Tensor renormalization group study of 3D principal chiral model

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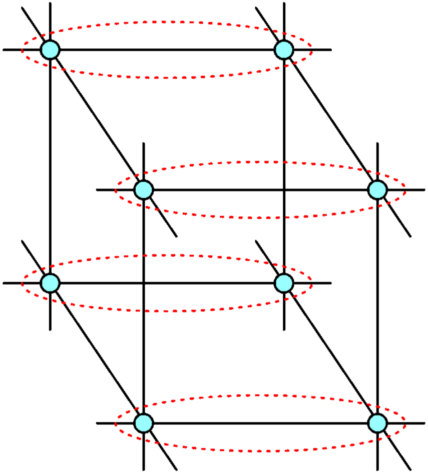
Tensor Networks

Some advantages

Tensor network methods are

- efficient for large systems with translational symmetries

  $\rightarrow$ CPU time $\sim \log V$



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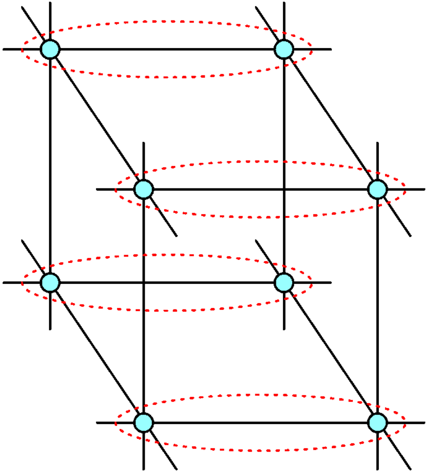

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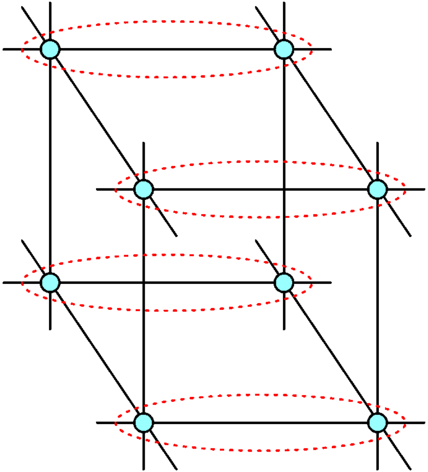
Seems ideal for

- large correlation lengths



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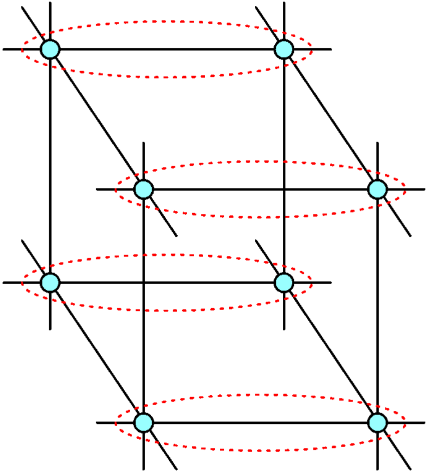

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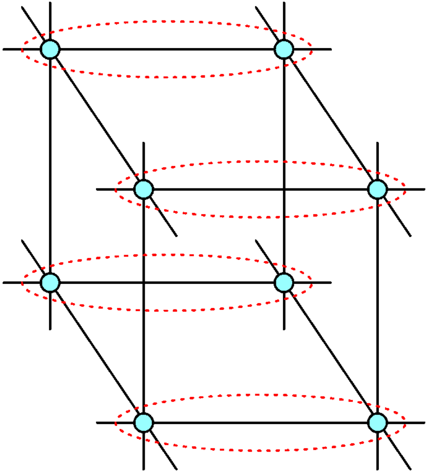

