Inclusive hadronic decay rate of the τ lepton from lattice QCD

Antonio Evangelista

in collaboration with

R. Frezzotti, G. Gagliardi, V. Lubicz, F. Sanfilippo, S. Simula, N. Tantalo

Extended Twisted Mass Collaboration

TOR VERGATA UNIVERSITY OF ROME
School of Mathematics, Physical and Natural Sciences

INFN ROMA TOR VERGATA

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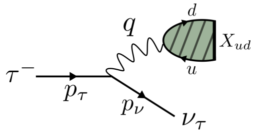
- inclusive hadronic τ lepton decay rate give access to the CKM matrix elements V_{ud} and V_{us}
- until now the main focus was on the strange-hadronic decays and the determination of V_{us}
  - standard OPE $\longrightarrow$ $|V_{us}| = 0.2184(21)$ — E. Gamiz et al – Nucl.Phys.B Proc.Suppl. 169 (2007) 85-89; A. Pich et al – Prog.Part.Nucl.Phys. 75 (2014) 41-85
  - data fitting OPE $\longrightarrow$ $|V_{us}| = 0.2219(22)$ — R.J. Hudspith – Phys.Lett.B 781 (2018) 206-212; K. Maltman et al – SciPost Phys.Proc. 1 (2019) 006
  - gen. dispersion integrals $\longrightarrow$ $|V_{us}| = 0.2240(18)$ — RBC and UKQCD – Phys.Rev.Lett. 121 (2018) 20, 202003; K. Maltman et al – SciPost Phys.Proc. 1 (2019) 006
  - $\mathrm{K}\ell 3 - \mathrm{K}/\pi\ell 2$ $\longrightarrow$ $|V_{us}| = 0.2248(6)$ — FLAG Review 2021 – Eur.Phys.J.C 82 (2022) 10, 869
  - CKM unitarity $\longrightarrow$ $|V_{us}| = 0.2277(13)$ — FLAG Review 2021 – Eur.Phys.J.C 82 (2022) 10, 869; HFLAV Collaboration – Phys.Rev.D 107 (2023) 5, 052008
  - $\tau \to \pi/K\nu_\tau$ exclusive $\longrightarrow$ $|V_{us}| = 0.2222(17)$ — HFLAV Collaboration – Phys.Rev.D 107 (2023) 5, 052008

In this work we perform for the first time a first–principles calculation of the inclusive hadronic decay rate of the τ lepton by using the Hansen-Lupo-Tantalo (HLT) method for spectral density reconstruction



- in the Fermi effective theory, the squared decay–amplitude of the τ–lepton in the ud–flavored channel

$$\left|\mathcal{A}\left(\tau \to X_{ud}\,\nu_\tau\right)\right|^2 = \frac{G_F^2\left|V_{ud}\right|^2}{2}\,\mathcal{L}^{\alpha\beta}(p,p_\nu)\sum_{X_{ud}}\langle 0|\;J^\alpha_{ud}(0)\left|X_{ud}(q)\right\rangle\left\langle X_{ud}(q)\right|\;J^\beta_{ud}(0)^\dagger\,|0\rangle$$
$$= \frac{G_F^2\left|V_{ud}\right|^2}{2}\,\mathcal{L}^{\alpha\beta}(p_\tau,p_\nu)\;\langle 0|\,J^\alpha_{ud}(0)\,(2\pi)^4\delta^{(4)}\left(\mathbb{P}-q\right)\;J^\beta_{ud}(0)^\dagger\,|0\rangle$$

- the leptonic tensor, evaluated in perturbation theory, reads

$$\mathcal{L}^{\alpha\beta}(p_\tau,p_\nu) = 4\left\{p_\tau^\alpha p_\beta^\nu + p_\tau^\beta p_\nu^\alpha - g^{\alpha\beta}p_\tau\cdot p_\nu\right\} - 4i\epsilon^{\alpha\beta\gamma\sigma}p_{\tau\,\gamma}p_{\nu\,\sigma}$$

- the hadronic tensor can be decomposed by relying on Lorentz covariance as

$$\rho^{\alpha\beta}(q) \equiv \langle 0|\,J^\alpha_{ud}(0)\,(2\pi)^4\delta^{(4)}\left(\mathbb{P}-q\right)J^\beta_{ud}(0)^\dagger\,|0\rangle$$
$$= q^\alpha q^\beta\,\rho_{\mathrm{L}}(q^2) + \left[q^\alpha q^\beta - g^{\alpha\beta}q^2\right]\rho_{\mathrm{T}}(q^2)$$

- the decay rate is obtained by integrating over phase space the squared amplitude

$$\Gamma_{ud}^{(\tau)} = \frac{G_F^2 |V_{ud}|^2}{4m_\tau} \int \frac{\mathrm{d}^3 p_\nu}{(2\pi)^3 2E_\nu} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} (2\pi)^4 \, \delta^{(4)} (p_\tau - p_\nu - q) \, \mathcal{L}^{\alpha\beta}(p_\tau, p_\nu) \, \rho_{\alpha\beta}(q) \,,$$

