Performance of two-level methods for the glueball spectrum in pure gauge theory.

FOR 5269

DESY.

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Glueballs in pure gauge

$\Lambda = \{n = (\vec{n}, n_t)\}$ 4D lattice

Consider $SU(3)$ Yang-Mills theory in $4D$:

$$S_g = \frac{\beta}{3} \sum_{n \in \Lambda} \sum_{\mu < \nu} \mathrm{Re} \left\{ \mathrm{Tr} \left[1 - U_{\mu\nu}(n) \right] \right\}$$

Plaquette

$$\langle W^\Gamma(t) W^\Gamma(0) \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[U] \, e^{-S_g[U]} \, W^\Gamma(t) W^\Gamma(0)$$

$$\approx \frac{1}{N} \sum_{i=1}^{N} W^\Gamma(U_i, t) \; W^\Gamma(U_i, 0) \; + \; \mathcal{O}\left(1/\sqrt{N}\right)$$

[B. Berg, A. Billoire, Nucl.Phys.B 221 (1983)]

$W^\Gamma(t) =$ Wilson loop

$\Gamma = \{$irrep$, P, C\}$ irrep $\in \{A_1, A_2, E, T_1, T_2\}$

Statistical Error *[G. Parisi Phys. Rept. 103 (1984) 203]*

- Generate N gauge configurations $U^{(i)}(\vec{x}, t)$
- Apply link smearing to $U^{(i)}(\vec{x}, t)$ to mitigate UV fluctuations
- compute $W_i^\Gamma(t) \equiv W^\Gamma(U^{(i)}, t) \quad \forall \quad i = 1, \ldots, N$
- compute $C^\Gamma(t) = \langle W^\Gamma(t) W^\Gamma(0) \rangle$

+ variational method (due to S/N ratio)…

Exponential loss of significance

$$\langle W^\Gamma(t)\ W^\Gamma(0)\rangle = |Z_0|^2 e^{-E_0 t} + |Z_1|^2 e^{-E_1 t} + \ldots \xrightarrow{t \gg 0} |Z_0|^2 e^{-E_0 t}$$

s. t. $E_0^\Gamma \longleftarrow \log\left(\frac{C(t)}{C(t+1)}\right)$

Problem S/N Ratio of Glueball correlation functions $C(t)\ /\ \sigma(t) \sim \sqrt{N} e^{-m_G t}\ /\ L^8$

[G. P. Lepage TASI (1989)]

! Extraction of E_0 from $C(t)$ at small t is contaminated by excited states ($E_1, \ \ldots$)

$W_k, W_l = W_1, \ \ldots, \ W_M$

Solution Determine E_0 at small t with a **variational method** $C_{kl}(t) = \langle W_k(t)\ W_l(0)\rangle$

[K. Wilson Talk @ Abingdon Meeting (1981)]

Solve $C(t)V(t) = C(t_{\text{ref}})\ \Lambda(t)\ V(t)$ ➡

$\Lambda(t) = [\lambda_1(t), \ldots, \lambda_M(t)]^T$ $\lambda_n \propto e^{-E_n(t-t_{\text{ref}})}$

$V(t) = [v_1(t), \ldots, v_M(t)]^T$ $E_n^\Gamma \longleftarrow \log\left(\frac{\lambda_n(t)}{\lambda_n(t+1)}\right)$

Different W_k constructed usually with combinations of **different smearing radii + loop shapes** *e.g. [A. Athenodorou, M. Teper (2020)]*

Multilevel algorithm: The Idea

We want to find an other variable B s. t. $\langle B \rangle = \langle A \rangle$ but $\langle B^2 \rangle_C \ll \langle A^2 \rangle_C$

Multihit method

SU(3) Polyakov loops for string tension

[G. Parisi et al., Phys.Lett. B 128 (1983)]

Example: <u>Ising model</u> $H = \frac{1}{2} \sum_{ik} J_{ik}\, s_i\, s_k$ $\langle s_i \rangle = \left\langle \tanh \left[\beta \sum_k J_{ik}\, s_k \right] \right\rangle$

$\langle s_i \rangle$ has smaller fluctuations than s_i

[H. B. Callen, Phys.Lett. 4B (1963)]

In *[G. Parisi et al., Phys.Lett. B 128 (1983)]* statistical errors σ was reduced

by 10 with just $\mathcal{O}(10)$ more measurements or “hits” of the Polyakov loops

Usually $\mathcal{O}(10^2)$ more statistics is required to achieve the same statistical errors.