Tensor Networks

Trouble

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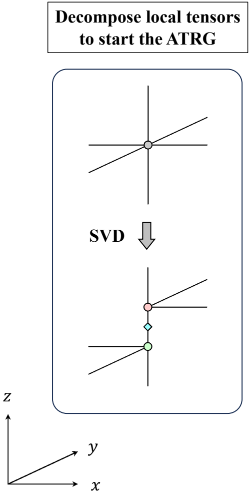
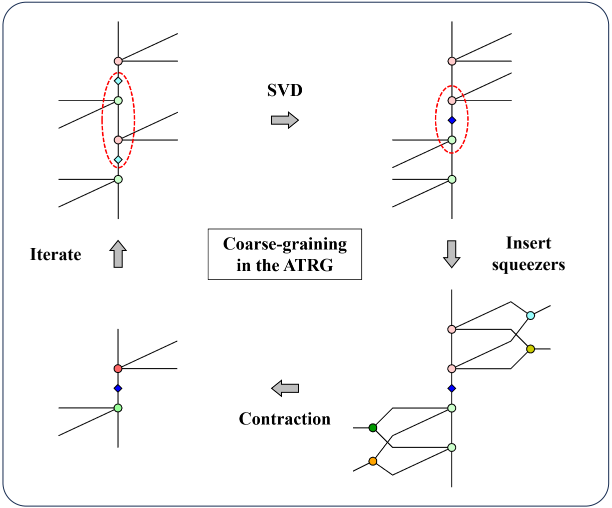
- Some slow-down in $D > 2$
- Some current approaches
  - Graph independent local truncation (GILT)[Hauru et al., 2018]
  - **Higher order tensor renormalization group** (HOTRG and cousins) [Xie et al., 2012]
  - Projected entangled pair states (PEPS)[Verstraete and Cirac, 2004]
  - Tree tensor networks [Shi et al., 2006, Gerster et al., 2014]
  - **Triad tensor renormalization group** [Kadoh and Nakayama, 2019]
  - **Anisotropic tensor renormalization group (ATRG)**[Adachi et al., 2020]



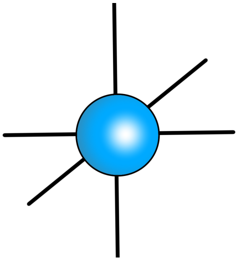
Decompose local tensors to start the ATRG
SVD
z
y
x



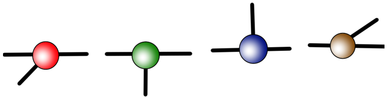
SVD
Iterate
Coarse-graining in the ATRG
Insert squeezers
Contraction

Triad TRG

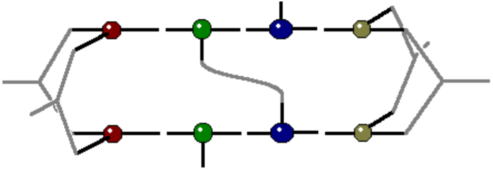
Briefly



3D tensor at a site



Same tensor split into four using SVD



Update using HOTRG method

Tensor Networks

Compare emerging methods

- Efficacy in higher dimensions

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- Generality

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- Checkable

Tensor Networks

Compare emerging methods

- Efficacy in higher dimensions
- Generality
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- Physically interesting

$\Longrightarrow$ $SU(2)$ principal chiral model

$\rightarrow$ Equivalent $O(4)$ NLSM

$\rightarrow$ Effective theory large T QCD? [Susskind, 1979, Pisarski and Wilczek, 1984, Toussaint, 1997, Bernard et al., 2000]

$\rightarrow$ Effective theory large T, g $SU(2)$ gauge?

$SU(2)$ principal chiral model

$$S = \frac{\beta}{2} \int d^3x \mathrm{Tr} \left[\sum_{\nu=1}^{3} \partial_\nu U(x)^\dagger \partial_\nu U(x) \right] \tag{1}$$

$U(x)$ are $SU(2)$ matrices

$$S = -\frac{\beta}{2} \sum_{n,\nu} \Re \left\{ \mathrm{Tr} \left[U(n) U(n+\hat{\nu})^\dagger \right] \right\} \tag{2}$$

n are the sites of the lattice

$$Z = \int \left(\prod_n dU(n) \right) e^{-S} \tag{3}$$

Local tensor from strong coupling

Expand the nearest-neighbor weight:

$$e^{-\frac{\beta}{2}\Re\mathrm{Tr}[U(x)U(x+\hat{\nu})]} = \sum_{r=0}^{\infty} F_r(\beta)\chi^r(U(x)U(x+\hat{\nu}))$$

