, \text{Re}(\Phi_1^\dagger \Phi_2)^2 + \text{Re}(\Phi_1^\dagger \Phi_2) \left[\eta_6 (\Phi_1^\dagger \Phi_1) + \eta_7 (\Phi_2^\dagger \Phi_2) \right]$$

- Explore phase space – physically realizable scenarios
- Tune bare couplings to match physical conditions

$$m_h / m_W \sim 1.6, \quad g^2(\mu = m_W) \sim 0.5$$

- Continuum extrapolation

$$V_{\text{2HDM}} =$$
$$\mu_{11}^2(\Phi_1^\dagger\Phi_1) + \mu_{22}^2(\Phi_2^\dagger\Phi_2) + \mu_{12}^2\,\mathrm{Re}(\Phi_1^\dagger\Phi_2)$$
$$+\,\eta_1(\Phi_1^\dagger\Phi_1)^2 + \eta_2(\Phi_2^\dagger\Phi_2) + \eta_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + \eta_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1)$$
$$+\,\eta_5\,\mathrm{Re}(\Phi_1^\dagger\Phi_2)^2 + \mathrm{Re}(\Phi_1^\dagger\Phi_2)\left[\eta_6(\Phi_1^\dagger\Phi_1) + \eta_7(\Phi_2^\dagger\Phi_2)\right]$$

- Most general case: $SU(2)$ global symmetry
- Previous lattice studies:
  - [Lewis and Woloshyn 2010]
  - [Wurtz, Lewis, and Steele 2009]

$$V_{\text{2HDM}} =$$
$$\mu_{11}^2(\Phi_1^\dagger\Phi_1) + \mu_{22}^2(\Phi_2^\dagger\Phi_2) + \mu_{12}^2\,\mathrm{Re}(\Phi_1^\dagger\Phi_2)$$
$$+\,\eta_1(\Phi_1^\dagger\Phi_1)^2 + \eta_2(\Phi_2^\dagger\Phi_2) + \eta_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + \eta_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1$$
$$+\,\eta_5\,\mathrm{Re}(\Phi_1^\dagger\Phi_2)^2 + \mathrm{Re}(\Phi_1^\dagger\Phi_2)\left[\eta_6(\Phi_1^\dagger\Phi_1) + \eta_7(\Phi_2^\dagger\Phi_2)\right]$$

- $O(4) \sim SU(2)_L \times SU(2)_R$ custodial symmetry
  - [Haber and O'Neil 2011]
  - $\eta_4 = \eta_5$

Global Symmetries

$$V_{\text{2HDM}} =$$
$$\mu_{11}^2(\Phi_1^\dagger\Phi_1) + \mu_{22}^2(\Phi_2^\dagger\Phi_2) + \mu_{12}^2\,\text{Re}(\Phi_1^\dagger\Phi_2)$$
$$+\,\eta_1(\Phi_1^\dagger\Phi_1)^2 + \eta_2(\Phi_2^\dagger\Phi_2) + \eta_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + \eta_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1)$$
$$+\,\eta_5\,\text{Re}(\Phi_1^\dagger\Phi_2)^2 + \text{Re}(\Phi_1^\dagger\Phi_2)\left[\eta_6(\Phi_1^\dagger\Phi_1) + \eta_7(\Phi_2^\dagger\Phi_2)\right]$$

- Discrete $\mathbb{Z}_2$ symmetries: $\mu_{12} = \eta_6 = \eta_7 = 0$
  - $\Phi_1 \longrightarrow -\Phi_1$
  - $\Phi_2 \longrightarrow -\Phi_2$
- Inert Model: $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric [Deshpande and Ma 1978]
  - $\mathbb{Z}_2$ and FNCF [Hou and Kikuchi 2018]
  - Dark matter model [Honorez et al. 2007]

Lattice action & Simulation details

Quaternion representation:

$$\Phi_n(x) = \frac{1}{\sqrt{2}} \sum_{\alpha=0}^{N} \theta_\alpha \phi_\alpha^{(n)}(x),$$

