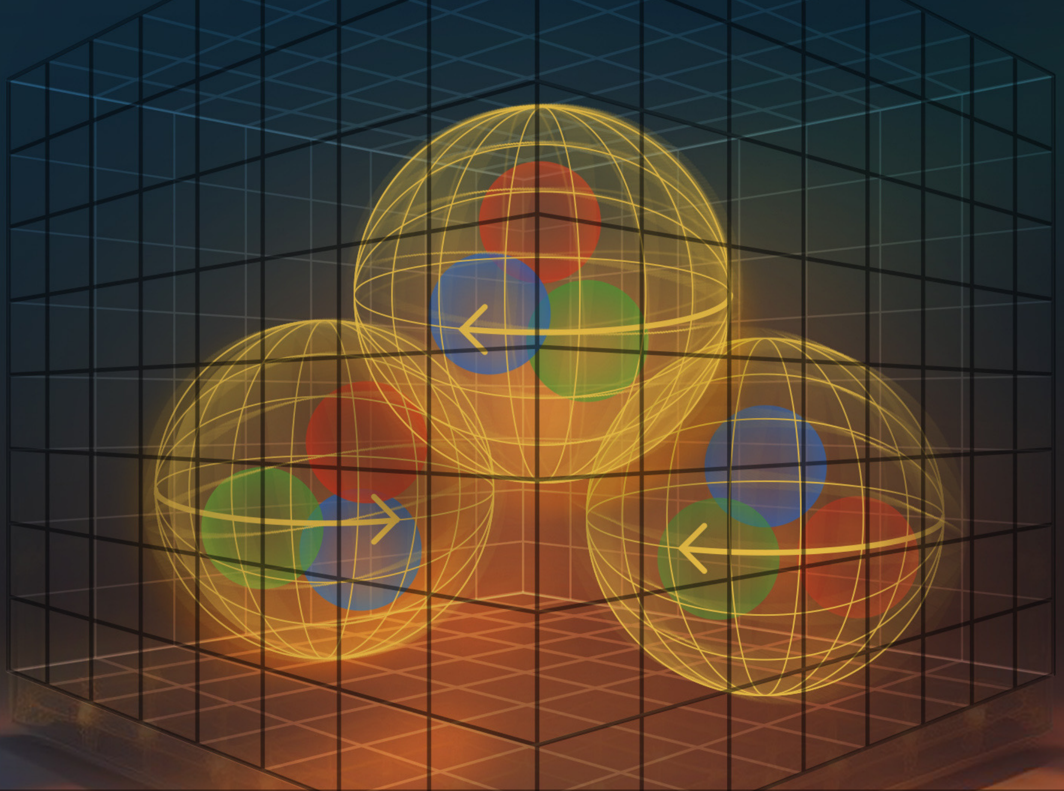
# Three Relativistic Spinning Particles in a Box



