Determination of the CP restoration temperature at $\theta = \pi$
in 4D SU(2) Yang-Mills theory through simulations at imaginary $\theta$

Mitsuaki Hirasawa$^1$

collaboration with
Kohta Hatakeyama$^2$, Masazumi Honda$^{3,4}$, Akira Matsumoto$^3$,Jun Nishimura$^{5,6}$, Atis Yosprakob$^7$.

$^1$ INFN Milano-Bicocca, $^2$ Hirosaki Univ., $^3$ YITP, $^4$ RIKEN iTHEMS, $^5$ KEK, $^6$ SOKENDAI, $^7$ Niigata Univ.
SU($N$) gauge theory with a $\theta$ term

- gauge theory with a $\theta$ term

$$Z = \int \mathcal{D}A_{\mu}e^{-S_{g} + i\theta Q}$$

**topological charge**

$$Q = \frac{1}{32\pi^{2}} \int d^{4}x \epsilon_{\mu\nu\rho\sigma} \text{Tr} \left [ F_{\mu\nu}F_{\rho\sigma} \right ]$$

($Q \in \mathbb{Z}$ on a 4d torus)

- periodicity for $\theta \to \theta + 2\pi n$ ($n \in \mathbb{Z}$)
- The theory has CP symmetry ($\theta \to -\theta$) not only at $\theta = 0$ but also $\theta = \pi$.
- Phase structure at $\theta = \pi$ is predicted by ’t Hooft anomaly matching.

possible phase diagrams \((\theta, T)\) for 4D SU(2) YM

- 2 possible phase diagrams

anomaly matching condition: \(T_{CP} \geq T_{dec}(\pi)\)

holographic analysis at large \(N\)

analysis based on SUSY (SU(2) SYM)

Which diagram is realized for SU(2) YM without SUSY?

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CP restoration at $\theta = \pi$

- behavior of $\langle Q \rangle$ near $\theta = \pi$

\[
\langle Q \rangle_{\theta=\pi-\epsilon} = -\langle Q \rangle_{\theta=-\pi-\epsilon} = -\langle Q \rangle_{\theta=-\pi+\epsilon}
\]

\[
\lim_{\epsilon \to 0} \lim_{V_s \to \infty} \frac{\langle Q \rangle_{\theta=\pi-\epsilon}}{V_s} = 0 : \text{CP}
\]

\[
\lim_{\epsilon \to 0} \lim_{V_s \to \infty} \frac{\langle Q \rangle_{\theta=\pi-\epsilon}}{V_s} \neq 0 : \text{CP}
\]

- order parameter for SSB of CP

Determination of the CP restoration temperature at $\theta = \pi$ in 4D SU(2) Yang-Mills theory through simulations at imaginary $\theta$
relation between $\langle Q \rangle$ and the topological charge distribution

- partition function

$$Z_\theta = \int dU e^{-S_g + i\theta Q} = Z_0 \int dq e^{i\theta q} \rho(q)$$

$\rho(q)$: distribution of the topological charge at $\theta = 0$

$$\rho(q) = \frac{1}{Z_0} \int dU \delta(q - Q) e^{-S_g}$$

- $\theta$ dependence of $\langle Q \rangle$

$$\langle Q \rangle = -i \frac{\partial}{\partial \theta} \log Z_\theta = \frac{\int dq q e^{i\theta q} \rho(q)}{\int dq e^{i\theta q} \rho(q)}$$

Behavior of $\langle Q \rangle$ near $\theta = \pi$ is determined by $\rho(q)$. 
Imaginary $\theta$ as a probe

- CP restoration at $\theta = \pi$ is related to the asymptotic behavior of $\rho(q)$.

$$\rho(q) \sim \begin{cases} \exp(-q \log q) : \text{instanton gas} & \text{(CP restored)} \\ \exp(-\frac{q^2}{2\chi_0 V}) : \text{large } N \text{ (low } T) & \text{(CP broken)} \end{cases}$$

- $\langle Q \rangle$ at imaginary $\theta$

$$\langle Q \rangle = \frac{\int dq e^{\tilde{\theta}q} \rho(q)}{\int dq e^{\tilde{\theta}q} \rho(q)} \quad \tilde{\theta} = i\theta$$

The asymptotic behavior is enhanced by $e^{\tilde{\theta}q}$.

We can observe the tail of $\rho(q)$ through $\langle Q \rangle$ at imaginary $\theta$.