$$\theta_0 = 1_{2\times 2}, \quad \theta_i = i\sigma_i$$

$$S_{\text{2HDM}} = S_{\text{Wilson}} + \sum_x \sum_{n=1}^{2} \left\{ \sum_\mu -2\kappa_n \operatorname{Tr}\left(\hat{\Phi}_n^\dagger U_\mu \hat{\Phi}_n(x+\hat{\mu})\right) + \operatorname{Tr}\left(\hat{\Phi}_n^\dagger \hat{\Phi}_n\right) + \hat{\eta}_n \left[\operatorname{Tr}\left(\hat{\Phi}_n^\dagger \hat{\Phi}_n\right) - 1\right]^2 \right\} + 2\mu^2 \operatorname{Tr}\left(\hat{\Phi}_1^\dagger \hat{\Phi}_2\right)$$
$$+ \hat{\eta}_3 \operatorname{Tr}\left(\hat{\Phi}_1^\dagger \hat{\Phi}_1\right) \operatorname{Tr}\left(\hat{\Phi}_2^\dagger \hat{\Phi}_2\right) + \hat{\eta}_4 \operatorname{Tr}\left(\hat{\Phi}_1^\dagger \hat{\Phi}_2\right)^2$$
$$+ 2 \operatorname{Tr}\left(\hat{\Phi}_1^\dagger \hat{\Phi}_2\right) \left[\hat{\eta}_6 \operatorname{Tr}\left(\hat{\Phi}_1^\dagger \hat{\Phi}_1\right) + \hat{\eta}_7 \operatorname{Tr}\left(\hat{\Phi}_2^\dagger \hat{\Phi}_2\right)\right],$$

- HMC code for GPU – [igit.ific.uv.es/alramos/latticegpu.jl]
- 24^4 Lattice

Observables

- Phase Diagram
  - Gauge invariant link $L_{\alpha,i} = 1/8V \sum_{x,\mu} \mathrm{Tr}\left[\alpha_i^\dagger(x) U_\mu(x) \alpha_i(x+\hat{\mu})\right]$

Observables

- Phase Diagram
  - Gauge invariant link $L_{\alpha,i} = 1/8V \sum_{x,\mu} \mathrm{Tr}\left[\alpha_i^\dagger(x) U_\mu(x) \alpha_i(x+\hat{\mu})\right]$
- Spectrum of the theory (Higgs, W-boson, Goldstone bosons, ...)

$$J^{PC} = 0^{++} \to H_k(\vec{x},t) = \sum_{\mu=1}^{3} \mathrm{Tr}\left(\Phi_k^\dagger(x) U_\mu(x) \Phi_k(x+\hat{\mu})\right)$$

$$J^{PC} = 1^{--} \to W_{kl,\mu}^a(\vec{x},t) = \mathrm{Tr}\left(\Phi_k^\dagger(x) U_\mu(x) \Phi_l(x+\hat{\mu}) \tau^a\right)$$

  - Gradient flow and Laplacian smearing for $U_\mu(x), \Phi(x)$

Observables

- Phase Diagram
  - Gauge invariant link $L_{\alpha,i} = 1/8V \sum_{x,\mu} \mathrm{Tr}\left[\alpha_i^\dagger(x)U_\mu(x)\alpha_i(x+\hat{\mu})\right]$
- Spectrum of the theory (Higgs, W-boson, Goldstone bosons, ...)

$$J^{PC} = 0^{++} \to H_k(\vec{x},t) = \sum_{\mu=1}^{3} \mathrm{Tr}\left(\Phi_k^\dagger(x)U_\mu(x)\Phi_k(x+\hat{\mu})\right)$$

$$J^{PC} = 1^{--} \to W_{kl,\mu}^a(\vec{x},t) = \mathrm{Tr}\left(\Phi_k^\dagger(x)U_\mu(x)\Phi_l(x+\hat{\mu})\tau^a\right)$$

  - Gradient flow and Laplacian smearing for $U_\mu(x), \Phi(x)$
- Non perturbative gauge running coupling – Gradient Flow
  - Flow gauge action density $\langle E(x,t)\rangle = -\frac{1}{4}\langle G_{\mu\nu}^a(x,t)G_{\mu\nu}^a(x,t)\rangle$
  - Perturbation theory [arXiv:1101.0963]

$$t^2\langle E(t)\rangle = \frac{9}{128\pi^2} g_{\bar{M}S}^2(\mu)(1+\mathcal{O}(g^2))\Big|_{\mu=1/\sqrt{8t}}$$

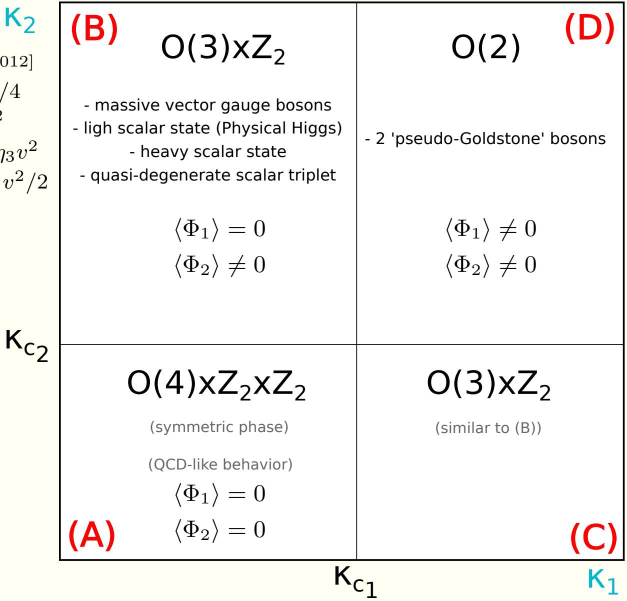
  - Gradient flow gauge running coupling

$$g_{GF}^2(\mu) \equiv \frac{128\pi^2}{9} t^2\langle E(t)\rangle\Big|_{t=1/8\mu^2}$$