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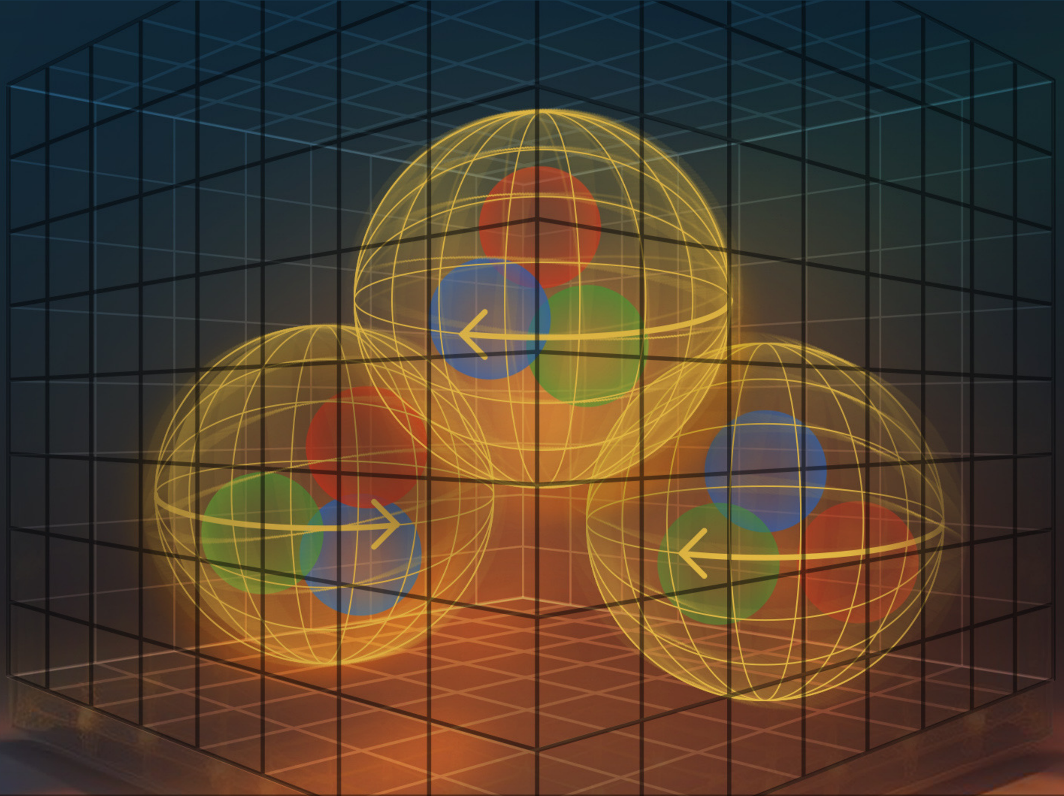


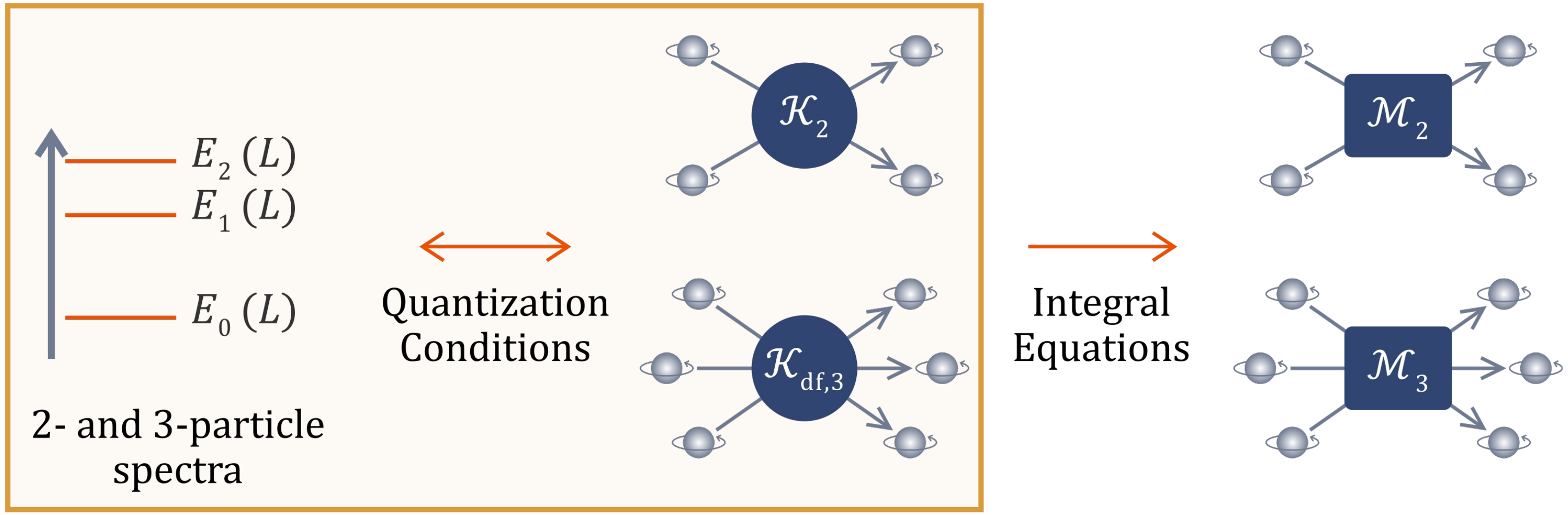
Based on work with Max Hansen, Fernando Romero-López, and Steve Sharpe

