Determination of the CP restoration temperature at $\theta = \pi$ in 4D SU(2) Yang-Mills theory through simulations at imaginary θ

Mitsuaki Hirasawa[1]

collaboration with

Kohta Hatakeyama[2], Masazumi Honda[3,4], Akira Matsumoto[3], Jun Nishimura[5,6], Atis Yosprakob[7].

[1] INFN Milano-Bicocca, [2] Hirosaki Univ., [3] YITP, [4] RIKEN iTHEMS, [5] KEK, [6] SOKENDAI, [7] Niigata Univ.

SU(N) gauge theory with a θ term

- gauge theory with a θ term

$$Z = \int \mathcal{D}A_\mu e^{-S_g + i\theta Q}$$

topological charge

$$Q = \frac{1}{32\pi^2}\int d^4x \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\left[F_{\mu\nu}F_{\rho\sigma}\right]$$

($Q \in \mathbb{Z}$ on a 4d torus)

- periodicity for $\theta \to \theta + 2\pi n \quad (n \in \mathbb{Z})$
- The theory has CP symmetry ($\theta \to -\theta$) not only at $\theta = 0$ but also $\theta = \pi$.
- Phase structure at $\theta = \pi$ is predicted by 't Hooft anomaly matching.

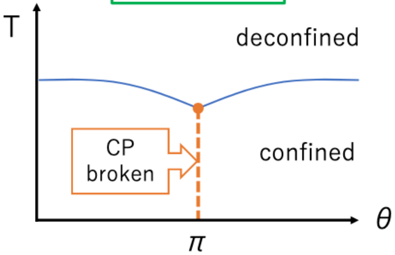
[D. Gaiotto, A.Kapustin, Z. Komargodski, N.Seiberg (2017)]

possible phase diagrams (θ, T) for 4D SU(2) YM

- 2 possible phase diagrams

anomaly matching condition: $T_{\rm CP} \geq T_{\rm dec}(\pi)$

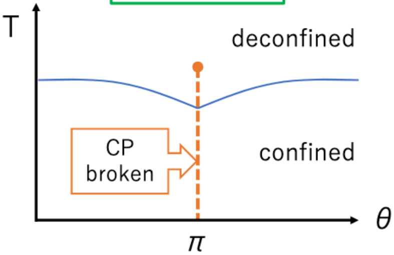
$T_{\rm CP} = T_{\rm dec}(\pi)$



holographic analysis at large N

[F. Bigazzi, A. L. Cotrone, R. Sisca (2015)]

$T_{\rm CP} > T_{\rm dec}(\pi)$



analysis based on SUSY (SU(2) SYM)

[S. Chen, K. Fukushima, H. Nishimura, Y. Tanizaki (2020)]

Which diagram is realized for SU(2) YM without SUSY?

CP restoration at $\theta = \pi$

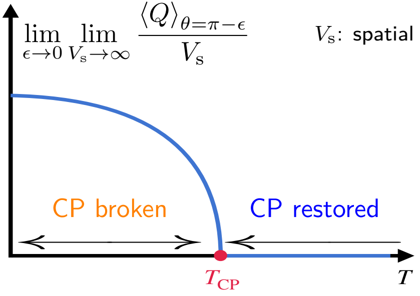
- behavior of $\langle Q \rangle$ near $\theta = \pi$

$$\langle Q \rangle_{\theta=\pi-\epsilon} \overset{\text{CP}}{=} -\langle Q \rangle_{\theta=-(\pi-\epsilon)} \underset{2\pi \text{ periodicity}}{=} -\langle Q \rangle_{\theta=\pi+\epsilon}$$

- order parameter for SSB of CP

$$\lim_{\epsilon \to 0} \lim_{V_s \to \infty} \frac{\langle Q \rangle_{\theta=\pi-\epsilon}}{V_s} \begin{cases} = 0 : \text{CP} \\ \neq 0 : \not{\text{CP}} \end{cases}$$

V_s: spatial volume



relation between $\langle Q \rangle$ and the topological charge distribution

- partition function

$$Z_\theta = \int dU e^{-S_g + i\theta Q} = Z_0 \int dq e^{i\theta q} \rho(q)$$

$\rho(q)$: distribution of the topological charge at $\theta = 0$

$$\rho(q) = \frac{1}{Z_0} \int dU \delta(q - Q) e^{-S_g}$$

- θ dependence of $\langle Q \rangle$

$$\langle Q \rangle = -i \frac{\partial}{\partial \theta} \log Z_\theta = \frac{\int dq q e^{i\theta q} \rho(q)}{\int dq e^{i\theta q} \rho(q)}$$

Behavior of $\langle Q \rangle$ near $\theta = \pi$ is determined by $\rho(q)$.