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$$T_{(r_1 m_1 n_1)(r_2 m_2 n_2)(r_3 m_3 n_3)(r_4 m_4 n_4)(r_5 m_5 n_5)(r_6 m_6 n_6)} = \sqrt{\prod_{p=1}^{6} F_{r_p}(\beta)}$$

$$\times \sum_{R_{12}=|r_1-r_2|}^{r_1+r_2} \sum_{R_{123}=|R_{12}-r_3|}^{R_{12}+r_3} \sum_{R_{56}=|r_5-r_6|}^{r_5+r_6} \sum_{M_{12},N_{12}} \sum_{M_{123},N_{123}} \sum_{M_{56},N_{56}}$$

$$\times C^{R_{12} M_{12}}_{r_1 m_1 r_2 m_2} C^{R_{12} N_{12}}_{r_1 n_1 r_2 n_2} C^{R_{123} M_{123}}_{R_{12} M_{12} r_3 m_3} C^{R_{123} N_{123}}_{R_{12} N_{12} r_3 n_3} C^{R_{123} M_{123}}_{r_4 m_4 R_{56} M_{56}} C^{R_{123} N_{123}}_{r_4 n_4 R_{56} N_{56}} C^{R_{56} M_{56}}_{r_5 m_5 r_6 m_6} C^{R_{56} N_{56}}_{r_5 n_5 r_6 n_6}$$

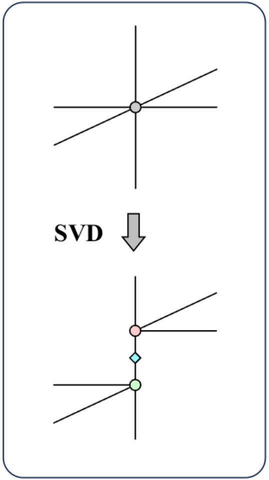
$$\times \frac{1}{2R_{123}+1}$$

Tensor formulations

ATRG

$$T_{(r_1 m_1 n_1)(r_2 m_2 n_2)(r_3 m_3 n_3)(r_4 m_4 n_4)(r_5 m_5 n_5)(r_6 m_6 n_6)}$$
$$\approx \sum_{\gamma} U_{(r_1 m_1 n_1)(r_2 m_2 n_2)(r_3 m_3 n_3)\gamma} \sigma_\gamma V^*_{(r_4 m_4 n_4)(r_5 m_5 n_5)(r_6 m_6 n_6)\gamma}$$

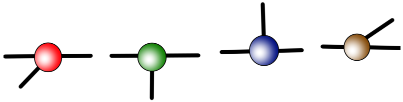
- SVD of initial tensor
- initial truncation



Tensor formulations

Triads

$$T_{ijklmn} = \sum_{a,b,c} A_{ika} B_{amb} C_{bnc} D_{clj}$$



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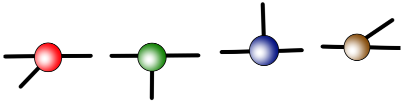


$$A_{(r_y,m_2,n_3),(r_x,m_1,n_3),(R,M,N)} = \sqrt{F_{r_x}(\beta) F_{r_y}(\beta)}\, C^{RM}_{r_x m_1 r_y m_3} C^{RN}_{r_x n_1 r_y n_3}$$

$$B_{(R,M,N),(r_z,m_5,n_5),(R',M',N')} = \frac{1}{\sqrt{d_{R'}}} \sqrt{F_{r_z}(\beta)}\, C^{R'M'}_{R,M,r_z m_5} C^{R'N'}_{R,N,r_z n_5}$$

$$C_{(R',M',N'),(r_{-z},m_6,n_6),(R'',M'',N'')} = \frac{1}{\sqrt{d_{R'}}} \sqrt{F_{r_{-z}}(\beta)}\, C^{R'M'}_{R'',M'',r_{-z} m_6} C^{R'N'}_{R'',N'',r_{-z} n_6}$$

$$D_{(R'',M'',N''),(r_{-x},m_2,n_2),(r_{-y},m_4,n_4)} = \sqrt{F_{r_{-x}}(\beta) F_{r_{-y}}(\beta)}\, C^{R''M''}_{r_{-x} m_2 r_{-y} m_4} C^{R''N''}_{r_{-x} n_2 r_{-y} n_4}$$