- by introducing $s \equiv q^2/m_\tau^2 = (p_\tau - p_\nu)^2/m_\tau^2 = (1 - 2\frac{p_\tau \cdot p_\nu}{m_\tau})$ we can write

$$R_{ud}^{(\tau)} \equiv \frac{\Gamma_{ud}^{(\tau)}}{\Gamma(\tau \to e\nu_\tau)} = 6\pi \, S_{EW} |V_{ud}|^2 \int_{s_h}^{1} \mathrm{d}s \, (1-s)^2 \, \left\{\rho_{\mathrm{L}}(s) + (1+2s)\, \rho_{\mathrm{T}}(s)\right\}$$

where $s_h = m_h^2/m_\tau^2$ and m_h the mass of the lightest final hadronic state

- on the lattice the primary data are Euclidean correlators that can be related to hadronic tensors by relying on Poincaré, gauge and discrete (P, C, ...) symmetries

$$C^{\alpha\beta}(t,\boldsymbol{q}) = \int \mathrm{d}^3x e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \langle 0|\, J^{\alpha}_{ud}(x)\, e^{-\mathbb{H}\,t} e^{i\boldsymbol{P}\cdot\boldsymbol{x}}\, J^{\beta}_{ud}(0)^{\dagger}\, |0\rangle =$$

$$= \langle 0|\, J^{\alpha}_{ud}(x)\, e^{-\mathbb{H}\,t}\, (2\pi)^3 \delta^{(3)}\left(\boldsymbol{P}-\boldsymbol{q}\right)\, J^{\beta}_{ud}(0)^{\dagger}\, |0\rangle =$$

$$= \int_0^{\infty} \frac{\mathrm{d}E}{2\pi}\, e^{-Et}\, \langle 0|\, J^{\alpha}_{ud}(0)\, (2\pi)^4\, \delta^{(4)}\left(\mathbb{P}-q\right) J^{\beta}_{ud}(0)^{\dagger}\, |0\rangle$$

$$= \int_0^{\infty} \frac{\mathrm{d}E}{2\pi}\, e^{-Et}\, \rho^{\alpha\beta}(E,\boldsymbol{q})$$

- ρ_{T} and ρ_{L} can be conveniently extracted from the correlators at zero spatial momentum according to

$$C_{\mathrm{I}}(t) \equiv \int_{E_0}^{\infty} \frac{\mathrm{d}E}{2\pi}\, e^{-Et}\, E^2\, \rho_{\mathrm{I}}(E^2)\,, \qquad \mathrm{I}=\mathrm{L},\mathrm{T}$$

$$C_{\mathrm{L}}(t) \equiv C^{00}(t,\boldsymbol{0})\,, \qquad C_{\mathrm{T}}(t) = \frac{1}{3}\sum_i C^{ii}(t,\boldsymbol{0})$$

the central object necessary for the computation of $R_{ud}^{(\tau)}$ are spectral densities $\rho_{\mathrm{L/T}}$

$\downarrow$

ideally the spectral densities are extracted from the correlators by performing an inverse Laplace transform

$\downarrow$

on the lattice the correlators can be evaluated only on a **finite set of points** and are affected by **numerical uncertainties**

On a finite volume spectral densities are δ–trains

$\downarrow$

numerically ill–posed problem

$\downarrow$

smeared spectral densities are well defined quantities at finite volume and a noise regulator is necessary

M. Hansen et al. – Phys.Rev.D 99 (2019) 9, 094508 J. Bulava et al. – JHEP 07 (2022) 034

C. Alexandrou et al. – Phys.Rev.Lett. 130 (2023) 24, 241901

$\downarrow$

Smeared kinematic factors play the role of the kernel entering in the inverse problem

Gambino and Hashimoto – Phys.Rev.Lett. 125 (2020) 3, 032001

- by taking into account the ElectroWeak short–distance correction S_{EW}

$$R_{ud}^{(\tau)} = 6\pi\, S_{EW}\, |V_{ud}|^2 \int_{s_h}^{\infty} \mathrm{d}s\,(1-s)^2\,\left[\rho_{\mathrm{L}}(s) + (1+2\,s)\,\rho_{\mathrm{T}}(s)\right]\theta(1-s)$$

$$= 12\pi\, S_{EW}\, |V_{ud}|^2 \int_{m_h}^{\infty} \frac{\mathrm{d}E}{E} \left(1-\frac{E^2}{m_\tau^2}\right)^2 \left[\frac{E^2}{m_\tau^2}\rho_{\mathrm{L}}(E^2) + \left(1+2\frac{E^2}{m_\tau^2}\right)\frac{E^2}{m_\tau^2}\rho_{\mathrm{T}}(E^2)\right]\theta\left(1-\frac{E}{m_\tau}\right)$$

$$= \frac{12\pi\, S_{EW}\, |V_{ud}|^2}{m_\tau^3} \int_{m_h}^{\infty} \mathrm{d}E \left[K_{\mathrm{L}}\left(\frac{E}{m_\tau}\right) E^2 \rho_{\mathrm{L}}(E^2) + K_{\mathrm{T}}\left(\frac{E}{m_\tau}\right) E^2 \rho_{\mathrm{T}}(E^2)\right]$$

- Where K_{T} and K_{L} are kinematical kernels

$$K_{\mathrm{L}}(x) \equiv \frac{1}{x}\left(1-x^2\right)^2\theta(1-x)\,, \qquad K_{\mathrm{T}}(x) \equiv \left(1+2x^2\right)K_{\mathrm{L}}(x)$$

- if $f(x)$ is a smooth function such that $f(x) \sim 0$ as $x \to \infty$ it can be approximated arbitrary well by a truncated series of decreasing exponentials