$O(4) \sim SU(2)_L \times SU(2)_R$ custodial symmetry – Inert Model

[Branco et al. 2012]

$$m_W^2 = g^2 v^2/4$$
$$m_h^2 = \eta_1 v^2$$
$$m_H^2 = m_A + \eta_3 v^2$$
$$m_T^2 = \mu_{11}^2 + \eta_1 v^2/2$$

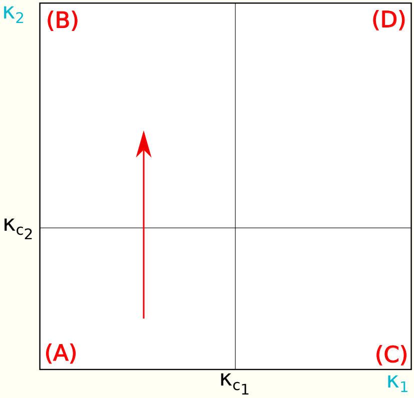


Tree Level:

$$\frac{1}{8} - \frac{\hat{\eta}_2}{4}$$

Exploring phase transitions – $O(4)$ Inert Model

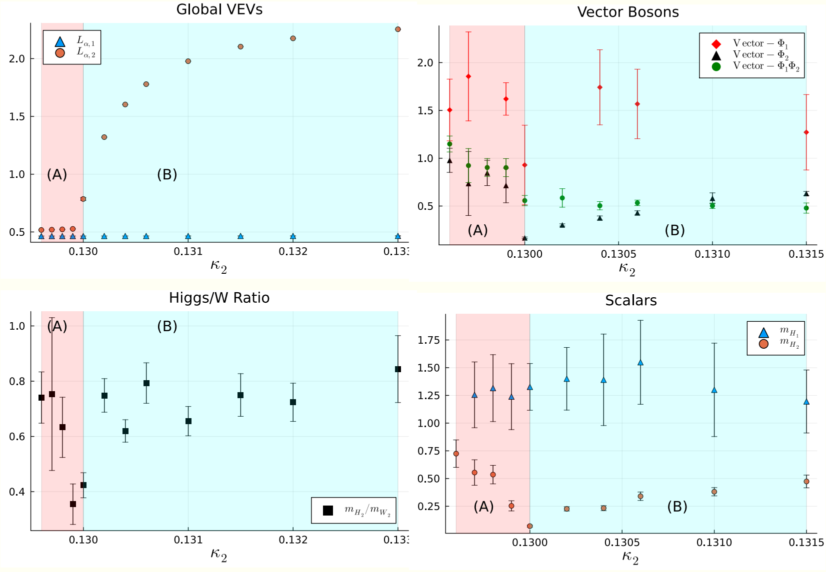
$$\text{(A)}: O(4) \times \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \text{(B)}: O(3) \times \mathbb{Z}_2$$



- $\beta = 6.0$
- $\kappa_1 = 0.127$
- $\hat{\eta}_1 = 0.003, \quad \hat{\eta}_2 = 0.001$
- $\hat{\eta}_3 = 0.0001$
- $\hat{\eta}_4 = 0.00005$

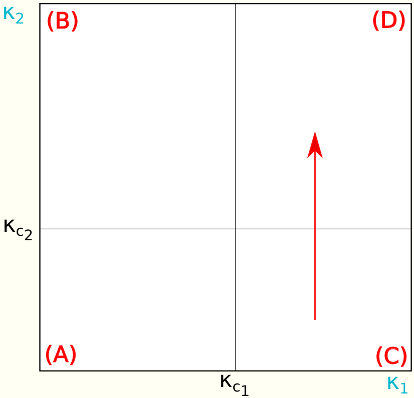
Exploring phase transitions – $O(4)$ Inert Model

$$\text{(A)}: O(4) \times \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \text{(B)}: O(3) \times \mathbb{Z}_2$$