(c.f. CP$^9$ case [V. Azcoiti, G. D. Carlo, A. Galante, V. Laliena (2002)])
behavior of $\langle Q \rangle$

- $\theta$ dependence of $\langle Q \rangle$

<table>
<thead>
<tr>
<th>model</th>
<th>$\langle Q \rangle / (\chi_0 V)$ at $\theta$</th>
<th>$\langle Q \rangle / (\chi_0 V)$ at $\tilde{\theta} \equiv i\theta$</th>
<th>CP at $\theta = \pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>instanton gas</td>
<td>$i \sin \theta$</td>
<td>$\sinh \tilde{\theta}$</td>
<td>restored</td>
</tr>
<tr>
<td>large N (low T)</td>
<td>$i\theta$</td>
<td>$\tilde{\theta}$</td>
<td>broken</td>
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We determine $T_{\text{CP}}$ through simulations of the theory at imaginary $\theta$.

The difference of $\rho(q)$ strongly affects the behavior of $\langle Q \rangle$ at imaginary $\theta$. 

Determination of the CP restoration temperature at $\theta = \pi$ in 4D SU(2) Yang-Mills theory through simulations at imaginary $\theta$
lattice setup

- gauge action: Wilson action
- topological charge: clover leaf definition + stout smearing
  
  \[ S = S_g(U) + S_\theta[\tilde{U}] \]
  \[ \tilde{U} : \text{smeared link} \]
  
  stout smearing step size: \( \rho = 0.09 \)
  stout smearing steps: \( N_\rho = 40 \)

- algorithm: HMC

\[ \text{[P. Di Vecchia, K. Fabricius, G. C. Rossi, G. Veneziano (1981)]} \]
\[ \text{[C. Morningstar, M. Peardon (2004)]} \]
previous work

- We determined $T_{cp}$ focusing on $T$ dependence of $\langle Q \rangle / (\chi_0 V)$ for fixed $\tilde{\theta}$.

- fitting by
  
  $f(x) = a + bx + cx^2 + dx^3$

$T_{CP}$ is identified as the peak position of $f'(x)$.

Determination of the CP restoration temperature at $\theta = \pi$ in 4D SU(2) Yang-Mills theory through simulations at imaginary $\theta$
previous work

\[
\frac{T_{\text{dec}}(\theta)}{T_{\text{dec}}(0)} \simeq 1 - R_2 \theta^2
\]

\[R_2 \sim 0.0178 \text{ in SU}(3) \text{ YM}\]

\[\rightarrow T_{\text{dec}}(0) > T_{\text{dec}}(\pi) \text{ in SU}(3) \text{ YM.}\]

Assuming that this is also the case with SU(2) YM, our result \(T_{\text{CP}} \sim 1.06 T_{\text{dec}}(0)\) implies \(T_{\text{CP}} > T_{\text{dec}}(\pi)\).
New analysis based on analytic continuation using $\tilde{\theta}$ dependence at each $T$

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our previous work

this work
New analysis based on analytic continuation

- analytic continuation $\tilde{\theta} \rightarrow \theta = i\tilde{\theta}$

<table>
<thead>
<tr>
<th>model</th>
<th>$\langle Q \rangle / V$ at $\theta$</th>
<th>$\langle Q \rangle / V$ at $\theta = i\theta$</th>
<th>CP at $\theta = \pi$</th>
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<tr>
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<tr>
<td>large N (low T)</td>
<td>$i\chi_0 \theta$</td>
<td>$\chi_0 \theta$</td>
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- We focus on the $\tilde{\theta}$ dependence of $\langle Q \rangle$
  1. $V_s \rightarrow \infty$ extrapolation for $\langle Q \rangle / V$ at each $\tilde{\theta}$
  2. fit the results for various $\tilde{\theta}$ to
    - hyperbolic sine series:
      $$f(\tilde{\theta}) = \sum_{n=1}^{3} a_n \sinh(n\tilde{\theta})$$
    - polynomial:
      $$g(\tilde{\theta}) = b_1 \tilde{\theta} + b_3 \tilde{\theta}^3$$
  3. analytic continuation $\tilde{\theta} \rightarrow \theta = i\tilde{\theta}$

\[ \lim_{\epsilon \rightarrow 0} \lim_{V_s \rightarrow \infty} \frac{\langle Q \rangle}{V_s} = \begin{cases} 
  i\chi_0 \sum_{n} \alpha_n \sin(n\theta) : CP \\
  i\chi_0 \sum_{n} \beta_{2n-1} \theta^{2n-1} : CP 
\end{cases} \]

at real $\theta$

Determination of the CP restoration temperature at $\theta = \pi$ in 4D SU(2) Yang-Mills theory through simulations at imaginary $\theta$
New analysis based on analytic continuation