Square error reduction achieved!

Multilevel algorithm: Applications

[M. Lüscher, P. Weisz, JHEP 09 (2001)]

Exploiting locality of pure SU(3) gauge theory for **Polyakov loops**, the authors factorise $\langle W(\mathscr{C})\rangle = \mathrm{tr}\left\{U(\mathscr{C})\right\}$

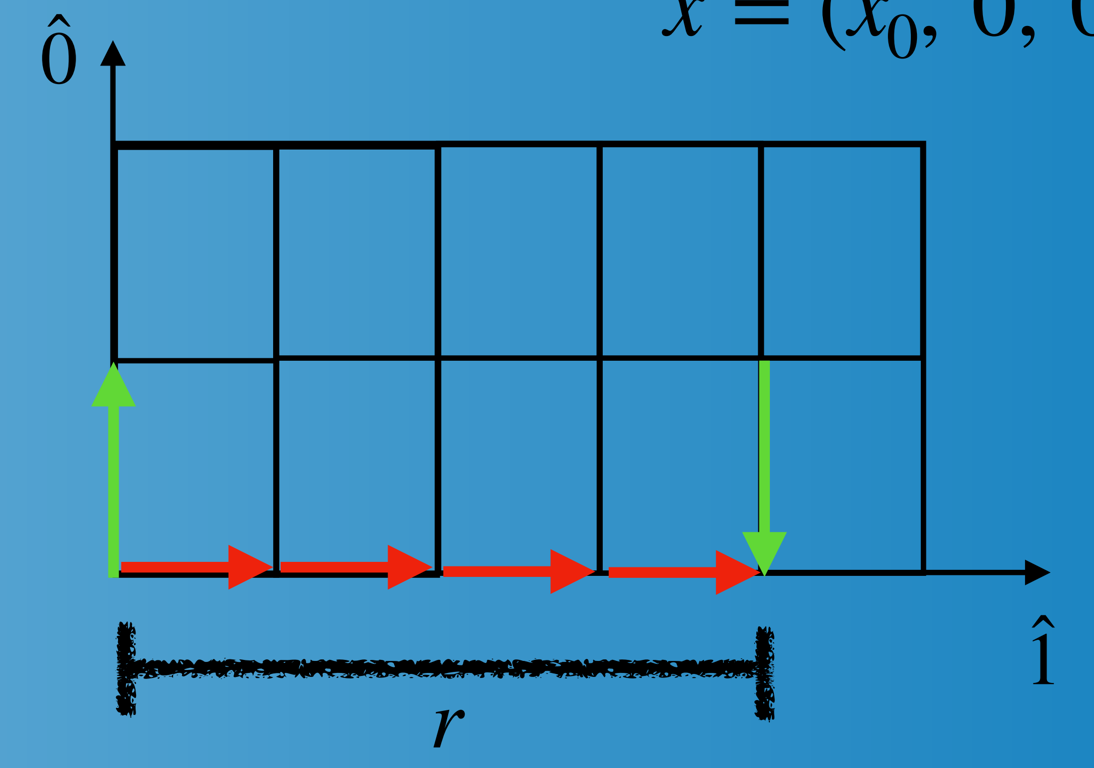
in terms of products of line operators $\mathbb{L}(x_0)$ and two-link $\mathbb{T}(x_0)$ operators