Imaginary θ as a probe

- CP restoration at $\theta = \pi$ is related to the asymptotic behavior of $\rho(q)$.

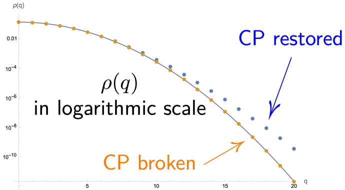
$$\rho(q) \sim \begin{cases} \exp\left(-q \log q\right) & : \text{instanton gas} \quad (\text{CP restored}) \\ \exp\left(-\frac{q^2}{2\chi_0 V}\right) & : \text{large } N \ (\text{low } T) \quad (\text{CP broken}) \end{cases}$$

[E. Witten (1979)]

- $\langle Q \rangle$ at imaginary θ

$$\langle Q \rangle = \frac{\int dq q e^{\tilde{\theta} q} \rho(q)}{\int dq e^{\tilde{\theta} q} \rho(q)} \qquad \tilde{\theta} \equiv i\theta$$

The asymptotic behavior is enhanced by $e^{\tilde{\theta} q}$.



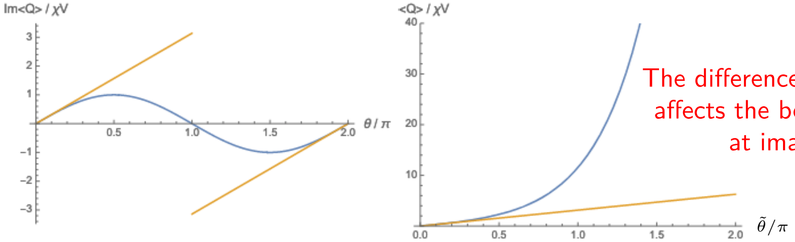
We can observe the tail of $\rho(q)$ through $\langle Q \rangle$ at imaginary θ.

(*c.f.* CP^9 case [V. Azcoiti, G. D. Carlo, A. Galante, V. Laliena (2002)])

behavior of $\langle Q \rangle$

- θ dependence of $\langle Q \rangle$

model	$\langle Q \rangle / (\chi_0 V)$ at θ	$\langle Q \rangle / (\chi_0 V)$ at $\tilde{\theta} \equiv i\theta$	CP at $\theta = \pi$
instanton gas	$i \sin\theta$	$\sinh\tilde{\theta}$	restored
large N (low T)	$i\theta$	$\tilde{\theta}$	broken



The difference of $\rho(q)$ strongly affects the behavior of <Q> at imaginary θ.

We determine $T_{\rm CP}$ through simulations of the theory at imaginary θ.

lattice setup

- gauge action: Wilson action
- topological charge: clover leaf definition + stout smearing

[P. Di Vecchia, K. Fabricius, G. C. Rossi, G. Veneziano (1981)]

[C. Morningstar, M. Peardon (2004)]

$$S = S_{\mathrm{g}}(U) + S_\theta[\tilde{U}] \qquad \tilde{U} \ : \text{smeared link}$$

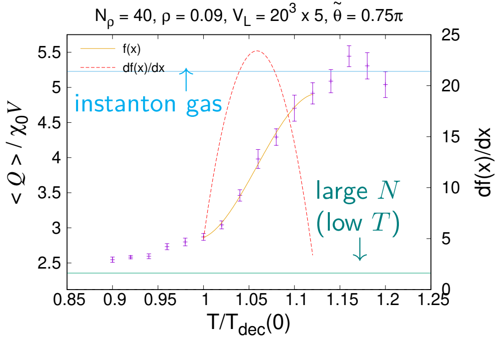
$$\text{stout smearing step size} : \rho = 0.09$$

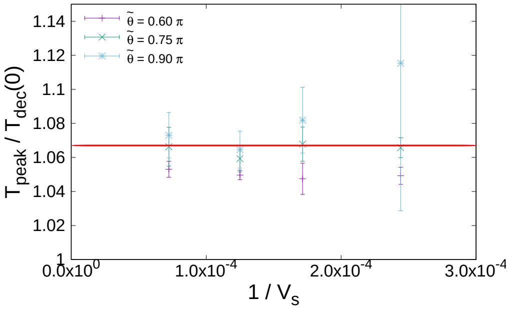
$$\text{stout smearing steps} : N_\rho = 40$$

- algorithm: HMC

[Matsumoto, Hatakeyama, Hirasawa, Honda, Nishimura, Yosprakob (PoS LATTICE2022 378)]

- We determined $T_{\rm cp}$ focusing on T dependence of $\langle Q \rangle / (\chi_0 V)$ for fixed $\tilde{\theta}$.