Exploring phase transitions – $O(4)$ Inert Model

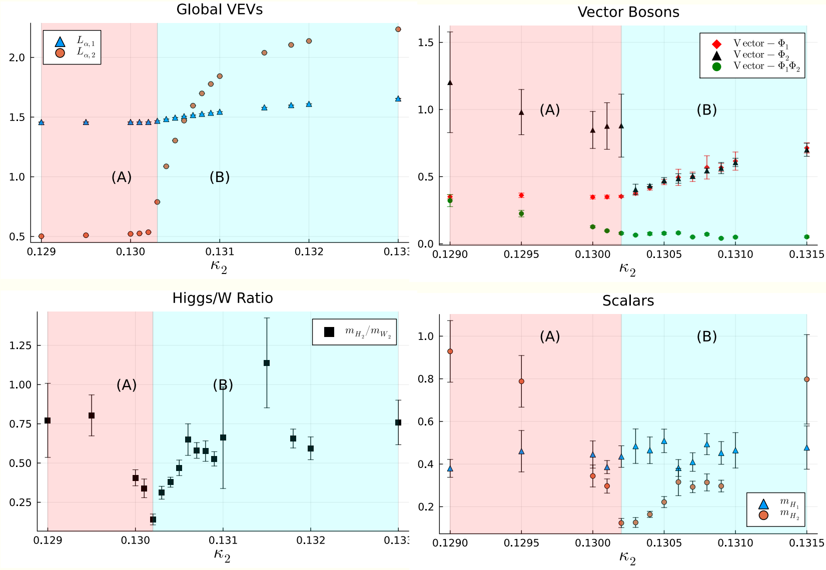
(C): $O(3) \times \mathbb{Z}_2 \longrightarrow$ (D): $O(2)$



- $\beta = 6.0$
- $\kappa_1 = 0.133$
- $\hat{\eta}_1 = 0.003, \quad \hat{\eta}_2 = 0.001$
- $\hat{\eta}_3 = 0.0001$
- $\hat{\eta}_4 = 0.00005$

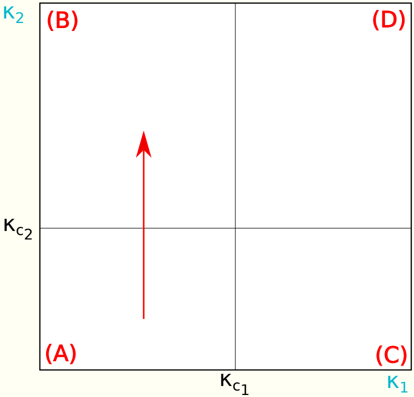
Exploring phase transitions – $O(4)$ Inert Model

$$\text{(C)}: O(3)\times\mathbb{Z}_2 \longrightarrow \text{(D)}: O(2)$$



$\mu_{12},\ \eta_6,\ \eta_7 - Z_2$ breaking terms

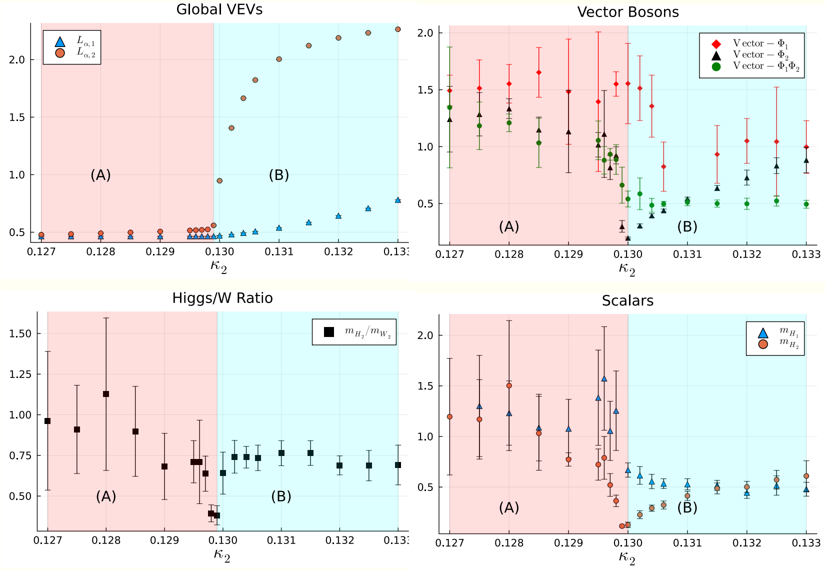
(A): $O(4) \longrightarrow$ (B): $O(3)$



- $\beta = 6.0$
- $\kappa_1 = 0.127$
- $\hat{\eta}_1 = 0.003, \quad \hat{\eta}_2 = 0.001$
- $\mu = 0.001$
- $\hat{\eta}_3 = 0.0001$
- $\hat{\eta}_4 = 0.00005$
- $\hat{\eta}_6 = 0.0005$
- $\hat{\eta}_7 = 0.0001$

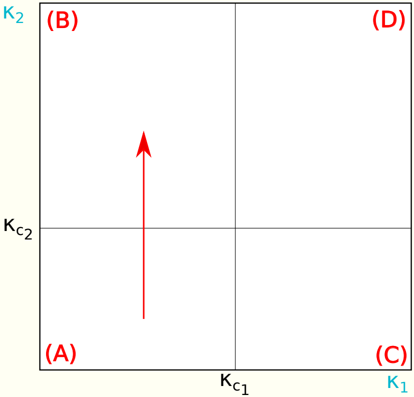
$\mu_{12},\ \eta_6,\ \eta_7 - Z_2$ breaking terms