$$\mathbb{T}(x)_{\alpha\beta\gamma\delta} = U_0(x)^*_{\alpha\beta}\; U_0(x+r\hat{1})_{\gamma\delta}$$

$$\mathbb{L}(x) = \left\{U_1(x)^* \;\cdots\; U_1(x+(r-a)\hat{1})^*\right\}$$

$$\langle W(\mathscr{C})\rangle = \left\langle \mathbb{L}(0)_{\alpha\gamma}\left\{\left[\mathbb{T}(0)\mathbb{T}(a)\right]\cdots\left[\mathbb{T}(t-2a)\mathbb{T}(t-a)\right]\right\}_{\alpha\beta\gamma\delta}\mathbb{L}(t)^*_{\beta\delta}\right\rangle$$

$x = (x_0, 0, 0, 0)$



$[\ldots]$ = Sublattice expectation values / measurements

$\langle\ldots\rangle$ = Full lattice expectation values

Exponential error reduction achieved!

e.g. $$\left[\mathbb{T}(0)\mathbb{T}(a)\right] = \frac{1}{\mathscr{Z}_{\mathrm{sub}}}\int \mathscr{D}[U]_{\mathrm{sub}}\;\mathbb{T}(0)\mathbb{T}(a)\; e^{-S[U]_{\mathrm{sub}}}$$

here sublattice consists of timeslices $[0, a]$

Similar idea applied for **Glueball spectrum**

$$\langle W(t)\;W(t_0)\rangle \longrightarrow \langle\left[W(t)\right]\;\left[W(t_0)\right]\rangle$$

SU(3) (3+1)D *[H. Meyer, JHEP 01 (2003)]*

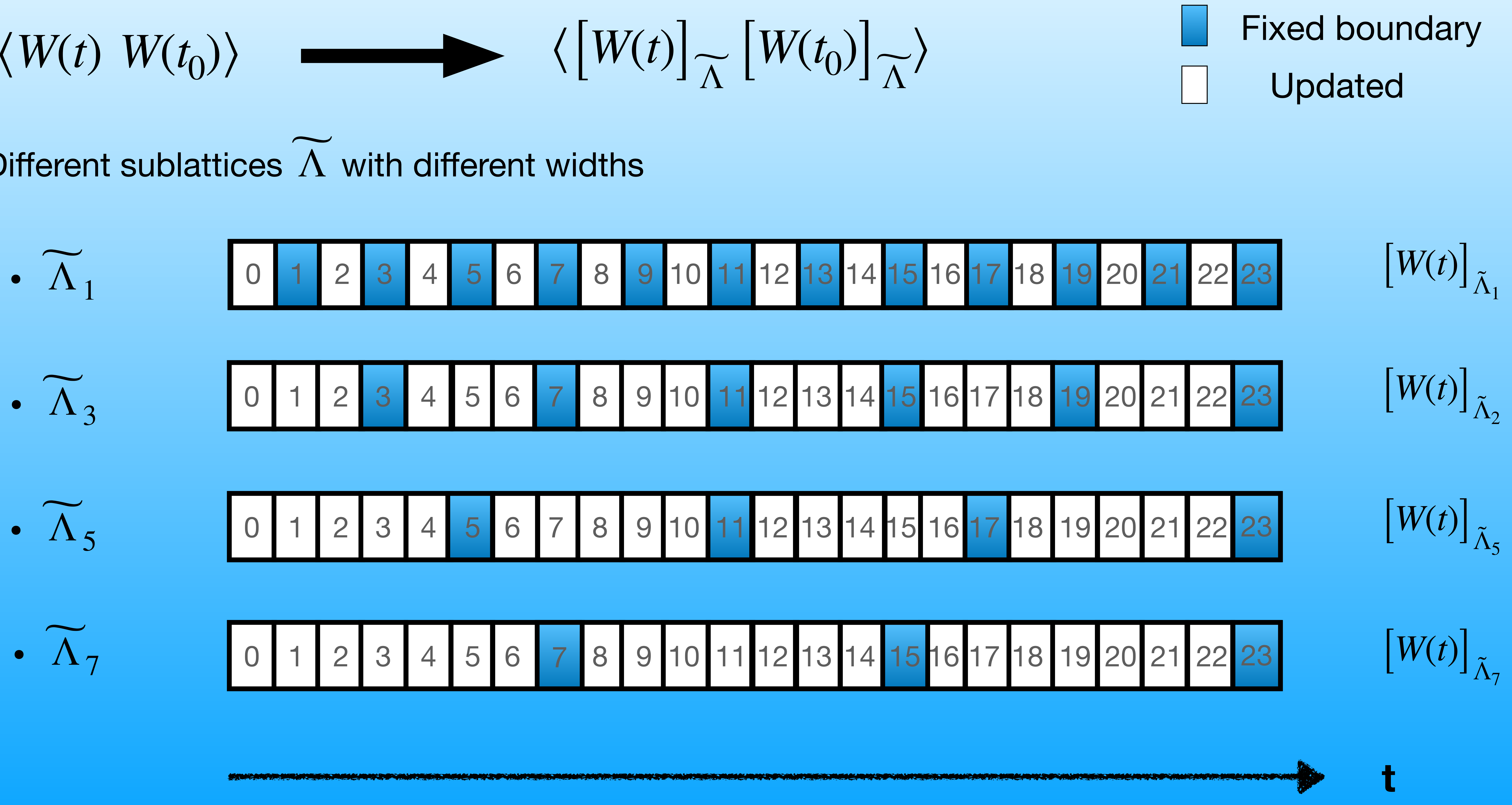
SU(2) (2+1)D *[H. Meyer, JHEP 01 (2004)]*