- fitting by

$$f(x) = a + bx + cx^2 + dx^3$$

$V_{\rm s} \equiv L_{\rm s}^3$: spatial volume

$$T_{\rm CP} \sim 1.06\ T_{\rm dec}(0)$$

$T_{\rm CP}$ is identified as the peak position of $f'(x)$.

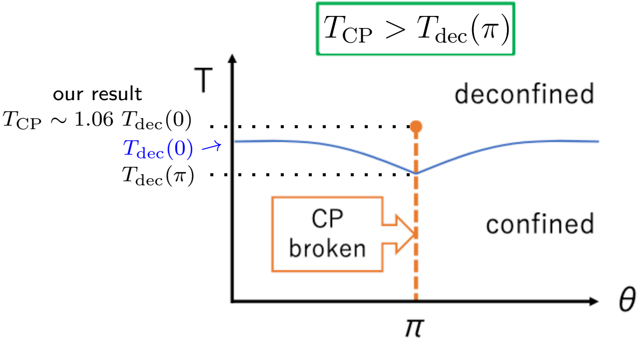
[Matsumoto, Hatakeyama, Hirasawa, Honda, Nishimura, Yosprakob (PoS LATTICE2022 378)]

$$\frac{T_{\rm dec}(\theta)}{T_{\rm dec}(0)} \simeq 1 - R_2\,\theta^2$$

$R_2 \sim 0.0178$ in SU(3) YM

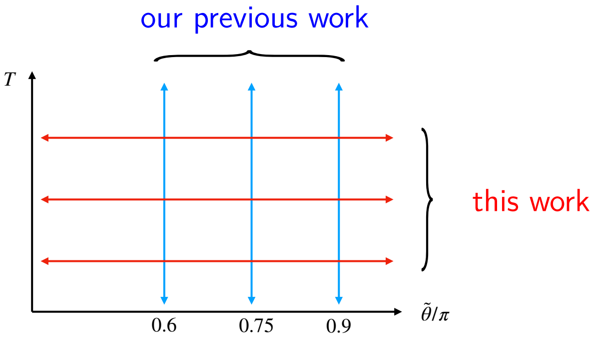
[M. D Elia, F. Negro (2013)]
[N. Otake, N. Yamada (2022)]

$\to T_{\rm dec}(0) > T_{\rm dec}(\pi)$ in SU(3) YM.



Assuming that this is also the case with SU(2) YM, our result $T_{\rm CP} \sim 1.06\, T_{\rm dec}(0)$ implies $T_{\rm CP} > T_{\rm dec}(\pi)$.

New analysis based on analytic continuation using $\tilde{\theta}$ dependence at each T



New analysis based on analytic continuation

- analytic continuation $\tilde{\theta} \to \theta = i\tilde{\theta}$

model	$\langle Q \rangle / V$ at θ	$\langle Q \rangle / V$ at $\tilde{\theta} \equiv i\theta$	CP at $\theta = \pi$
instanton gas	$i\chi_0 \sin\theta$	$\chi_0 \sinh\tilde{\theta}$	restored
large N (low T)	$i\chi_0\theta$	$\chi_0\tilde{\theta}$	broken

- We focus on the $\tilde{\theta}$ dependence of $\langle Q \rangle$
  1. $V_\mathrm{s} \to \infty$ extrapolation for $\langle Q \rangle / V$ at each $\tilde{\theta}$
  2. fit the results for various $\tilde{\theta}$ to
     - hyperbolic sine series: $f(\tilde{\theta}) = \sum_{n=1}^{3} a_n \sinh(n\tilde{\theta})$
     - polynomial: $g(\tilde{\theta}) = b_1\tilde{\theta} + b_3\tilde{\theta}^3$
  3. analytic continuation $\tilde{\theta} \to \theta = i\tilde{\theta}$

c.f.