arXiv:2303.10219v2 (JHEP)

# Introduction

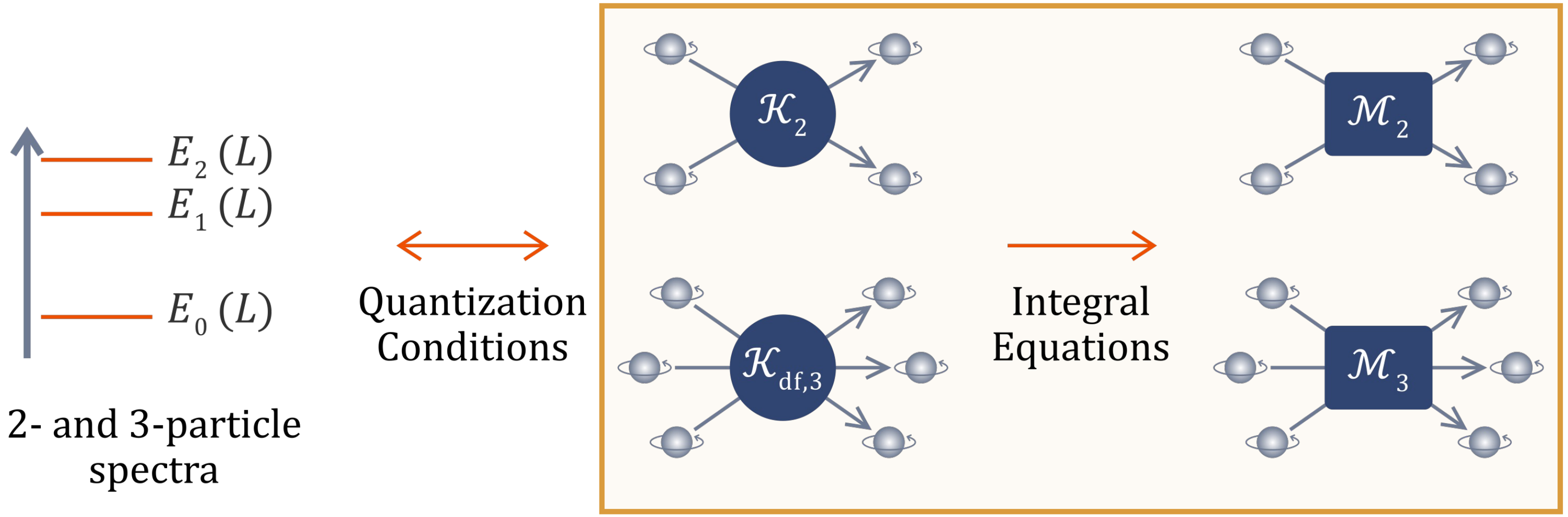
- Why study three relativistic fermions in finite volume?
- Lattice QCD calculations of two-baryon scattering amplitudes are rapidly progressing
    - Includes determination of masses and other properties of light nuclei and hypernuclei with heavy quark masses
- Less progress for the three-nucleon interaction
    - Important for nuclei near neutron driplines
    - For nuclear saturation
    - For determining neutron star equation of state
- Step on the path to studying the Roper: $N(1440) \to \pi N + \pi\pi N$
- A formalism relating finite-volume energies to infinite-volume quantities is needed