(A): $O(4) \longrightarrow$ (B): $O(3)$



Tuning bare couplings – $O(4)$ Inert Model

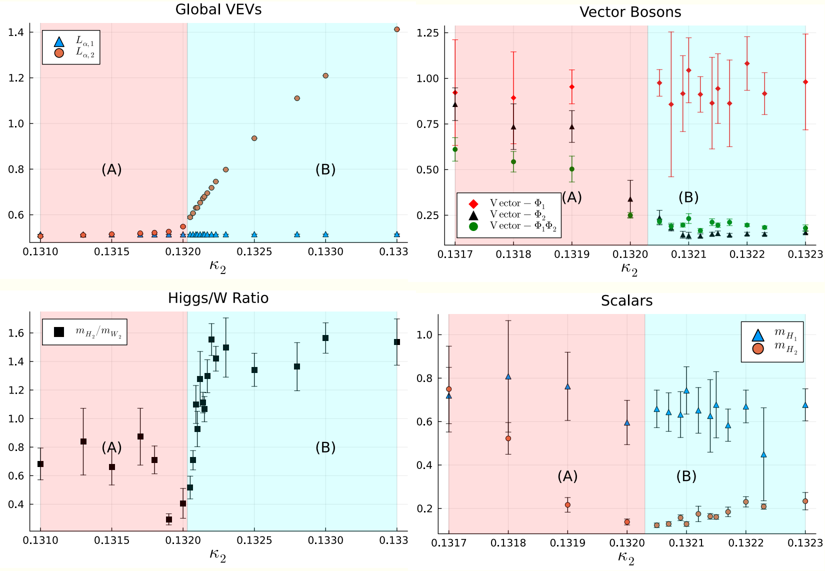
$$\text{(A)}: O(4) \times \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \text{(B)}: O(3) \times \mathbb{Z}_2$$



- $\beta = 6.0$
- $\kappa_1 = 0.1308$
- $\hat{\eta}_1 = 0.003, \quad \hat{\eta}_2 = 0.004$
- $\hat{\eta}_3 = 0.0001$
- $\hat{\eta}_4 = 0.00005$

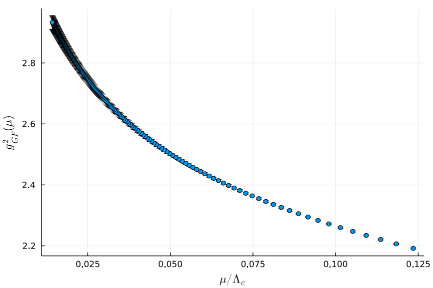
Tuning bare couplings – $O(4)$ Inert Model

$$(\mathrm{A}): O(4) \times \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow (\mathrm{B}): O(3) \times \mathbb{Z}_2$$



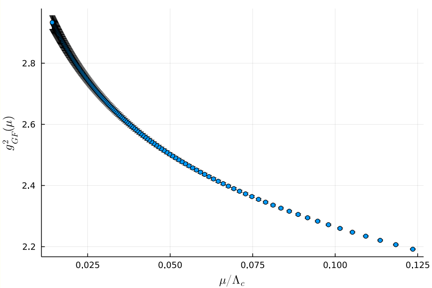
Running coupling $g^2_{GF}(\mu)$ & Phase Structure

- Confinement Phase

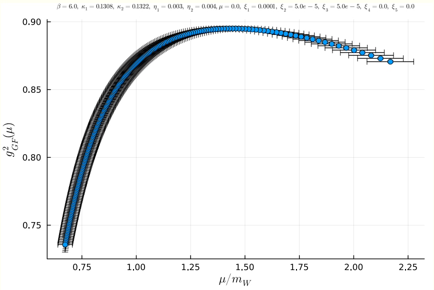


Running coupling $g^2_{GF}(\mu)$ & Phase Structure

- Confinement Phase



- Higgs Phase



$$\beta_{SU(N)+Scalars} = \mu \frac{\mathrm{d}g}{\mathrm{d}\mu} = -\frac{b_0 g^3}{16\pi^2} + \mathcal{O}\left(g^5\right), \quad b_0 = \frac{11N - n_s}{3}$$