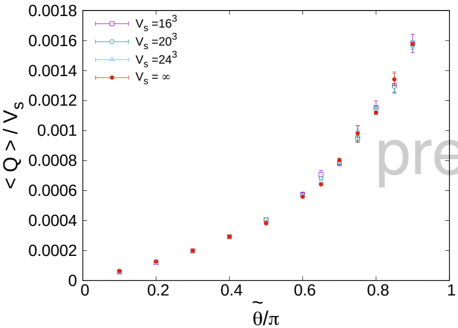
$$\lim_{\epsilon\to 0}\lim_{V_\mathrm{s}\to\infty}\frac{\langle Q \rangle}{V_\mathrm{s}} = \begin{cases} i\chi_0 \sum_n \alpha_n \sin(n\theta) : \mathrm{CP} \\ i\chi_0 \sum_n \beta_{2n-1}\theta^{2n-1} : \not{\mathrm{CP}} \end{cases}$$

at real θ

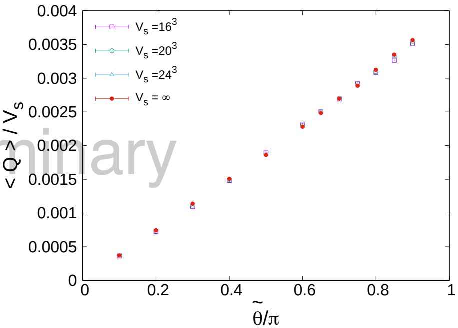
New analysis based on analytic continuation

1. $V_s \to \infty$ extrapolation at each $\tilde{\theta}$

$T = 1.2 T_{\rm dec}(0)$



$T = 0.9 T_{\rm dec}(0)$



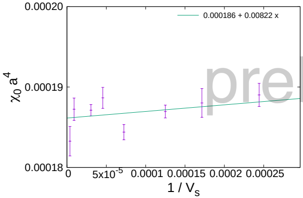
2. fitting the $V_s \to \infty$ extrapolated results to
   - hyperbolic sine series: $f(\tilde{\theta}) = \sum_{n=1}^{3} a_n \sinh(n\tilde{\theta})$
   - polynomial: $g(\tilde{\theta}) = b_1\tilde{\theta} + b_3\tilde{\theta}^3$

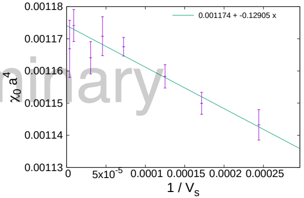
   We can impose a constraint on the fitting parameters using $\left.\frac{\partial\langle Q\rangle}{\partial\tilde{\theta}}\right|_{\tilde{\theta}=0} = \chi_0$.

   - $a_1 + 2a_2 + 3a_3 = \chi_0$
   - $b_1 = \chi_0$

- the infinite volume extrapolation for χ_0

$T = 1.2T_{\rm dec}(0)$ $\qquad$ $T = 0.9T_{\rm dec}(0)$





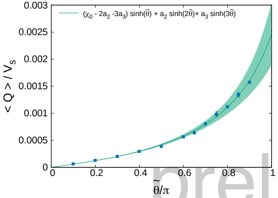
analysis using analytic continuation

2. fitting the $V_s \to \infty$ extrapolated results

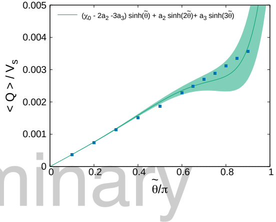
hyperbolic sine series

$$f(\tilde{\theta}) = \sum_{n=1}^{3} a_n \sinh(n\tilde{\theta})$$

$T = 1.2T_{\rm dec}(0)$



$T = 0.9T_{\rm dec}(0)$



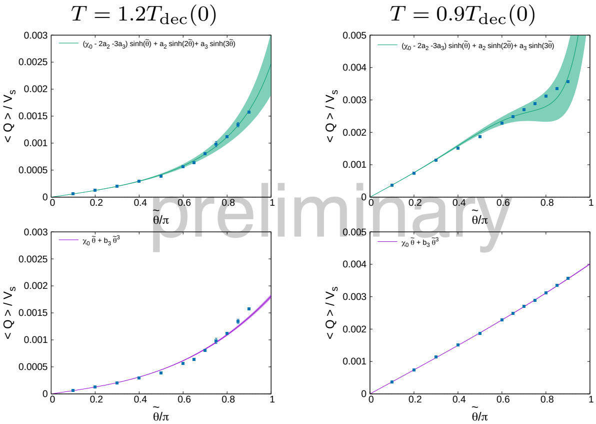
2. fitting the $V_s \to \infty$ extrapolated results