# Introduction to Three-Particle Formalisms




- Quantization condition relates finite-volume 2- and 3-particle spectra to 2- and 3-particle K-matrices
  - $\mathcal{K}_{\text{df},3}$ is a real, infinite-volume K matrix, smooth apart from 3-particle resonance poles
  - Parametrize $\mathcal{K}_2$ and $\mathcal{K}_{\text{df},3}$ in effective-range-like expansion about threshold
  - Determine parameters by fitting spectrum

# Introduction to Three-Particle Formalisms




- Integral equations relate two- and three-particle K-matrices to the two- and three-particle scattering amplitudes, $\mathcal{M}_2$ and $\mathcal{M}_3$.
- Formalism exists for arbitrary choice of spinless particles

# Introduction to Three-Particle Formalisms

- Three approaches used in derivation:
  - Relativistic Field Theory (RFT) [M.T. Hansen & S.R. Sharpe 1408.5933]
  - Non-Relativistic Effective Field Theory (NREFT)

    [H.W. Hammer, J.Y. Pang & A. Rusetsky, 1706.07700 & 1707.02176]
  - Finite-volume Unitarity (FVU) [M. Mai & M. Döring, 1709.08222]
- Formally equivalent up to technical details
- We use the RFT approach

# Overview of RFT Approach

- QC3 always has same form, though matrix structure varies

$$\det\left[F_3^{-1}(E,\boldsymbol{P},L) + \mathcal{K}_{\mathrm{df},3}(E^*)\right] = 0$$

- $F_3$ contains three quantities we will revisit

$$F_3 = \frac{F}{3} - F\,\frac{1}{\mathcal{K}_{2,L}^{-1} + F + G}\,F$$

- With the range of validity

$$\sqrt{4m_N^2 - m_\pi^2} + m_N < E_3^* < 3m_N + m_\pi$$

- QC3 derived by determining (power-law) volume dependence of 3-particle correlation function
    - Calculations to all orders in an expansion using a generic relativistic EFT
    - Volume dependence arises (in part) from 3-particle cuts: $F$ and $G$

# Extension to Spin-1/2 Particles

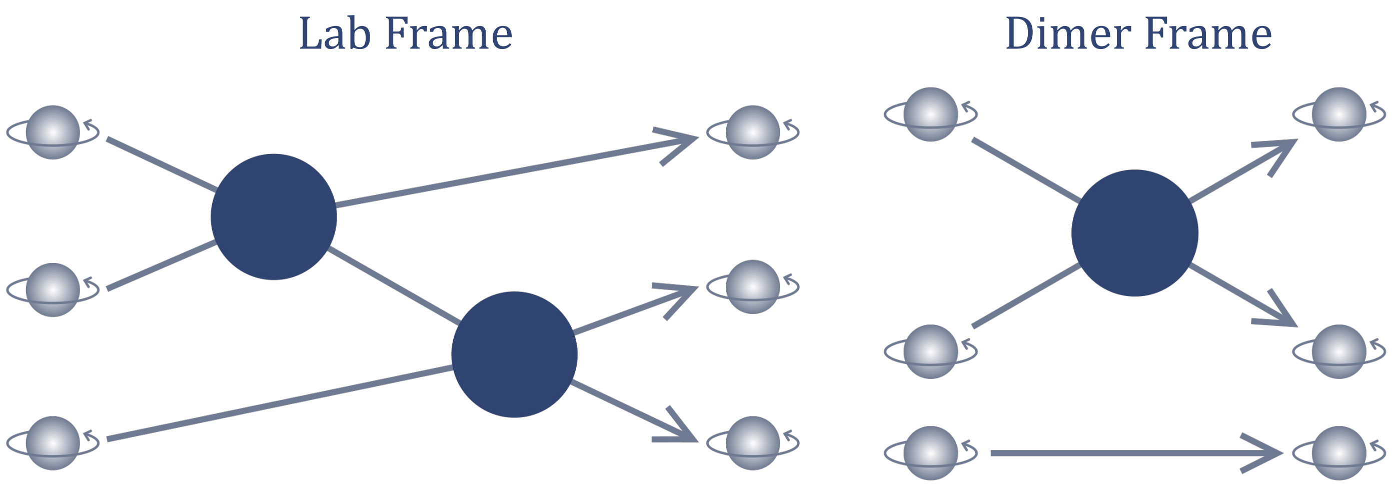
- How does QC3 change when we incorporate spin?
    - Additional matrix indices for spin
    - Antisymmetrization of states as a result of Fermi statistics
    - Wigner rotations resulting from boosts imply total spin is not conserved
        - In the non-relativistic limit, total spin is conserved