Our simulations

$SU(3)$ Yang-Mills, $V = 24^4$, $\beta = 6.2$

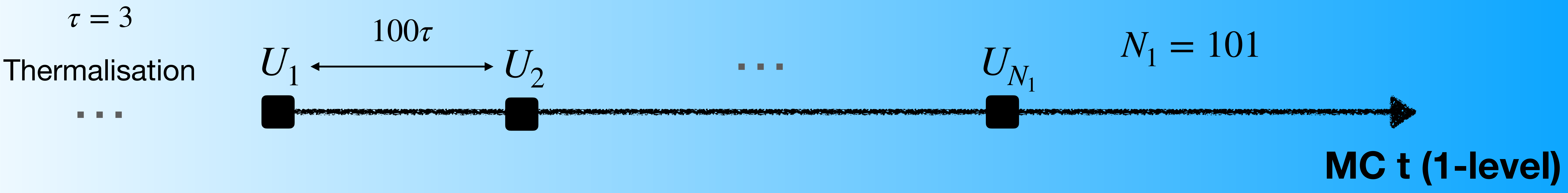
$$\langle W(t)\ W(t_0)\rangle \longrightarrow \langle [W(t)]_{\widetilde{\Lambda}} [W(t_0)]_{\widetilde{\Lambda}} \rangle$$