1. $V_s \to \infty$ extrapolation at each $\tilde{\theta}$

$$T = 1.2T_{\text{dec}}(0)$$

$$T = 0.9T_{\text{dec}}(0)$$
New analysis based on analytic continuation

- fitting the $V_s \to \infty$ extrapolated results to
  - hyperbolic sine series: $f(\tilde{\theta}) = \sum_{n=1}^{3} a_n \sinh(n\tilde{\theta})$
  - polynomial: $g(\tilde{\theta}) = b_1 \tilde{\theta} + b_3 \tilde{\theta}^3$

We can impose a constraint on the fitting parameters using

$$\frac{\partial \langle Q \rangle}{\partial \tilde{\theta}} \bigg|_{\tilde{\theta}=0} = \chi_0.$$ 

- the infinite volume extrapolation for $\chi_0$

$$T = 1.2 T_{\text{dec}}(0)$$

$$T = 0.9 T_{\text{dec}}(0)$$

Determination of the CP restoration temperature at $\theta = \pi$ in 4D SU(2) Yang-Mills theory through simulations at imaginary $\theta$
analysis using analytic continuation

fitting the $V_s \to \infty$ extrapolated results

hyperbolic sine series

$$f(\tilde{\theta}) = \sum_{n=1}^{3} a_n \sinh(n\tilde{\theta})$$
Analysis using analytic continuation

2 fitting the $V_s \to \infty$ extrapolated results

Hyperbolic sine series

$$f(\tilde{\theta}) = \sum_{n=1}^{3} a_n \sinh(n\tilde{\theta})$$

Polynomial

$$g(\tilde{\theta}) = b_1 \tilde{\theta} + b_3 \tilde{\theta}^3$$
analysis using analytic continuation

fitting the \( V_s \rightarrow \infty \) extrapolated results

hyperbolic sine series
\[
f(\tilde{\theta}) = \sum_{n=1}^{3} a_n \sinh(n\tilde{\theta})
\]

polynomial
\[
g(\tilde{\theta}) = b_1 \tilde{\theta} + b_3 \tilde{\theta}^3
\]
analytic continuation \( \tilde{\theta} \to \theta = i\tilde{\theta} \)

\[
\lim_{V_s \to \infty} \frac{\langle Q \rangle}{V_s} \text{ against } \tilde{\theta}
\]

\[
\lim_{V_s \to \infty} \text{Im} \frac{\langle Q \rangle}{V_s} \text{ against } \theta
\]

no gap at \( \theta = \pi \)
\( \rightarrow \) CP restored

gap at \( \theta = \pi \)
\( \rightarrow \) CP broken

We are trying to use this analysis to determine \( T_{CP} \).
We studied the phase structure for 4D SU(2) YM at $\theta = \pi$.

CP restoration/breaking at $\theta = \pi$ is related to the tail of the topological charge distribution at $\theta = 0$.

→ We used imaginary $\theta$ to probe for the tail of the distribution.
  
  (c.f. [V. Azcoiti, G. D. Carlo, A. Galante, V. Laliena (2002)])

→ Our previous work

We focused on the $T$ dependence of $\langle Q \rangle / (\chi_0 V)$

$T_{CP} \sim 1.06 T_{dec}(\theta = 0)$

Our results suggested $T_{CP} > T_{dec}(\theta = \pi)$ for SU(2)

unlike large-$N$ result ($T_{CP} = T_{dec}(\theta = \pi)$).

[Matsumoto, Hatakeyama, Hirasawa, Honda, Nishimura, Yosprakob (PoS LATTICE2022 378)]
summary

- this work: analytic continuation focusing on $\tilde{\theta}$ dependence of $\langle Q \rangle$

<table>
<thead>
<tr>
<th>temperature</th>
<th>appropriate fitting function</th>
<th>$\langle Q \rangle$ at $\theta = \pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T = 1.2T_{\text{dec}}(0)$</td>
<td>$f(\tilde{\theta}) = \sum_{n=1}^{3} a_n \sinh(n\tilde{\theta})$</td>
<td>$\langle Q \rangle = 0$ (CP restored)</td>
</tr>
<tr>
<td>$T = 0.9T_{\text{dec}}(0)$</td>
<td>$g(\tilde{\theta}) = b_1\tilde{\theta} + b_3\tilde{\theta}^3$</td>
<td>$\langle Q \rangle \neq 0$ (CP broken)</td>
</tr>
</tbody>
</table>

This result is consistent with the results of our previous study.