# Defining Spin States

- Now we define our spin-1/2 states
  - Start with stationary spin state in center of mass frame, then boost to moving frame
  - $\left|\boldsymbol{p}, s\, m_s\right\rangle = U\left(L\left(\beta_p\right)\right)\left|\mathbf{0}, s\, m_s\right\rangle \equiv |\boldsymbol{p}, m_s(\boldsymbol{p})\rangle$ for spin-1/2
- With our boost defined as follows
  - $L\left(\beta_p\right) = R\left(\theta_p, \hat{n}_p\right) \cdot L\left(\beta_p \hat{z}\right) \cdot R\left(\theta_p, \hat{n}_p\right)^{-1}$
- This object rotates as a 2-component spinor
  - $U(R)\left|\boldsymbol{p}, s\, m_s\right\rangle = \left|R\boldsymbol{p}, s\, m_s{}'\right\rangle \mathcal{D}^{(s)}_{m_s{}', m_s}(R)$
- In the lab frame, we have
  - $|\boldsymbol{k}, m_s(\boldsymbol{k})\rangle \otimes |\boldsymbol{a}, m_s(\boldsymbol{a})\rangle \otimes |\boldsymbol{b}, m_s(\boldsymbol{b})\rangle$

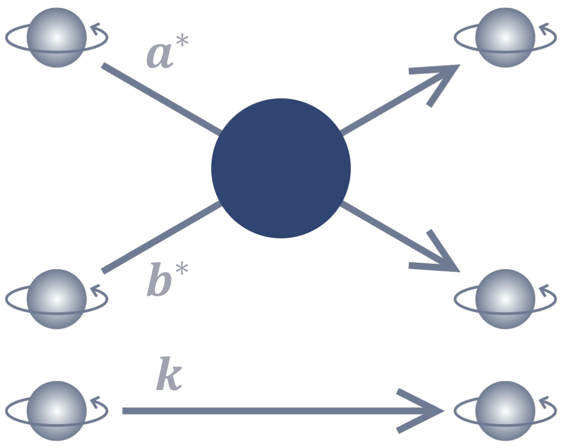
# Picking a Frame

- What basis should we use?
- Two natural bases in the problem
    - A natural choice for defining $G$ — the lab frame in which total spin is conserved
    - A natural choice for defining $\mathcal{K}_2$ — the (two-particle C.O.M.) dimer frame
- Transforming between the two requires Wigner matrices



# The Dimer Frame

- To combine spins with orbital angular momentum of interacting pair, boost to find $\boldsymbol{a}^*$ and $\boldsymbol{b}^*$
  - Start with stationary spin state in center of mass frame, boost to moving frame
  - $|\boldsymbol{a}^*, m_s(\boldsymbol{a}^*)\rangle = U\big(L(\boldsymbol{\beta}_{a^*})\big)\big|\boldsymbol{0}, m_s\big\rangle$ and $|\boldsymbol{b}^*, m_s(\boldsymbol{b}^*)\rangle = U\big(L(\boldsymbol{\beta}_{b^*})\big)\big|\boldsymbol{0}, m_s\big\rangle$
- In the dimer-axis frame, we have
  - $|\boldsymbol{k}, m_s(\boldsymbol{k})\rangle \otimes |\boldsymbol{a}^*, m_s(\boldsymbol{a}^*)\rangle \otimes |\boldsymbol{b}^*, m_s(\boldsymbol{b}^*)\rangle$
  - This is a natural basis for $\mathcal{K}_2$ and for derivation of QC3
- Orbital and spin axes aligned in dimer-frame, not lab-frame
- Wigner rotation relates spin components
  - $|\boldsymbol{a}^*, m_s(\boldsymbol{a})\rangle = |\boldsymbol{a}^*, m_s{}'(\boldsymbol{a}^*)\rangle\, \mathcal{D}(R_a)_{m_s{}', m_s}$