- by introducing a smeared version of the θ–function

$$\Theta_\sigma(x) = \frac{1}{1+e^{-x/\sigma}}\,, \qquad K_{\rm I}^\sigma(x) \simeq \tilde{K}_{\rm I}^\sigma(x) = \sum_{n=1}^{n_{max}} g_{\rm I}(n;\sigma)\, e^{-axn}$$

- by using this approximation we can trade the integral with a sum over the correlators

$$R_{ud}^{(\tau,I)}(\sigma) \propto \int_{m_h}^{\infty} \mathrm{d}E\, K_{\rm I}^\sigma\left(\frac{E}{m_\tau}\right) E^2\, \rho_{\rm I}(E^2)$$

$$= \sum_{n=1}^{n_{max}} g_{\rm I}(n;\sigma) \int_{m_h}^{\infty} \mathrm{d}E\, e^{-aEn}\, E^2 \rho_{\rm I}(E^2)$$

$$= 2\pi \sum_{n=1}^{n_{max}} g_{\rm I}(n,\sigma)\, C_{\rm I}(na)$$

- the $\boldsymbol{g}$ coefficients are obtained by minimizing the functional at a value $\lambda^{\star}$ in the **statistically dominated regime**

$$W_{\mathrm{I}}^{\alpha}[\boldsymbol{g}] \equiv \frac{A_{\mathrm{I}}^{\alpha}[\boldsymbol{g}]}{A_{\mathrm{I}}^{\alpha}[\mathbf{0}]} + \lambda B_{\mathrm{I}}[\boldsymbol{g}]$$

$$A_{\mathrm{I}}^{\alpha}[\boldsymbol{g}] = \int_{E_{min}}^{E_{max}} \mathrm{d}E\, e^{aE\alpha} \left| \sum_{n=1}^{n_{\max}} g_{\mathrm{I}}(n) e^{-aEn} - K_{\mathrm{I}}^{\sigma}\left(\frac{E}{m_{\tau}}\right) \right|^2$$

$$B_{\mathrm{I}}[\boldsymbol{g}] = B_{\text{norm}} \sum_{n_1,n_2=1}^{\max} g_{\mathrm{I}}(n_1)\, g_{\mathrm{I}}(n_2)\, \mathrm{Cov}_{\mathrm{I}}(an_1, an_2)$$

- the residual error on the reconstruction: from the spread of the values of $R_{ud}^{(\tau,I)}$ at $\lambda^{\star}$ and at $\lambda^{\star\star}$, with the latter defined by

$$\frac{B_{\mathrm{I}}[\boldsymbol{g}_{\mathbf{I}}^{\boldsymbol{\lambda}^{\star\star}}]}{A_{\mathrm{I}}[\boldsymbol{g}_{\mathbf{I}}^{\boldsymbol{\lambda}^{\star\star}}]} = \kappa\, \frac{B_{\mathrm{I}}[\boldsymbol{g}_{\mathbf{I}}^{\boldsymbol{\lambda}^{\star}}]}{A_{\mathrm{I}}[\boldsymbol{g}_{\mathbf{I}}^{\boldsymbol{\lambda}^{\star}}]}\,, \qquad \kappa = 10$$

- we can measure the quality of the kernel reconstruction by evaluating

$$d_I[\boldsymbol{g}_{\boldsymbol{I}}^{\boldsymbol{\lambda}}] \equiv \sqrt{\frac{A_I^0[\boldsymbol{g}_{\boldsymbol{I}}^{\boldsymbol{\lambda}}]}{A_I^0[\mathbf{0}]}}$$

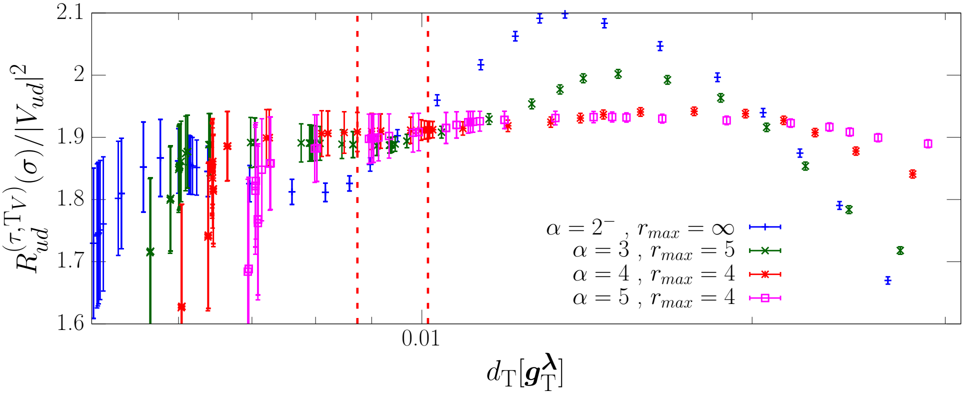
ensemble	β	V/a^4	a (fm)	am_ℓ	M_π (MeV)	L (fm)
B64	1.778	$64^3 \cdot 128$	0.07957 (13)	0.00072	140.2 (0.2)	5.09
B96	1.778	$96^3 \cdot 192$	0.07957 (13)	0.00072	140.2 (0.2)	7.64
C80	1.836	$80^3 \cdot 160$	0.06821 (13)	0.00060	136.7 (0.2)	5.46
D96	1.900	$96^3 \cdot 192$	0.05692 (12)	0.00054	140.8 (0.2)	5.46

ensemble	N_{conf}	N_{sources}	Z_V	Z_A
B64	776	$\sim 10^3$	0.706379 (24)	0.74294 (24)
B96	602	$\sim 10^3$	0.706405 (17)	0.74267 (17)
C80	401	$\sim 10^3$	0.725404 (19)	0.75830 (16)
D96	373	$\sim 10^3$	0.744108 (12)	0.77395 (12)