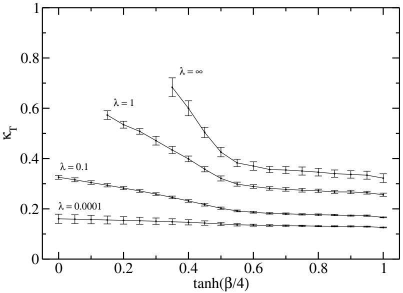
Conclusions & Outlook

- Explore phase structure for most general 2HDM
- Confirmed predictions – symmetry breaking pattern & spectrum
- Find physically realizable scenarios
  - Interpret new states
- Effect of quartic couplings – larger values [Hou and Kikuchi 2018]
- New observables
  - Mixed gauge-invariant links
  - Interpolators – distinguish scalar states
- Physical conditions – tune bare couplings
  - ‘Continuum’ extrapolation

Lagrangian and Phase structure - Single Higgs

$$S = S_{\mathrm{YM}}[U;\beta] + \sum_x \left\{ \sum_\mu -2\kappa \,\mathrm{Tr}\left(\hat{\Phi}^\dagger(x) U_\mu(x) \hat{\Phi}(x+\hat{\mu})\right) + \mathrm{Tr}\left(\hat{\Phi}^\dagger(x)\hat{\Phi}(x)\right) + \lambda \left[\mathrm{Tr}\left(\hat{\Phi}^\dagger(x)\hat{\Phi}(x)\right) - 1\right]^2 \right\}$$

- $\beta = 4/g^2$
- $a^2\mu^2 = \frac{1-2\eta-8\kappa}{\kappa}$
- $\lambda = \eta/\kappa^2$
- Confinement & Higgs
- Analytically connected



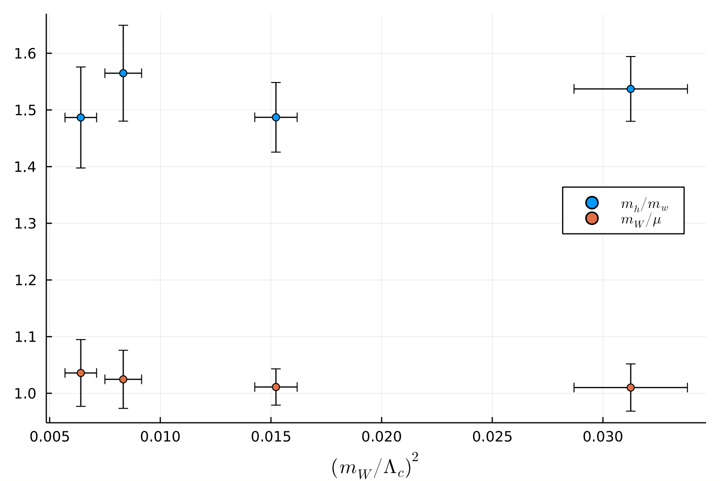
[M. Wurtz et al, Phys.Rev.D 79 (2009) 074501]

Physical conditions & Continuum limit – Single Higgs theory

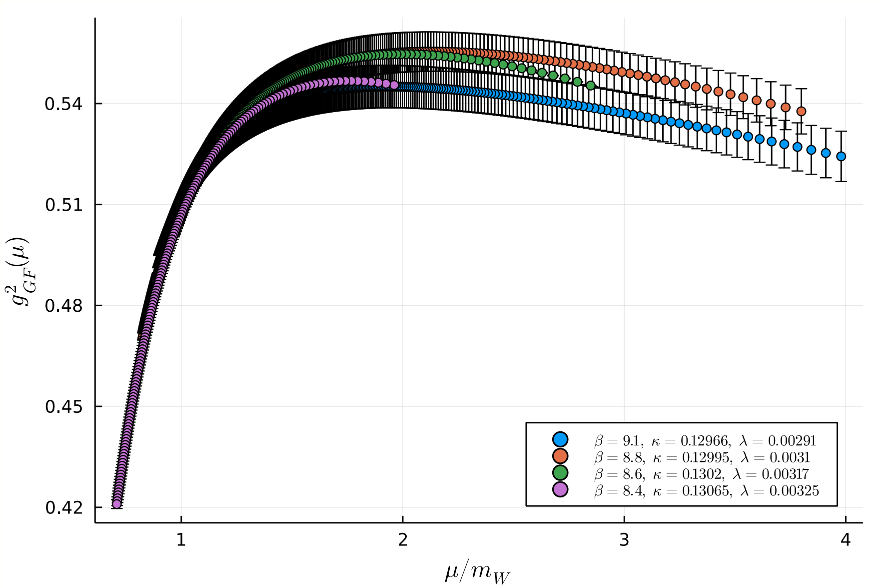
- $m_H/m_W \approx 1.5$
- $g^2_{GF}(\mu = m_W) = 0.5, \quad m_W/\mu = 1.0$