Observables

The free energy density:

$$F \equiv \frac{1}{V} \log(Z)$$

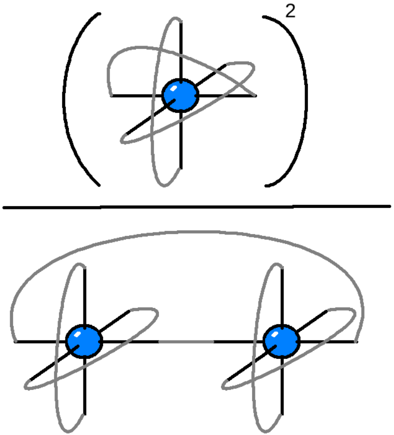
The average action density:

$$\langle s \rangle \equiv -\frac{\partial}{\partial \beta} F$$

Action susceptibility

$$\chi_s = V(\langle s^2 \rangle - \langle s \rangle^2) \equiv \frac{\partial^2}{\partial \beta^2} F$$

$X \equiv$



[Gu and Wen, 2009]

Results

Average action varying r_{max}

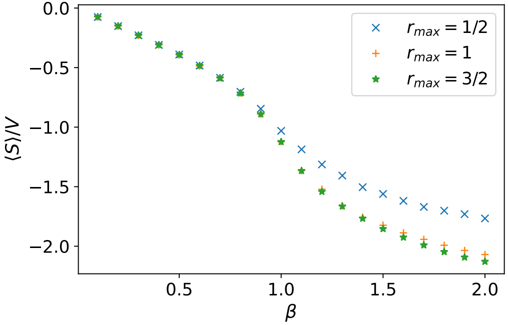


Figure: ATRG, $D = 40,\ V = 1024^3$

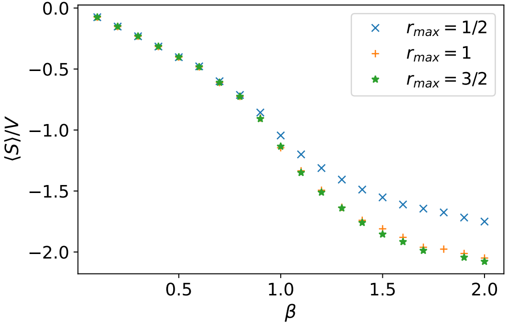


Figure: Triad, $D = 40,\ V = 1024^3$

Results

X varying $r_{\max}$ and β_c

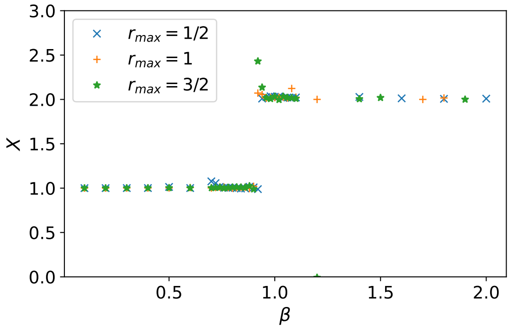


Figure: Triad, $D = 40$, $V = 1024^3$
$r_{\max} = 1/2$: $\beta_c = 0.935(5)$
$r_{\max} = 1$: $\beta_c = 0.915(5)$
$r_{\max} = 3/2$: $\beta_c = 0.915(5)$

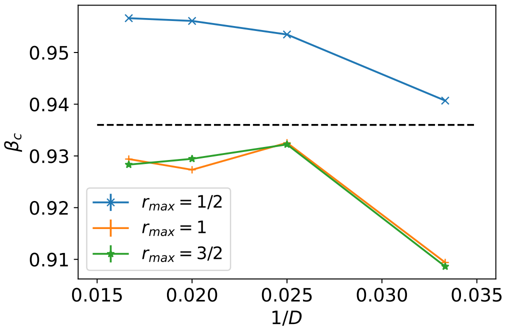


Figure: β_c from ATRG using X, $V = 1024^3$,
$\beta_c = 0.9360(1)$ [López-Contreras et al., 2022]