- We are now studying the intermediate $T$ region.
  - hyperbolic sine fitting works at $T \geq T_{CP}$
  - polynomial fitting works at $T \leq T_{CP}$

- A similar study for the SU(3) YM is ongoing.
  ($T_{CP} = T_{\text{dec}}(\pi) < T_{\text{dec}}(0)$ is expected like large-$N$.)

($T_{CP} = T_{\text{dec}}(\pi) < T_{\text{dec}}(0)$ is expected like large-$N$.)

Determination of the CP restoration temperature at $\theta = \pi$ in 4D SU(2) Yang-Mills theory through simulations at imaginary $\theta$
Thank you for listening!
predicted phase structure for 4D SU(N) at $\theta = \pi$

't Hooft anomaly matching condition

At least either CP or $Z_2$ is broken.


possible scenario

- SSB of CP
- SSB of $Z_2$
- gapless (CFT)
- topological QFT

large $N$: Both CP restoration and deconfining phase transition occur at the same temperature.

small $N$ (i.e. $N=2$): not determined.

[F. Bigazzi, A. L. Cotrone, R. Sisca (2015)]
predicted phase structure for 4D SU(2) YM at $\theta = \pi$

- confinement/deconfinement in SU(2) YM at $\theta = \pi$
  - deconfinement at high temperature (1-loop analysis)
    
  - confinement and CP breaking at $T = 0$ (non-perturbative)
    
    [R. Kitano, R. Matsudo, N. Yamada, M. Yamazaki (2021)]

- constraint from the anomaly matching condition
  - "mixed 't Hooft anomaly between CP symmetry and center symmetry"
  - CP restoration does not occur in the confined phase.
    
    $T_{CP} \geq T_{dec(\theta=\pi)}$
    

- large $N : T_{CP} = T_{dec(\theta=\pi)}$. 
  
  [F. Bigazzi, A. L. Cotrone, R. Sisca (2015)]

- small $N$ (i.e. $N = 2$): ?
### Simplified models for CP restoration at $\theta = \pi$

- We consider two different types of models.

<table>
<thead>
<tr>
<th>model</th>
<th>free energy $F(\theta) \equiv - \log Z_{\theta}$</th>
<th>CP at $\theta = \pi$</th>
</tr>
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<tbody>
<tr>
<td>instanton gas</td>
<td>$\chi_0 V (1 - \cos \theta)$</td>
<td>restored</td>
</tr>
<tr>
<td>large $N$ (low $T$)</td>
<td>$\frac{1}{2} \chi_0 V \min \left( \theta - 2\pi n \right)^2$</td>
<td>broken</td>
</tr>
</tbody>
</table>

$$\langle Q \rangle := -i \frac{\partial}{\partial \theta} \log Z$$

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**Figure:**

- Graph showing $F(\theta) / \chi V$.
- Graph showing $\rho(q)$ in logarithmic scale.

- CP restored
- CP broken

*E. Witten (1979)*
numerical results ($T$ dependence of $\langle Q \rangle / (\chi_0 V)$)

- $T$ dependence of $\langle Q \rangle / (\chi_0 V)$ at $\theta / \pi = 0.75i$

- fitting by
  $$f(x) = a + bx + cx^2 + dx^3$$
  
  $T_{CP}$ is identified as a peak position of the derivative $f'(x)$.

- $f'(x)$ has a peak at $T \sim 1.06T_{dec}(0)$. 

- $N_p = 40$, $\rho = 0.09$, $V_L = 20^3 \times 5$, $\tilde{\theta} = 0.75\pi$

- large $N$ (low $T$)

- instanton gas
numerical results (volume dependence)

- results for various $\tilde{\theta}$ and volume dependence

The transition temperature seems to converge around $1.06 T_{\text{dec}}(0)$.

The order of the transition is the second or higher.

(c.f. 1st order case : The peak increases linearly in $V_s$.)
detail of the fitting

\[ T = 1.2T_{\text{dec}}(0) \]

\[ T = 0.9T_{\text{dec}}(0) \]

\[
\langle Q \rangle / V_s \quad \text{against} \quad \tilde{\theta}
\]

\[
\text{Im} \langle Q \rangle / V_s \quad \text{against} \quad \theta
\]

parameters

\[ a_2 = -0.0000027 \]
\[ a_3 = 0.0000002 \]
\[ b_3 = 0.0000394 \]

\[ a_2 = -0.0001314 \]
\[ a_3 = 0.0000043 \]
\[ b_3 = 0.0000103 \]