Physical conditions & Continuum limit – Single Higgs theory

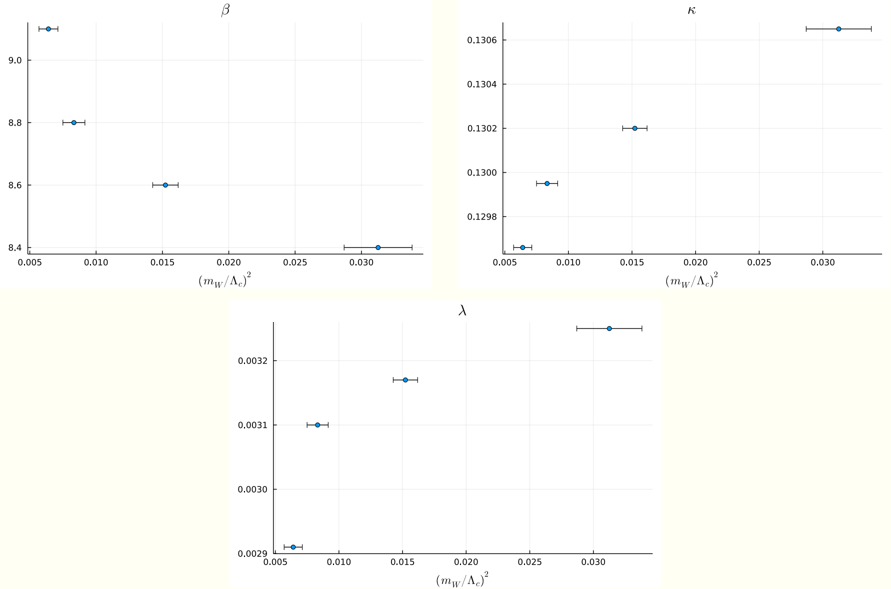
- $m_H/m_W \approx 1.5$
- $g^2_{GF}(\mu = m_W) = 0.5, \quad m_W/\mu = 1.0$



Running coupling $g^2_{GF}(\mu)$ – Single Higgs theory

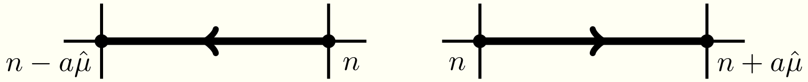


Bare couplings



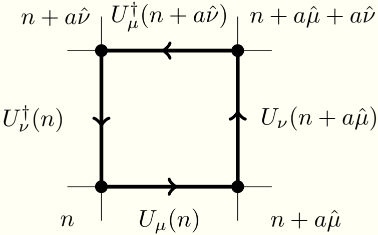
Gauge fields on the Lattice – Wilson Action

Group value fields/links $U_\mu(x) = \exp(iA_\mu(x))$



$U_{-\mu} = U^\dagger_\mu(n - a\hat{\mu})$ $\qquad$ $U_\mu(n)$

$$U_{\mu\nu}(x) = U_\mu(x)U_\nu(x+\hat{\mu})U^*_\mu(x+\hat{\nu})U^*_\nu(x)$$



Wilson Action – $\beta = 4/g^2$

$$S_{\mathrm{YM}}[U;\beta] = \beta \sum_x \sum_{\mu>\nu} \left[1 - \frac{1}{2}\,\mathrm{Re}\,\mathrm{Tr}\,U_{\mu\nu}(x)\right]$$

Scale setting and the Continuum Limit

- All observables in the lattice are dimensionless
- Lattice spacing a is undetermined
- Choose Bare couplings $\rightarrow$ obtain dimensionless predictions
  - $\mu_1 = am_1,\ \mu_2 = am_2, \ldots$
- Scale setting:
  - Choose reference quantity m_1
  - Determine lattice spacing/inverse cutoff $a = \mu_1/m_1$
- Continuum Limit $a \rightarrow 0$
  - Repeat simulations – Tune bare couplings such that $\mu_i \rightarrow 0$
  - Keep physics constant – dimensionless ratios $\leftrightarrow$ couplings

$$\frac{m_H}{m_W},\ g^2(\mu)_{\mu=m_W}, \ldots$$

  - Continuum extrapolation

Fields & Couplings – Lattice action

$$\Phi_n(x) = \frac{\hat{k}_n}{a}\hat{\Phi}_n(x), \qquad a^2\mu_{nn}^2 = \frac{1-2\hat{\eta}_n - 8\hat{k}_n}{\hat{k}_n}, \qquad \eta_n = \frac{\hat{\eta}_n}{\hat{k}_n^2}.$$

$$a^2\mu_{12}^2 = \hat{\mu}_{12}^2, \qquad \eta_n = \frac{\hat{\eta}_n}{\hat{k}_1\hat{k}_2}, \quad n = 3, 4, 5$$

$$\eta_6 = \frac{\hat{\eta}_6}{\hat{k}_1^{3/2}\hat{k}_2^2}, \qquad \eta_7 = \frac{\hat{\eta}_7}{\hat{k}_1^{1/2}\hat{k}_2^3}.$$

Lattice interpolators

$$C(t) = \left\langle O(t)O^{\dagger}(0)\right\rangle = \sum_n \left|\langle 0|\,\hat{O}\,|n\rangle\right|^2 e^{-tE_n}$$

