Three Relativistic Spinning Particles in a Box
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Based on work with Max Hansen, Fernando Romero-López, and Steve Sharpe
arXiv:2303.10219v2 (JHEP)
Introduction

- Why study three relativistic fermions in finite volume?
- Lattice QCD calculations of two-baryon scattering amplitudes are rapidly progressing
  - Includes determination of masses and other properties of light nuclei and hypernuclei with heavy quark masses
- Less progress for the three-nucleon interaction
  - Important for nuclei near neutron driplines
  - For nuclear saturation
  - For determining neutron star equation of state
- Step on the path to studying the Roper: $N(1440) \to \pi N + \pi\pi N$
- A formalism relating finite-volume energies to infinite-volume quantities is needed
Introduction to Three-Particle Formalisms

- Quantization condition relates finite-volume 2- and 3-particle spectra to 2- and 3-particle $K$-matrices
  - $K_{df,3}$ is a real, infinite-volume $K$ matrix, smooth apart from 3-particle resonance poles
  - Parametrize $K_2$ and $K_{df,3}$ in effective-range-like expansion about threshold
  - Determine parameters by fitting spectrum
Introduction to Three-Particle Formalisms

- Integral equations relate two- and three-particle K-matrices to the two- and three-particle scattering amplitudes, $\mathcal{M}_2$ and $\mathcal{M}_3$.
- Formalism exists for arbitrary choice of spinless particles.
Introduction to Three-Particle Formalisms

• Three approaches used in derivation:
  • Relativistic Field Theory (RFT) [M.T. Hansen & S.R. Sharpe 1408.5933]
  • Non-Relativistic Effective Field Theory (NREFT)
    [H.W. Hammer, J.Y. Pang & A. Rusetsky, 1706.07700 & 1707.02176]
  • Finite-volume Unitarity (FVU) [M. Mai & M. Döring, 1709.08222]

• Formally equivalent up to technical details
• We use the RFT approach
Overview of RFT Approach

- QC3 always has same form, though matrix structure varies
  \[ \det[F_3^{-1}(E, P, L) + \mathcal{K}_{df3}(E^*)] = 0 \]

- \( F_3 \) contains three quantities we will revisit
  \[ F_3 = \frac{F}{3} - F \frac{1}{\mathcal{K}_{2L}^{-1} + F + G} \]

- With the range of validity
  \[ \sqrt{4m_N^2 - m_\pi^2} + m_N < E_3^* < 3m_N + m_\pi \]

- QC3 derived by determining (power-law) volume dependence of 3-particle correlation function
  - Calculations to all orders in an expansion using a generic relativistic EFT
  - Volume dependence arises (in part) from 3-particle cuts: \( F \) and \( G \)
Extension to Spin-1/2 Particles

- How does QC3 change when we incorporate spin?
  - Additional matrix indices for spin
  - Antisymmetrization of states as a result of Fermi statistics
  - Wigner rotations resulting from boosts imply total spin is not conserved
    - In the non-relativistic limit, total spin is conserved
Defining Spin States

- Now we define our spin-1/2 states
  - Start with stationary spin state in center of mass frame, then boost to moving frame
    - \(|\mathbf{p}, s m_s\rangle = U \left(L(\beta_p)\right)|0, s m_s\rangle \equiv |\mathbf{p}, m_s(p)\rangle\) for spin-1/2
  - With our boost defined as follows
    - \(L(\beta_p) = R(\theta_p, \hat{n}_p) \cdot L(\beta_p \hat{z}) \cdot R(\theta_p, \hat{n}_p)^{-1}\)
  - This object rotates as a 2-component spinor
    - \(U(R)|\mathbf{p}, s m_s\rangle = |\mathbf{R} p, s m_s'\rangle \mathcal{D}^{(s)}_{m_s', m_s}(R)\)
  - In the lab frame, we have
    - \(|\mathbf{k}, m_s(k)\rangle \otimes |\mathbf{a}, m_s(a)\rangle \otimes |\mathbf{b}, m_s(b)\rangle\)
What basis should we use?

Two natural bases in the problem:
- A natural choice for defining $G$ — the lab frame in which total spin is conserved
- A natural choice for defining $\mathcal{K}_2$ — the (two-particle C.O.M.) dimer frame

Transforming between the two requires Wigner matrices
The Dimer Frame

- To combine spins with orbital angular momentum of interacting pair, boost to find $a^*$ and $b^*$
  - Start with stationary spin state in center of mass frame, boost to moving frame
    - $|a^*, m_s(a^*)\rangle = U(L(\beta_{a^*}))|0, m_s\rangle$ and $|b^*, m_s(b^*)\rangle = U(L(\beta_{b^*}))|0, m_s\rangle$
  - In the dimer-axis frame, we have
    - $|k, m_s(k)\rangle \otimes |a^*, m_s(a^*)\rangle \otimes |b^*, m_s(b^*)\rangle$
    - This is a natural basis for $\mathcal{K}_2$ and for derivation of QC3
- Orbital and spin axes aligned in dimer-frame, not lab-frame
- Wigner rotation relates spin components
  - $|a^*, m_s(a)\rangle = |a^*, m_s'(a^*)\rangle \mathcal{D}(R_a)m_s', m_s$
Incorporating Spin in $G$

\[
\det[F_{3}^{-1}(E, P, L) + \mathcal{K}_{df,3}(E^{*})] = 0 \quad \text{where} \quad F_{3} = \frac{F}{3} - F \frac{1}{\mathcal{K}_{2L}^{-1} + F} G
\]