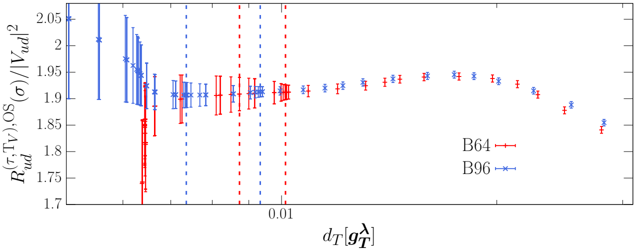
- $N_f = 2 + 1 + 1$ flavours of Wilson-Clover twisted-mass fermions at maximal twist.
- bare quark masses at (very close to) the physical pion–-mass point
- two regularizations: twisted mass (tm) and Osterwalder-Seiler (OS)
- vector and axial currents renormalization constants computed by employing hadronic methods

- $n_{\max} = T/2a$
- $E_{\min} = 0.05 m_\tau \simeq 90$ MeV
- $r_{\max} \equiv aE_{max} = \infty$ with $\alpha = 2^-$
- $r_{\max} \in [4,5,6]$ with $\alpha \in [3,4,5]$

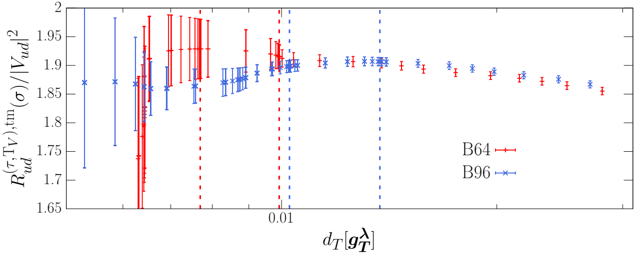
- at increasing α the stability improves $\longrightarrow$ compatibility of the results in a wide range of $d\,[\boldsymbol{g}]$



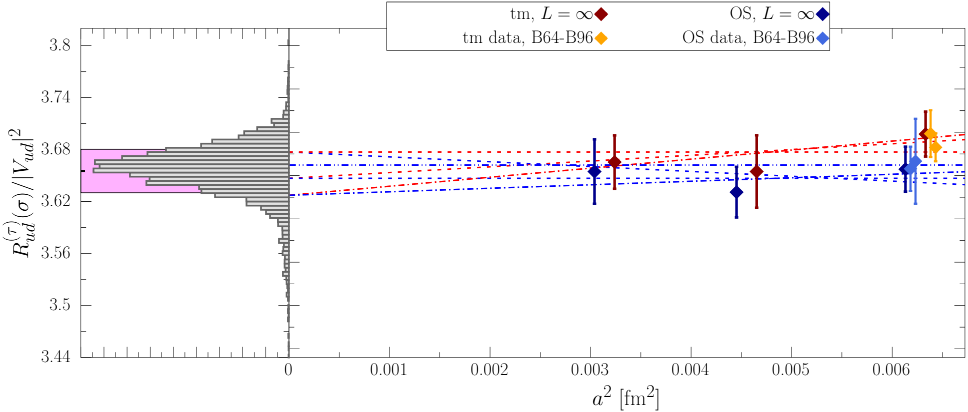
- results at fixed lattice spacing but different volumes are compatible within the errors $(\alpha = 4, r_{max} = 4, \sigma = 0.004)$



- any difference of the results of the two volumes is considered as source of systematic effect

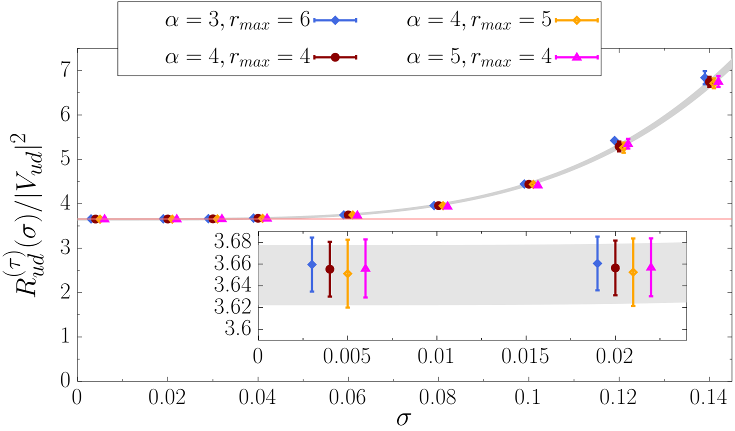


- combined fits: constant or linear in a^2; averaged with a Bayesian model procedure $(\alpha = 4, r_{max} = 4, \sigma = 0.004)$