Different sublattices $\widetilde{\Lambda}$ with different widths



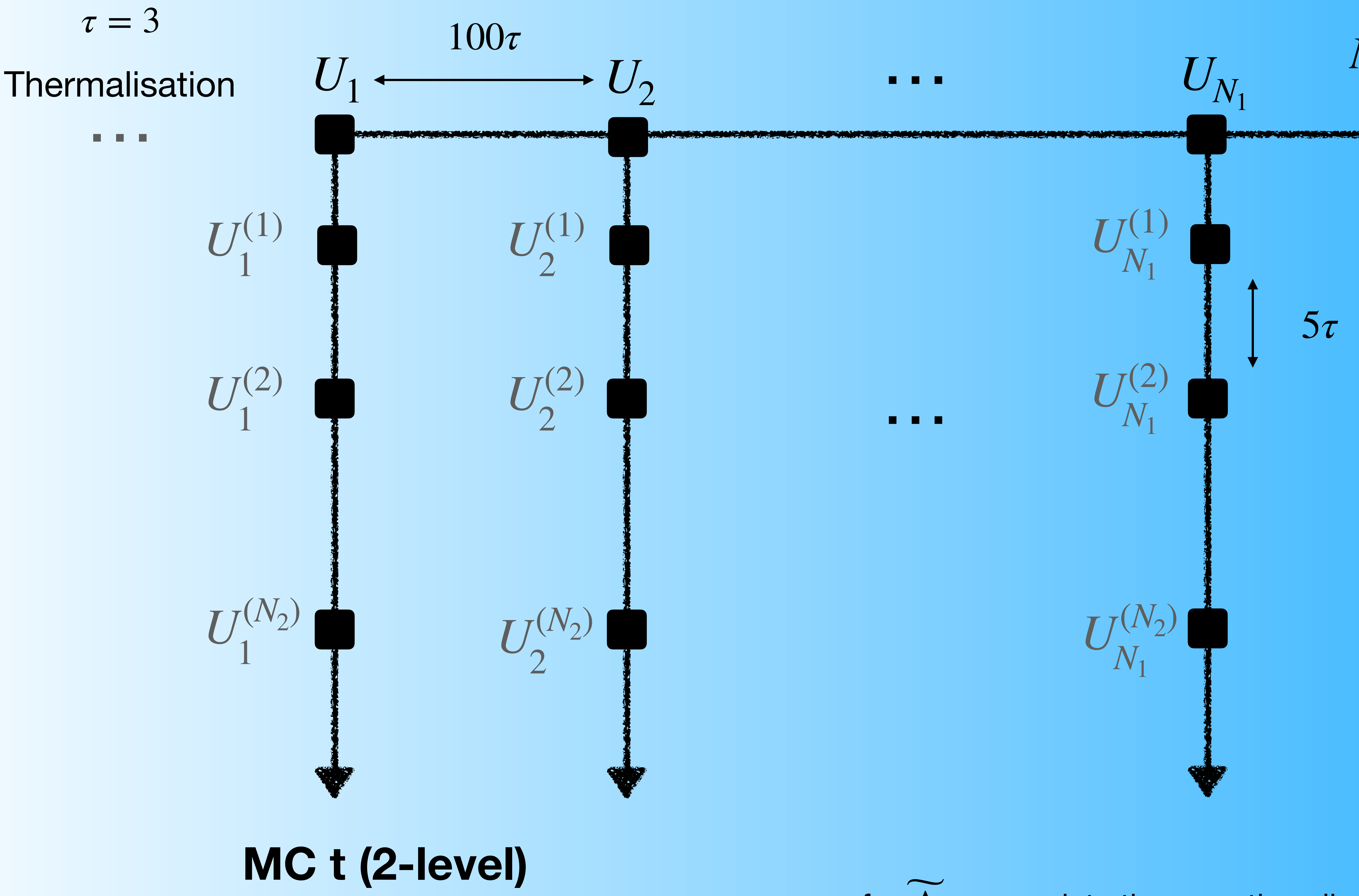
2-level HMC Algorithm

We generate standard (full lattice updates) $N_1 = 101$ thermalised configurations



2-level HMC Algorithm

We generate standard (full lattice updates)
$N_1 = 101$ thermalised configurations



$N_1 = 101$

MC t (1-level)

$\forall \ U_i = U_1, \ \ldots, \ U_{N_1}$

we perform additional MC
with a specific sublattice

$\left(\widetilde{\Lambda}_1, \widetilde{\Lambda}_3, \widetilde{\Lambda}_5, \widetilde{\Lambda}_7\right)$

compute $\left[W(t)\right]_{\widetilde{\Lambda}_k}$

$\left\langle \left[W(t)\right]_{\widetilde{\Lambda}_k} \left[W(t_0)\right]_{\widetilde{\Lambda}_k} \right\rangle$