- $G$ arises for 3-particle cuts in which the spectator is switched
- Spin components are conserved in the lab frame
  \[
  \Delta_{L,\alpha\beta}(b) = i \frac{(\not{\psi} + m)_{\alpha\beta}}{b^2 - m^2 + i\epsilon} + R_{L,\alpha\beta}(b)
  \]
- In the lab frame, we have
  \[
  [G^{\text{lab}}]_{p\ell',m'_{s};k\ell m m_{s}}(E, P, L) = -\delta_{m'_{s}(p),m_{s}(p)}\delta_{m'_{s}(k),m_{s}(k)}\delta_{m'_{s}(b),m_{s}(b)} i \frac{H(p)H(k)}{4\omega_p \omega_k L^6} \frac{4\pi \mathcal{Y}_{\ell'}(k_{p}^*) \mathcal{Y}_{\ell m}^*(p_{k}^*)}{q_{2,\ell}^* q_{2,\ell}} \frac{1}{b^2 - m^2} \]
- To transform to the dimer-axis frame, Wigner matrices are needed for each pair of spins
  \[
  G_{p\ell' m'_{s};k\ell m m_{s}} = \mathcal{D}^{(p,k)\dagger}_{m'_{s} m_{s}} G^{\text{lab}}_{p\ell' m'_{s};k\ell m m_{s}} \mathcal{D}^{(k,p)}_{m'_{s} m_{s}}
  \]
Incorporating Spin in $F$

$$\det[F_3^{-1}(E, P, L) + \mathcal{K}_{df3}(E^*)] = 0 \quad \text{where} \quad F_3 = \frac{F}{\mathcal{K}_{df3}^{-1}(E^*) + F + G}$$

- $F$ arises for 3-particle cuts in which the spectator is fixed

$$[F_{lab}]_{k'l'\ell'm'm'k\ell'mm}(E, P, L) \equiv \delta_{m'_s m_s} \delta_{k'k} \frac{iH(k)}{2\omega_k L^3} \frac{1}{2} \sum_a -p.v. \int_a \frac{4\pi \mathcal{Y}_{\ell'm'}(a^*_k) \mathcal{Y}_{\ell m}(a_k)}{2\omega_\alpha(b^2 - m^2)} \frac{1}{(q^*_2, k)_{\ell + \ell'}}$$

- Where the Kronecker delta enforces spin conservation in the lab frame

$$\delta_{m'_s m_s} = \delta_{m'_s(k)m_s(k)} \delta_{m'_s(a)m_s(a)} \delta_{m'_s(b)m_s(b)}$$

- Wigner D-matrices cancel when transforming to dimer-axis frame

$$F = F_{lab}$$
Incorporating Spin in $\mathcal{K}_2$

\[
\det[F_3^{-1}(E, P, L) + \mathcal{K}_{df,3}(E^*)] = 0 \quad \text{where} \quad F_3 = \frac{F}{3} - F_{\mathcal{K}_2^{-1}(L)} + F + G F
\]

- $\mathcal{K}_2$ incorporates 2-particle interactions
  - Natural to express in the dimer-axis
- Spin components are conserved in the lab frame

\[
[K_2]_{k'\ell' m' m_s^*; k\ell m m_s^*}(E, P) = i \delta_{k'k} 2\omega_k L^3 \mathcal{K}_2(\ell' m' m_s^*; \ell m m_s^*)(E^*_2, k)
\]

\[
\mathcal{K}_2(\ell' m' m_s^*; \ell m m_s^*)(E^*_2, k) = \delta_{m'_s(k) m_s(k)} \mathcal{K}_2[\ell' m'_s(a^*) m'_s(b^*)], [\ell m m_s(a^*) m_s(b^*)](E^*_2, k)
\]
Threshold Expansion for $\mathcal{K}_{df,3}$

- Have collected all the quantities entering $F_3 = \frac{F}{3} - F \frac{1}{\mathcal{K}_{2,L} + F + G} F$

- Final term appearing in quantization condition, $\det[F_3^{-1}(E, P, L) + \mathcal{K}_{df,3}(E^*)] = 0$, is $\mathcal{K}_{df,3}$

- To implement quantization condition, need a parameterization of $\mathcal{K}_{df,3}$
  - Analogous to effective range expansion for $\mathcal{K}_2$
Threshold Expansion for $\mathcal{K}_{df,3}$

- Start with nucleon field operator $\mathcal{N}$
- Write down all operators of form $(\mathcal{N} \bar{\mathcal{N}})^3$ with any gamma matrix structure and derivatives
  - Lorentz and parity invariant, requiring even numbers of derivatives
- Expand each in powers of 3-momentum using non-relativistic Dirac spinor expansion
- Results in two independent terms up to order $p^2$

$$\mathcal{K}_A = \bar{A} \left[ (\chi_{k'}^\dagger \sigma \cdot k' \sigma \cdot k \chi_k)(\chi_{a'}^\dagger \chi_a)(\chi_{b'}^\dagger \chi_b) \right] \quad \mathcal{K}_B = \bar{A} \left[ k' \cdot k(\chi_{k'}^\dagger \chi_k)(\chi_{a'}^\dagger \chi_a)(\chi_{b'}^\dagger \chi_b) \right]$$
Conclusion and Future Steps

• Including spin in the three-particle formalism requires
  • Additional matrix structures
  • Antisymmetrization and changed signs
  • Momentum-dependent Wigner rotations
• The first step toward implementation of QC3 has begun for toy model interactions
• In future the formalism should be generalized to incorporate
  • 3 nucleons at arbitrary isospin
  • \(N\pi\pi\) at maximal isospin
  • \(N\pi\pi + N\pi\) to understand the Roper
Thank You

Questions?