hyperbolic sine series

$$f(\tilde{\theta}) = \sum_{n=1}^{3} a_n \sinh(n\tilde{\theta})$$

polynomial

$$g(\tilde{\theta}) = b_1\tilde{\theta} + b_3\tilde{\theta}^3$$



analysis using analytic continuation

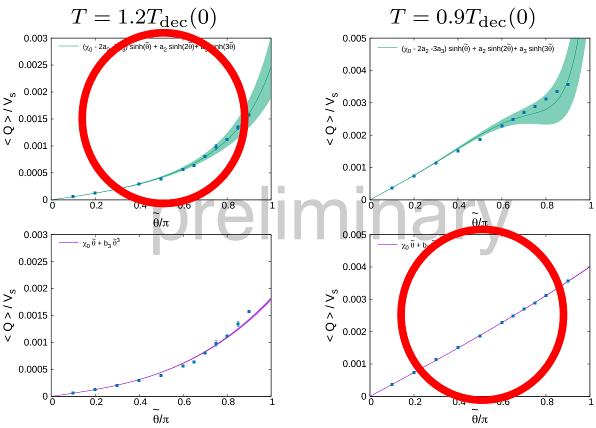
2. fitting the $V_s \to \infty$ extrapolated results

hyperbolic sine series

$$f(\tilde{\theta}) = \sum_{n=1}^{3} a_n \sinh(n\tilde{\theta})$$

polynomial

$$g(\tilde{\theta}) = b_1\tilde{\theta} + b_3\tilde{\theta}^3$$



analysis using analytic continuation

3. analytic continuation $\tilde{\theta} \to \theta = i\tilde{\theta}$

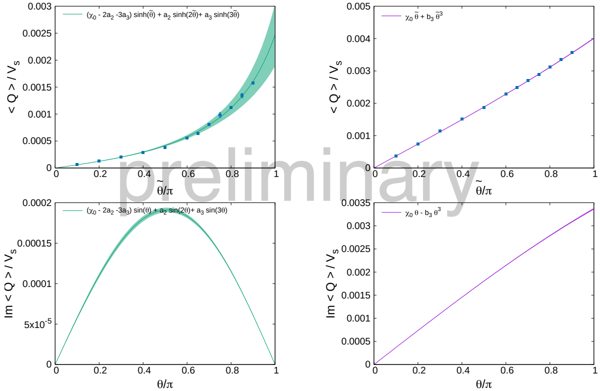
$T = 1.2T_{\text{dec}}(0)$ $\quad$ $T = 0.9T_{\text{dec}}(0)$

$$\lim_{V_s\to\infty} \langle Q \rangle / V_s$$

against $\tilde{\theta}$

$$\lim_{V_s\to\infty} \mathrm{Im} \langle Q \rangle / V_s$$

against θ



no gap at $\theta = \pi$ $\to$ CP restored

gap at $\theta = \pi$ $\to$ CP broken

We are trying to use this analysis to determine T_{CP}.

summary

- We studied the phase structure for 4D SU(2) YM at $\theta = \pi$.
- CP restoration/breaking at $\theta = \pi$ is related to the tail of the topological charge distribution at $\theta = 0$.

$\rightarrow$ We used imaginary θ to probe for the tail of the distribution.

(*c.f.* [V. Azcoiti, G. D. Carlo, A. Galante, V. Laliena (2002)])

- our previous work

  We focused on the T dependence of $\langle Q \rangle / (\chi_0 V)$

  $\rightarrow T_{\mathrm{CP}} \sim 1.06 T_{\mathrm{dec}}(\theta = 0)$

  Our results suggested $T_{\mathrm{CP}} > T_{\mathrm{dec}}(\theta = \pi)$ for SU(2)

  unlike large-N result ($T_{\mathrm{CP}} = T_{\mathrm{dec}}(\theta = \pi)$).