- Choose $\hat{O}(t)$ with the correct quantum numbers – non-unique choices (may use this to improve the signal)

$$J^{PC} = 0^{++} \rightarrow H(\vec{x},t) = \sum_{\mu=1}^{3} \mathrm{Tr}\left(\Phi^{\dagger}(x)U_{\mu}(x)\Phi(x+\hat{\mu})\right)$$

$$J^{PC} = 1^{--} \rightarrow W_{i,\mu}(\vec{x},t) = \mathrm{Tr}\left(\Phi^{\dagger}(x)U_{\mu}(x)\Phi(x+\hat{\mu})\tau_i\right)$$

- Apply Smearing – improve overlap with ground state

$$\begin{aligned}\phi^{(n+1)}(x) &= (1+r_s\nabla)\phi^{(n)}(x)\\ &= \phi^{(n)}(x) + r_s\sum_{\mu=1}^{3}\left(\tilde{U}_{\mu}(x)\phi^{(n)}(x+\hat{\mu}) - 2\phi^{(n)}(x) + \tilde{U}^{\dagger}_{\mu}(x-\hat{\mu})\phi^{(n)}(x-\hat{\mu})\right)\end{aligned}$$

- Effective mass $m_{eff} = -\log\frac{C(t+1)}{C(t)}$

RGEs for the Inert Model

$$8\pi^2\frac{\mathrm{d}\lambda_1}{\mathrm{d}t}=(N+4)\lambda_1^2+N\lambda_3^2+2\lambda_3\lambda_4+\lambda_4^2+\lambda_5^2-\frac{3(N^2-1)}{N}\lambda_1 g^2+\frac{3(N-1)(N^2+2N-2)}{4N^2}g^4$$
$$8\pi^2\frac{\mathrm{d}\lambda_2}{\mathrm{d}t}=(N+4)\lambda_2^2+N\lambda_3^2+2\lambda_3\lambda_4+\lambda_4^2+\lambda_5^2-\frac{3(N^2-1)}{N}\lambda_2 g^2+\frac{3(N-1)(N^2+2N-2)}{4N^2}g^4$$
$$8\pi^2\frac{\mathrm{d}\lambda_3}{\mathrm{d}t}=[(N+1)\lambda_3+\lambda_3](\lambda_1+\lambda_2)+2\lambda_3^2+\lambda_4^2+\lambda_5^2-\frac{3(N^2-1)}{N}\lambda_3 g^2+\frac{3(N^2+2)}{4N^2}g^4$$
$$8\pi^2\frac{\mathrm{d}\lambda_4}{\mathrm{d}t}=\lambda_4(\lambda_1+\lambda_2)+4\lambda_3\lambda_4+N\lambda_4^2+(N+2)\lambda_5^2-\frac{3(N^2-1)}{N}\lambda_4 g^2+\frac{3(N^2+2)}{4N^2}g^4$$
$$8\pi^2\frac{\mathrm{d}\lambda_5}{\mathrm{d}t}=\lambda_5[(\lambda_1+\lambda_2)+4\lambda_3+2(N+1)\lambda_4-\frac{3(N^2-1)}{N}\lambda_5 g^2$$

$O(4)$ condition:

$$8\pi^2\frac{\mathrm{d}(\eta_4-\eta_5)}{\mathrm{d}t}=2(\eta_4^2+2\eta_5^2-3\eta_4\eta_5)+(\eta_4-\eta_5)\left[2\eta_1+2\eta_2+4\eta_3-9/2g^2\right].$$

Gradient Flow

- Diffusion equation for Gauge Fields – Flow time $[t] = -2$

$$\frac{\mathrm{d}B_\mu}{\mathrm{d}t} = D_\nu G_{\nu\mu}, \quad B_\mu(x, t=0) = A_\mu(x)$$

- Smoothing property – radius $= \sqrt{8t}$
- Gauge invariant observables at $t > 0$ require no extra renormalization
  - Gauge action density $\langle E(x,t)\rangle = -\frac{1}{4}\left\langle G^a_{\mu\nu} G^a_{\mu\nu}\right\rangle$
  - $t^2 \langle E(t)\rangle$ is renormalized and dimensionless
  - Define scale t_0 by $t_0^2 \langle E(t_0)\rangle = c$
  - Perturbation theory

$$t^2 \langle E(t)\rangle = \frac{9}{128\pi^2} g^2_{\bar{M}S}(\mu)(1 + \mathcal{O}(g^2)) \Big|_{\mu = 1/\sqrt{8t}}$$

  - Renormalized Coupling

$$g^2_{GF}(\mu) \equiv \frac{128\pi^2}{9} t^2 \langle E(t)\rangle \Big|_{t = 1/8\mu^2}$$