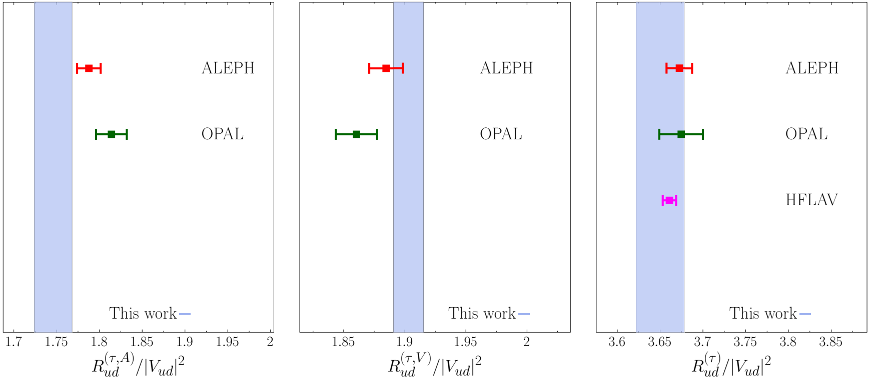
- in the infinite–volume limit the **hadronic** spectral densities are expected to be regular at $E = m_\tau$

$$\int_0^\infty \mathrm{d}E\, E^2 \rho_I(E^2) \left[K_I^\sigma\left(\frac{E}{m_\tau}\right) - K_I\left(\frac{E}{m_\tau}\right) \right] = \mathcal{O}(\sigma^4) \quad \longrightarrow \quad R_{ud}^{(\tau,I)}(\sigma) = R_I + A_I \sigma^4$$



- no σ–dependence for $\sigma < 0.04$
- totally compatible results from different algorithm parameters

	$R_{ud}^{(\tau)}/\|V_{ud}\|^2$	$\|V_{ud}\|$	$\Delta_{V-A}^{(\tau)}$
this work	3.650 (28)	0.9752 (39)	0.042 (5)
Hardy & Towner	-	0.97373 (31)	-
HFLAV	3.660 (8)	-	-
ALEPH	3.672 (15)	-	0.026 (7)
OPAL	3.675 (18)	-	0.013 (7)



- by reconstructing the relevant spectral densities from lattice correlators we performed for the first time a first–principles calculation of the inclusive hadronic decay rate of the τ lepton in the $\bar{u}d$ flavour–channel

- our theoretical results for $|V_{ud}|$ are compatible within errors with the values obtained from nuclear β decay
  Hardy & Towner – Phys.Rev.C 102 (2020) 4, 045501

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  HFLAV Collaboration – Phys.Rev.D 107 (2023) 5, 052008

- we reached an error of $\mathcal{O}(1\%)$, comparable with the magnitude of the isospin–breaking effects

Next to do:

- extend our study to the inclusive process $\tau \to X_{us}\,\nu_\tau$

- compute the leading isospin breaking effects

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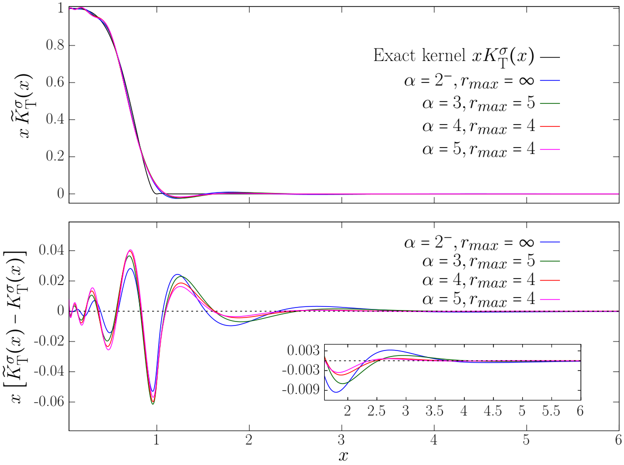
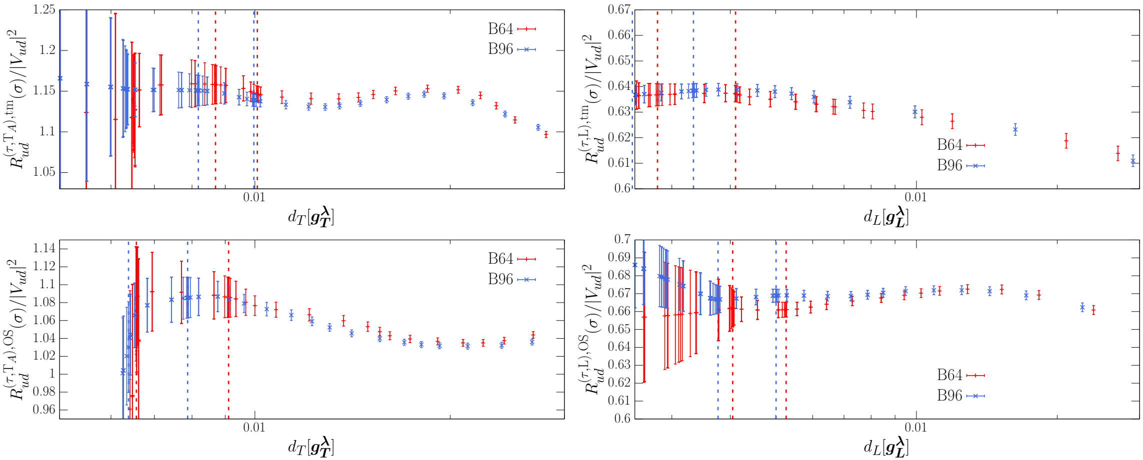
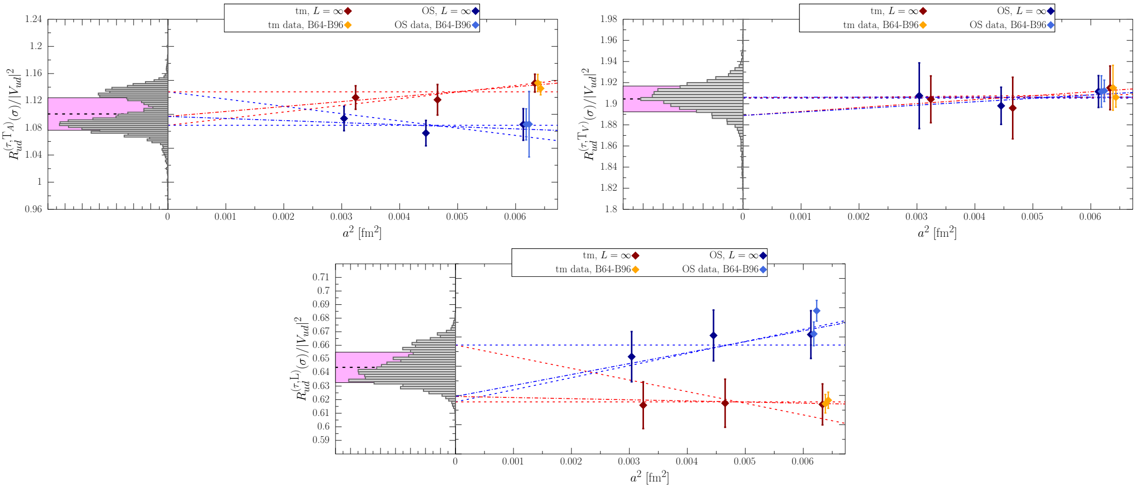
Thanks for the attention !