[Matsumoto, Hatakeyama, Hirasawa, Honda, Nishimura, Yosprakob (PoS LATTICE2022 378)]

summary

- this work: analytic continuation focusing on $\tilde{\theta}$ dependence of $\langle Q \rangle$

temperature	appropriate fitting function	$\langle Q \rangle$ at $\theta = \pi$
$T = 1.2 T_{\rm dec}(0)$	$f(\tilde{\theta}) = \sum_{n=1}^{3} a_n \sinh(n\tilde{\theta})$	$\langle Q \rangle = 0$ (CP restored)
$T = 0.9 T_{\rm dec}(0)$	$g(\tilde{\theta}) = b_1 \tilde{\theta} + b_3 \tilde{\theta}^3$	$\langle Q \rangle \neq 0$ (CP broken)

This result is consistent with the results of our previous study.

- We are now studying the intermediate T region.
  (· hyperbolic sine fitting works at $T \geq T_{\rm CP}$
  · polynomial fitting works at $T \leq T_{\rm CP}$) ?

- A similar study for the SU(3) YM is ongoing.
  ($T_{\rm CP} = T_{\rm dec}(\pi) < T_{\rm dec}(0)$ is expected like large-N.)

Thank you for listening!

predicted phase structure for 4D SU(N) at $\theta = \pi$

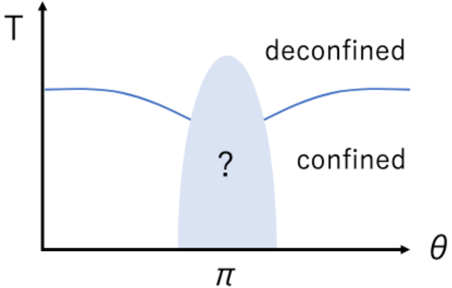
't Hooft anomaly matching condition

At least either CP or Z_2 is broken.

[D. Gaiotto, A.Kapustin, Z. Komargodski, N.Seiberg (2017)]

possible scenario

- SSB of CP
- SSB of Z_2
- gapless (CFT)
- topological QFT



- large N : Both CP restoration and deconfining phase transition occur at the same temperature. [F. Bigazzi, A. L. Cotrone, R. Sisca (2015)]
- small N (*i.e.* N=2): not determined.

predicted phase structure for 4D SU(2) YM at $\theta = \pi$

- confinement/deconfinement in SU(2) YM at $\theta = \pi$
  - deconfinement at high temperature (1-loop analysis)

    [D.J. Gross, R.D. Pisarski, L.G. Yaffe (1981)], [N. Weiss (1981)]
  - confinement and CP breaking at $T = 0$ (non-perturbative)

    [R. Kitano, R. Matsudo, N. Yamada, M. Yamazaki (2021)]

- constraint from the anomaly matching condition

  "mixed 't Hooft anomaly between CP symmetry and center symmetry"
  - CP restoration does not occur in the confined phase.

    $$T_{\mathrm{CP}} \geq T_{\mathrm{dec}(\theta=\pi)}$$

    [D. Gaiotto, A.Kapustin, Z. Komargodski, N.Seiberg (2017)]

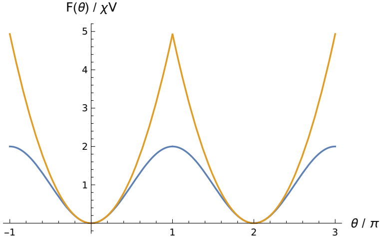
    - large N : $T_{\mathrm{CP}} = T_{\mathrm{dec}(\theta=\pi)}$. [F. Bigazzi, A. L. Cotrone, R. Sisca (2015)]
    - small N (*i.e.* $N = 2$): ?

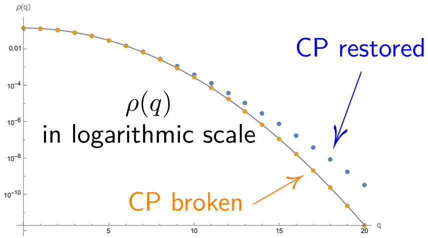
Simplified models for CP restoration at $\theta = \pi$

- We consider two different type of models. $\langle Q \rangle := -i \frac{\partial}{\partial\theta} \log Z$

model	free energy $F(\theta) \equiv -\log Z_\theta$	CP at $\theta = \pi$
instanton gas	$\chi_0 V(1 - \cos\theta)$	restored $\left(\lim_{\epsilon\to 0} F'(\pi+\epsilon) = \lim_{\epsilon\to 0} F'(\pi-\epsilon)\right)$
large N (low T)	$\frac{1}{2}\chi_0 V \min_n (\theta - 2\pi n)^2$	broken $\left(\lim_{\epsilon\to 0} F'(\pi+\epsilon) \neq \lim_{\epsilon\to 0} F'(\pi-\epsilon)\right)$