Results

Convergence of X, high-temperature, $\beta = 0.8$

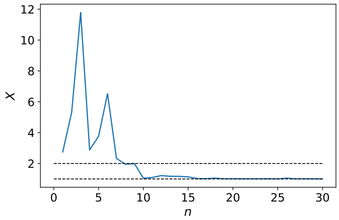


Figure: $r_{max} = 1/2$

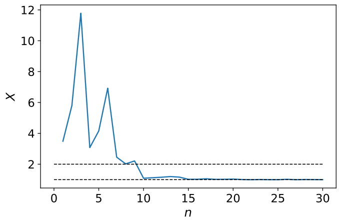


Figure: $r_{max} = 1$

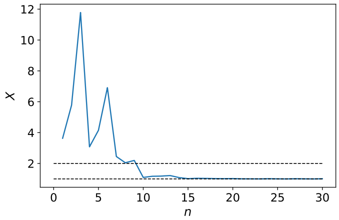


Figure: $r_{max} = 3/2$

Results

Convergence of X, low-temperature, $\beta = 1.1$

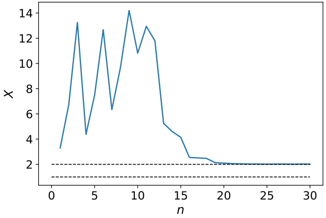


Figure: $r_{max} = 1/2$

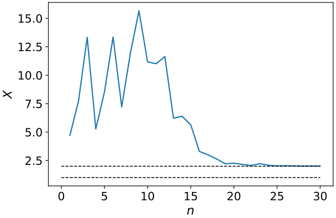


Figure: $r_{max} = 1$

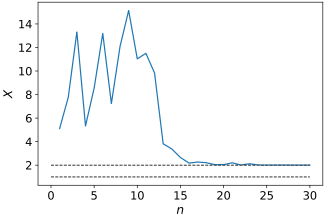


Figure: $r_{max} = 3/2$

Results

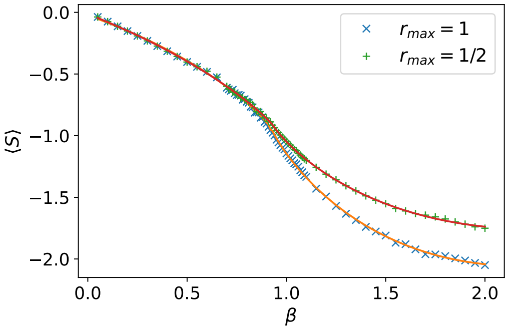


Figure: Triad low-high temp fit, $D = 40$, $V = 1024^3$
$r_{max} = 1/2$: $\beta_c = 0.933(1)$
$r_{max} = 1$: $\beta_c = 0.9232(1)$
$r_{max} = 3/2$: $\beta_c = 0.92295(4)$

$$\langle S \rangle / V = A + B|\beta - \beta_c| + C|\beta - \beta_c|^{1-\alpha}$$

- $\alpha = -0.247(6)$ [Toldin et al., 2003]
- $\beta_c = 0.9360(1)$ [López-Contreras et al., 2022]

Results

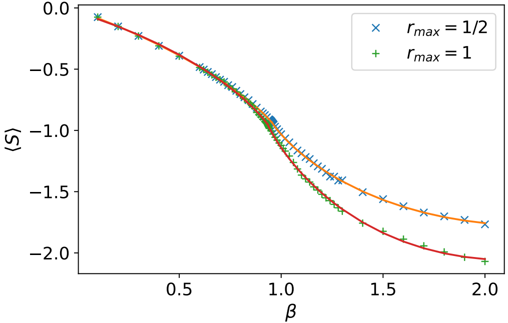


Figure: ATRG low-high temp fit, $D = 40$, $V = 1024^3$

- Fix β_c from X
- Fit for α

$r_{\max} = 1/2$: $\alpha = -0.21259(1)$
$r_{\max} = 1$: $\alpha = -0.2151(1)$
$r_{\max} = 3/2$: $\alpha = -0.22676(4)$

Closing remarks

Conclusions:

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Future work:

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Thank you!

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