Backup

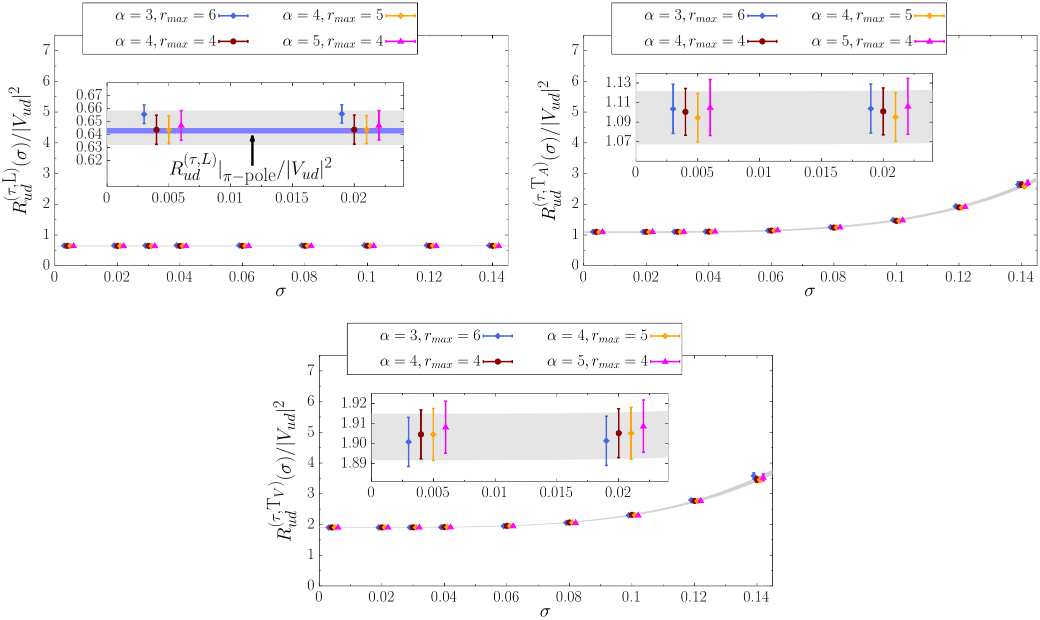
- the reconstruction of the kinematical kernel is less precise around the θ–function discontinuity
- values of $\alpha > 2$ improve (& finiteness of $r_{\max}$ does not harm) the kernel approximation accuracy at high energies