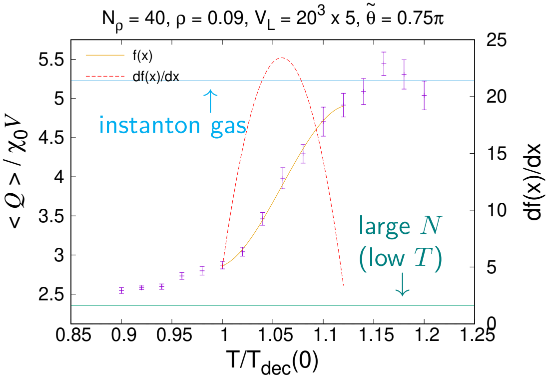
[E. Witten (1979)]





numerical results (T dependence of $\langle Q \rangle / (\chi_0 V)$)

- T dependence of $\langle Q \rangle / (\chi_0 V)$ at $\theta/\pi = 0.75i$



- fitting by

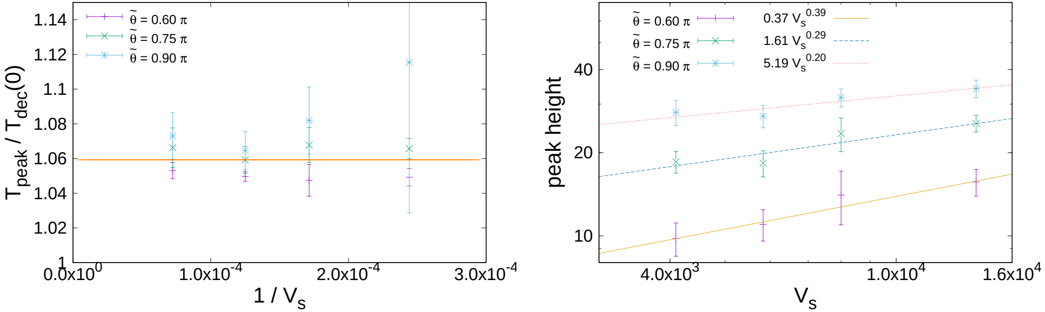
$$f(x) = a + bx + cx^2 + dx^3$$

$T_{\rm CP}$ is identified as a peak position of the derivative $f'(x)$.

- $f'(x)$ has a peak at $T \sim 1.06 T_{\rm dec}(0)$.

numerical results (volume dependence)

- results for various $\tilde{\theta}$ and volume dependence



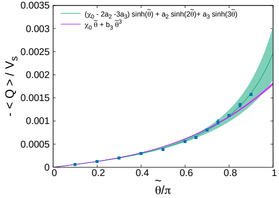
- The transition temperature seems to converge around $1.06 T_{\rm dec}(0)$.
- The order of the transition is the second or higher.

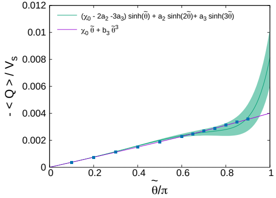
  (*c.f.* 1st order case : The peak increases linearly in $V_{\rm s}$.)

detail of the fitting

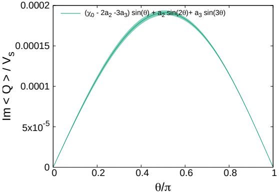
$T = 1.2T_{\rm dec}(0)$ | $T = 0.9T_{\rm dec}(0)$

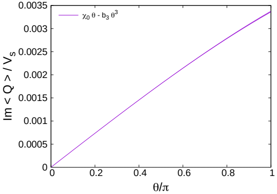
$\langle Q \rangle / V_s$ against $\tilde{\theta}$





$\mathrm{Im}\langle Q \rangle / V_s$ against θ





parameters

$T = 1.2T_{\rm dec}(0)$	$T = 0.9T_{\rm dec}(0)$
$a_2 = -0.0000027$	$a_2 = -0.0001314$
$a_3 = 0.0000002$	$a_3 = 0.0000043$
$b_3 = 0.0000394$	$b_3 = 0.0000103$