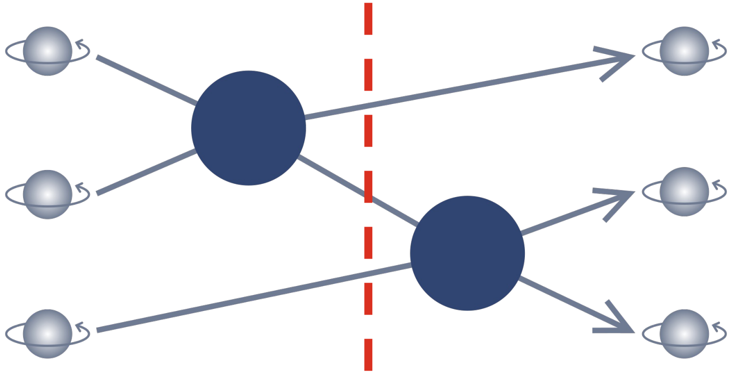


# Incorporating Spin in $G$

$$\det\left[F_3^{-1}(E,\boldsymbol{P},L)+\mathcal{K}_{\mathrm{df},3}(E^*)\right]=0 \quad \text{where} \quad F_3=\frac{F}{3}-F\frac{1}{\mathcal{K}_{2,L}^{-1}+F+G}F$$

- $G$ arises for 3-particle cuts in which the spectator is switched
- Spin components are conserved in the lab frame

$$\Delta_{L,\alpha\beta}(b)=i\frac{(\not{b}+m)_{\alpha\beta}}{b^2-m^2+i\epsilon}+R_{L,\alpha\beta}(b)$$



- In the lab frame, we have

$$[\mathbf{G}^{\mathsf{lab}}]_{p\ell'm'\boldsymbol{m}_s';k\ell m\boldsymbol{m}_s}(E,\boldsymbol{P},L)\equiv-\delta_{m_s'(\boldsymbol{p}),m_s(\boldsymbol{p})}\delta_{m_s'(\boldsymbol{k}),m_s(\boldsymbol{k})}\delta_{m_s'(\boldsymbol{b}),m_s(\boldsymbol{b})}\frac{i}{4\omega_p\omega_kL^6}\frac{H(\boldsymbol{p})H(\boldsymbol{k})}{b^2-m^2}\frac{4\pi\mathcal{Y}_{\ell'm'}(\boldsymbol{k}_p^*)\mathcal{Y}_{\ell m}^*(\boldsymbol{p}_k^*)}{q_{2,p}^{*\ell'}\,q_{2,k}^{*\ell}}$$

- To transform to the dimer-axis frame, Wigner matrices are needed for each pair of spins

$$\mathbf{G}_{p\ell'm'\boldsymbol{m}_s'^*;k\ell m\boldsymbol{m}_s^*}=\mathcal{D}^{(p,k)\dagger}_{\boldsymbol{m}_s'^*\boldsymbol{m}_s''}\,\mathbf{G}^{\mathsf{lab}}_{p\ell'm'\boldsymbol{m}_s'';k\ell m\boldsymbol{m}_s'''}\,\mathcal{D}^{(k,p)}_{\boldsymbol{m}_s'''\boldsymbol{m}_s^*}$$