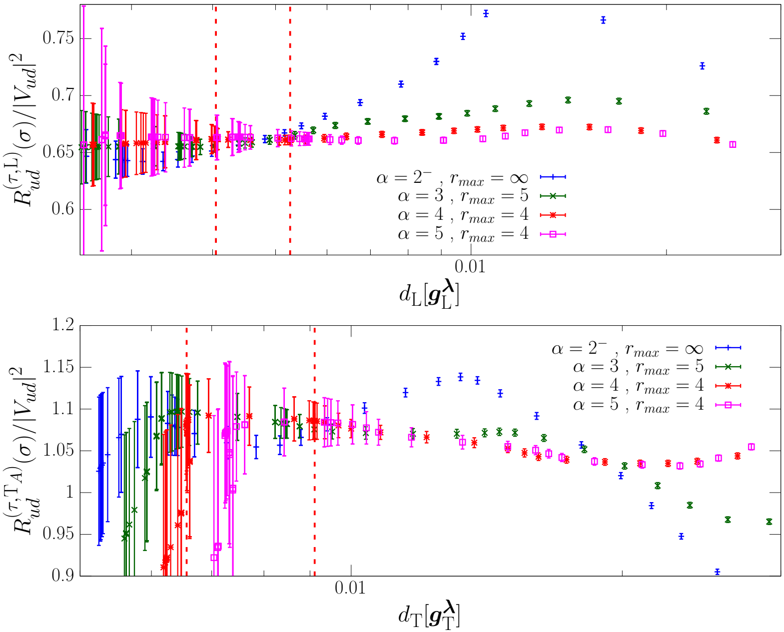

$R_{ud}^{(\tau,\mathrm{T}_A),\mathrm{tm}}(\sigma)/|V_{ud}|^2$
$d_T[\boldsymbol{g_T^\lambda}]$
$R_{ud}^{(\tau,\mathrm{L}),\mathrm{tm}}(\sigma)/|V_{ud}|^2$
$d_L[\boldsymbol{g_L^\lambda}]$
$R_{ud}^{(\tau,\mathrm{T}_A),\mathrm{OS}}(\sigma)/|V_{ud}|^2$
$d_T[\boldsymbol{g_T^\lambda}]$
$R_{ud}^{(\tau,\mathrm{L}),\mathrm{OS}}(\sigma)/|V_{ud}|^2$
$d_L[\boldsymbol{g_L^\lambda}]$
B64
B96
0.01



tm, $L=\infty$
OS, $L=\infty$
tm data, B64-B96
OS data, B64-B96
$R_{ud}^{(\tau,\mathrm{T}_A)}(\sigma)/|V_{ud}|^2$
$R_{ud}^{(\tau,\mathrm{T}_V)}(\sigma)/|V_{ud}|^2$
$R_{ud}^{(\tau,\mathrm{L})}(\sigma)/|V_{ud}|^2$
a^2 [fm^2]



$\alpha = 3, r_{max} = 6$
$\alpha = 4, r_{max} = 5$
$\alpha = 4, r_{max} = 4$
$\alpha = 5, r_{max} = 4$
$R_{ud}^{(\tau,\mathrm{L})}(\sigma)/|V_{ud}|^2$
$R_{ud}^{(\tau,L)}|_{\pi-\mathrm{pole}}/|V_{ud}|^2$
$R_{ud}^{(\tau,\mathrm{T}_A)}(\sigma)/|V_{ud}|^2$
$R_{ud}^{(\tau,\mathrm{T}_V)}(\sigma)/|V_{ud}|^2$
σ



0.75
0.7
0.65
0.6
$R_{ud}^{(\tau,\mathrm{L})}(\sigma)/|V_{ud}|^2$
$\alpha = 2^-$, $r_{max} = \infty$
$\alpha = 3$, $r_{max} = 5$
$\alpha = 4$, $r_{max} = 4$
$\alpha = 5$, $r_{max} = 4$
0.01
$d_\mathrm{L}[\boldsymbol{g}_\mathrm{L}^{\boldsymbol{\lambda}}]$
1.2
1.15
1.1
1.05
1
0.95
0.9
$R_{ud}^{(\tau,\mathrm{TA})}(\sigma)/|V_{ud}|^2$
$\alpha = 2^-$, $r_{max} = \infty$
$\alpha = 3$, $r_{max} = 5$
$\alpha = 4$, $r_{max} = 4$
$\alpha = 5$, $r_{max} = 4$
0.01
$d_\mathrm{T}[\boldsymbol{g}_\mathrm{T}^{\boldsymbol{\lambda}}]$

$$R_{ud}^{(\tau)}(\sigma) = \int_0^{+\infty} \mathrm{d}x\,(1-x)^2\,\tilde{\rho}(x)\,\Theta_\sigma\,(1-x)\;, \qquad\qquad x = E/m_\tau$$

$$\tilde{\rho}\,(x) = 12\pi\,S_{EW}\,|V_{ud}|^2\,x(1+x)^2\left[\rho_{\mathrm{L}}(m_\tau^2 x^2) + (1+2x^2)\,\rho_{\mathrm{T}}(m_\tau^2 x^2)\right]$$

By noticing the properties

$$\Theta_\sigma(x) = \Theta_1\left(\frac{x}{\sigma}\right)\;, \qquad \Theta_1(x)+\Theta_1(-x) = 1\,, \qquad x^p\,\partial_x^q\,\left[1-\Theta_1(x)\right] \overset{x\mapsto\infty}{=} \mathcal{O}\Big(e^{-x}\Big) \quad \forall\,p,q\in\mathbb{N}\,,$$

$$\begin{aligned}
\Delta R_{ud}^{(\tau)}(\sigma) &\equiv R_{ud}^{(\tau)}(\sigma) - R_{ud}^{(\tau)} \\
&= \int_0^\infty \mathrm{d}x\,\left\{\Theta_\sigma\,(1-x) - \theta(1-x)\right\}(1-x)^2\tilde{\rho}(x) \\
&= \int_0^\infty \mathrm{d}x\,\left\{\Theta_1\left(\frac{1-x}{\sigma}\right) - \theta\left(\frac{1-x}{\sigma}\right)\right\}(1-x)^2\tilde{\rho}(x) \\
&= -\sigma^3\int_{-\infty}^{\frac{1}{\sigma}} \mathrm{d}y\,\left\{\theta\,(y) - \Theta_1\,(y)\right\}\,y^2\tilde{\rho}(1-\sigma y)\;, \qquad y = (1-x)/\sigma
\end{aligned}$$

By splitting the integral appearing in the last line of the previous equation and by relying on the properties of the smeared θ–function,

$$\Delta R^{(\tau)}_{ud}(\sigma) = -\sigma^3 \left\{ \int_0^{\frac{1}{\sigma}} \mathrm{d}y \, \left[1 - \Theta_1(y)\right] \, y^2 \tilde{\rho}(1-\sigma y) - \int_{-\infty}^{0} \mathrm{d}y \, \Theta_1(y) \, y^2 \tilde{\rho}(1-\sigma y) \right\} ,$$

$$= -\sigma^3 \left\{ \int_0^{\infty} \mathrm{d}y \, \left[1 - \Theta_1(y)\right] \, y^2 \tilde{\rho}(1-\sigma y) - \int_0^{\infty} \mathrm{d}y \, \Theta_1(-y) \, y^2 \tilde{\rho}(1+\sigma y) \right\} + \mathcal{O}\left(e^{-\frac{1}{\sigma}}\right)$$

$$= \sigma^3 \int_0^{\infty} \mathrm{d}y \, \left[1 - \Theta_1(y)\right] \, y^2 \left[\tilde{\rho}(1+\sigma y) - \tilde{\rho}(1-\sigma y)\right] + \mathcal{O}\left(e^{-\frac{1}{\sigma}}\right) .$$

- until now we only assumed that $\tilde{\rho}(x)$ is a tempered distribution: it grows at most as a power in the $x \mapsto \infty$ limit
- the behaviour of $\Delta R^{(\tau)}_{ud}(\sigma)$ w.r.t. σ is strongly dependent upon the behaviour of $\tilde{\rho}(x)$ around $x = 1$.

If we assume that $\tilde{\rho}(x) = \tilde{\rho}_{\text{reg}}(x)$ is a $\mathcal{C}_\infty$ regular function at $x = 1$, the asymptotic expansion of $\Delta R(\sigma)$ can readily be obtained by using the Taylor series expansion of the spectral density in the previous equation

$$\tilde{\rho}_{\text{reg}}(1+\sigma y) - \tilde{\rho}_{\text{reg}}(1-\sigma y) = 2\sum_{n=0}^{\infty} \frac{\tilde{\rho}^{(2n+1)}(1)}{(2n+1)!}(\sigma y)^{2n+1}$$

and by defining the numeric coefficients

$$C_{\Theta}^{n} = \int_0^\infty \mathrm{d}y\ \left[1 - \Theta_1\left(y\right)\right]\ y^n$$

We have

$$\Delta R_{\text{reg}}(\sigma) = 2\sigma^4 \sum_{n=0}^{\infty} \frac{\tilde{\rho}^{(2n+1)}(1)}{(2n+1)!} C_{\Theta}^{2n+3} \sigma^{2n} = \mathcal{O}\left(\sigma^4\right)$$

- we fitted the same dataset in N different ways obtaining $\{x_k\}_{k=1,\ldots,N}$ values
- their average $\bar{x}$ and final uncertainty σ_x are given by

$$\bar{x} = \sum_{k=1}^{N} \omega_k x_k \ , \qquad \sigma_x^2 = \sigma_{x,stat}^2 + \sum_{k=1}^{N} \omega_k (x_k - \bar{x})^2 \ , \qquad \sum_{k=1}^{N} \omega_k = 1$$

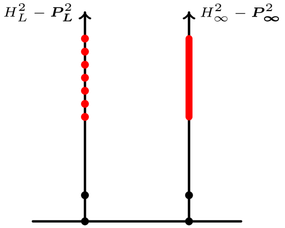
  where ω_k is the weight associated to the $k-$th fit, and $\sigma_{x,stat}$ is the statistical error of $\bar{x}$.
- we choose the weights according to the Akaike Information Criterion (AIC)

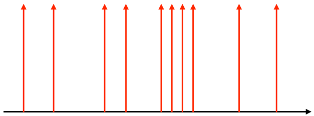
$$\omega_k \propto \exp\left\{ - \left(\chi_k^2 + 2N_{par}^k - N_{meas}^k \right) /2 \right\}$$

  where χ_k^2, N_{par}^k and N_{meas}^k are respectively the chi-squared, the total number of fit parameters, and the total number of measurements of the $k-$th fit.

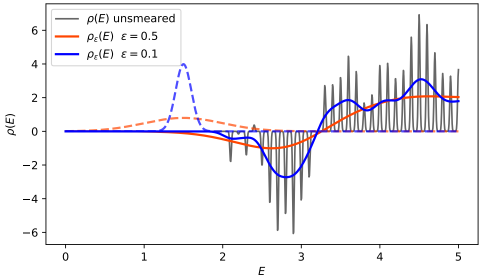
- in the finite volume the spectral densities are distributions since the Hamiltonian has a discrete spectrum

$$\rho_L(E,\boldsymbol{q}) = \langle 0|\, J(0)(2\pi)^4\delta^{(3)}(\boldsymbol{P}-\boldsymbol{q})\delta\left(E-H_L\right) J^\dagger(0)\,|0\rangle_L = \sum_n c_n(L)\delta(\omega-\omega_n(L))$$





- integrals of smeared spectral densities can be studied at finite volumes



- smeared spectral densities are smooth functions and studying their infinte volume limit is a well posed problem