# Incorporating Spin in $F$

$$\det\left[F_3^{-1}(E,\boldsymbol{P},L)+\mathcal{K}_{\mathrm{df},3}(E^*)\right]=0 \quad \text{where} \quad F_3=\frac{F}{3}-F\frac{1}{\mathcal{K}_{2,L}^{-1}+F+G}F$$

- $F$ arises for 3-particle cuts in which the spectator is fixed

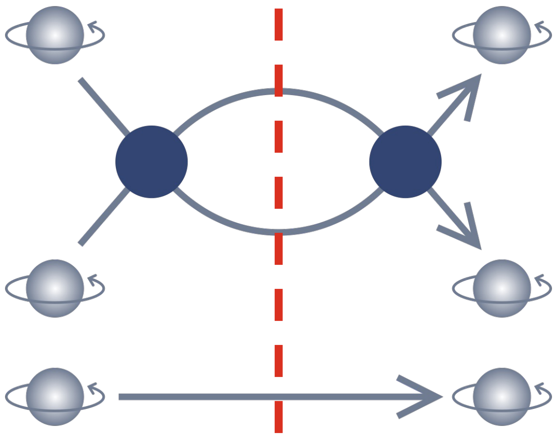
$$[\mathbf{F}^{\mathrm{lab}}]_{k'\ell'm'\boldsymbol{m}'_s;k\ell m\boldsymbol{m}_s}(E,\boldsymbol{P},L)\equiv\delta_{\boldsymbol{m}'_s\boldsymbol{m}_s}\delta_{k'k}\frac{iH(\boldsymbol{k})}{2\omega_k L^3}\frac{1}{2}\left[\frac{1}{L^3}\sum_{\boldsymbol{a}}-\text{p.v.}\int_{\boldsymbol{a}}\right]\frac{4\pi\mathcal{Y}_{\ell'm'}(\boldsymbol{a}_k^*)\mathcal{Y}^*_{\ell m}(\boldsymbol{a}_k^*)}{2\omega_a(b^2-m^2)}\frac{1}{(q^*_{2,k})^{\ell+\ell'}}$$

- Where the Kronecker delta enforces spin conservation in the lab frame

$$\delta_{\boldsymbol{m}'_s\boldsymbol{m}_s}=\delta_{m'_s(\boldsymbol{k})m_s(\boldsymbol{k})}\delta_{m'_s(\boldsymbol{a})m_s(\boldsymbol{a})}\delta_{m'_s(\boldsymbol{b})m_s(\boldsymbol{b})}$$

- Wigner D-matrices cancel when transforming to dimer-axis frame

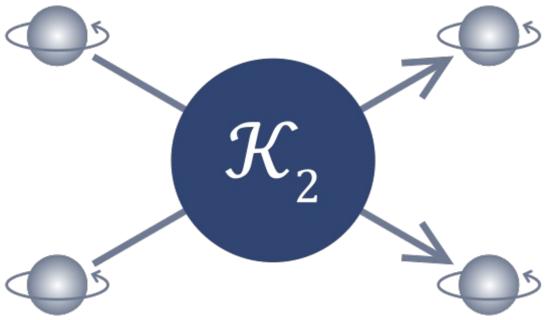
$$\mathbf{F}=\mathbf{F}^{\mathrm{lab}}$$



# Incorporating Spin in $\mathcal{K}_2$

$$\det\left[F_3^{-1}(E,\boldsymbol{P},L)+\mathcal{K}_{\mathrm{df},3}(E^*)\right]=0 \quad \text{where} \quad F_3=\frac{F}{3}-F\frac{1}{\mathcal{K}_{2,L}^{-1}+F+G}F$$

- $\mathcal{K}_2$ incorporates 2-particle interactions
    - Natural to express in the dimer-axis
- Spin components are conserved in the lab frame

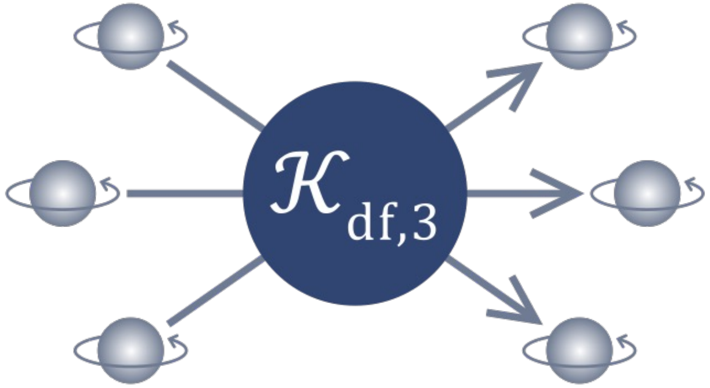



$$[\mathbf{K}_2]_{k'\ell'm'\boldsymbol{m}_s'^*;k\ell m\boldsymbol{m}_s^*}(E,\boldsymbol{P})=i\delta_{k'k}2\omega_k L^3\mathcal{K}_2^{(\ell'm'\boldsymbol{m}_s'^*,\ell m\boldsymbol{m}_s^*)}(E_{2,k}^*)$$

$$\mathcal{K}_2^{(\ell'm'\boldsymbol{m}_s'^*,\ell m\boldsymbol{m}_s^*)}(E_{2,k}^*)=\delta_{m_s'(\boldsymbol{k})m_s(\boldsymbol{k})}\ \mathcal{K}_2^{[\ell'm'm_s'(\boldsymbol{a}'^*)m_s'(\boldsymbol{b}'^*)],\,[\ell m m_s(\boldsymbol{a}^*)m_s(\boldsymbol{b}^*)]}(E_{2,k}^*)$$

# Threshold Expansion for $\mathcal{K}_{\text{df},3}$

- Have collected all the quantities entering $F_3 = \frac{F}{3} - F \frac{1}{\mathcal{K}_{2,L}^{-1} + F + G} F$
- Final term appearing in quantization condition, $\det\left[F_3^{-1}(E, \boldsymbol{P}, L) + \mathcal{K}_{\text{df},3}(E^*)\right] = 0$, is $\mathcal{K}_{\text{df},3}$
- To implement quantization condition, need a parameterization of $\mathcal{K}_{\text{df},3}$
    - Analogous to effective range expansion for $\mathcal{K}_2$



# Threshold Expansion for $\mathcal{K}_{\text{df},3}$

- Start with nucleon field operator $\mathcal{N}$
- Write down all operators of form $(\mathcal{N}\overline{\mathcal{N}})^3$ with any gamma matrix structure and derivatives
    - Lorentz and parity invariant, requiring even numbers of derivatives
- Expand each in powers of 3-momentum using non-relativistic Dirac spinor expansion
- Results in two independent terms up to order $\boldsymbol{p}^2$

$$\mathcal{K}_A = \overline{\mathcal{A}}\left[(\chi^\dagger_{k'}\,\boldsymbol{\sigma}\cdot\boldsymbol{k'}\,\boldsymbol{\sigma}\cdot\boldsymbol{k}\,\chi_k)(\chi^\dagger_{a'}\chi_a)(\chi^\dagger_{b'}\chi_b)\right] \qquad \mathcal{K}_B = \overline{\mathcal{A}}\left[\boldsymbol{k'}\cdot\boldsymbol{k}(\chi^\dagger_{k'}\chi_k)(\chi^\dagger_{a'}\chi_a)(\chi^\dagger_{b'}\chi_b)\right]$$

# Conclusion and Future Steps

- Including spin in the three-particle formalism requires
  - Additional matrix structures
  - Antisymmetrization and changed signs
  - Momentum-dependent Wigner rotations
- The first step toward implementation of QC3 has begun for toy model interactions
- In future the formalism should be generalized to incorporate
  - 3 nucleons at arbitrary isospin
  - $N\pi\pi$ at maximal isospin
  - $N\pi\pi + N\pi$ to understand the Roper

# Thank You

## Questions?