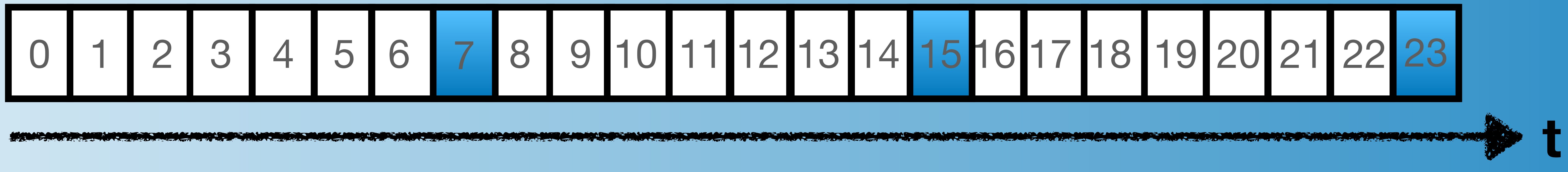
e.g. for $\widetilde{\Lambda}_1$ we update the even timeslices and fix the odd (sublattice boundaries)

2-level setups

$N_1 = 101$

$N_2 \in \{1,\ 4,\ 10,\ 30,\ 50,\ 100,\ 200,\ 1000\}$

- $\widetilde{\Lambda}_7$



$\langle [W(t)]_{\widetilde{\Lambda}_7}\ [W(t_0)]_{\widetilde{\Lambda}_7} \rangle$

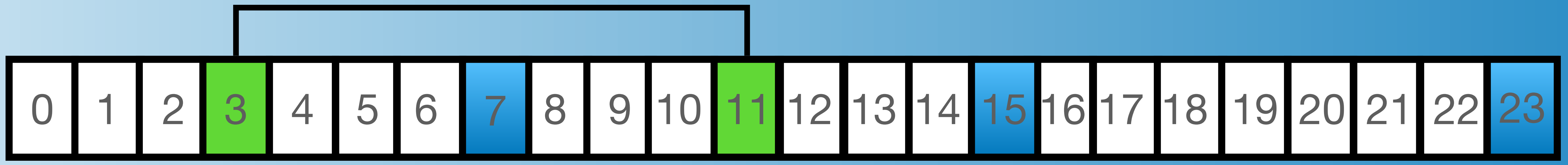
$t = t_0 + \Delta t$

$$\Sigma_{\text{rel}}(t, t_0, N_2) = \frac{\sigma(t, t_0, N_2)}{\sigma(t, t_0, N_2 = 1)}$$

Relative error wrt $N_2 = 1$

$\Delta t = 8$

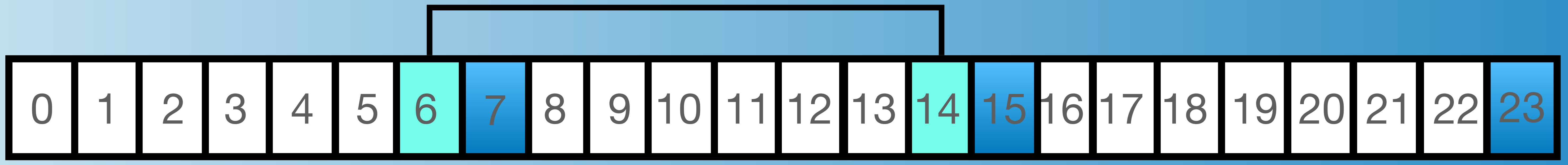
$\Sigma_{\text{rel}}(t = 11,\ t_0 = 2,\ N_2)$



Best multilevel performance, i.e. $\Sigma_{\text{rel}} \sim \mathcal{O}(1\ /\ N_2)$

$\Delta t = 8$

$\Sigma_{\text{rel}}(t = 14,\ t_0 = 6,\ N_2)$



Poor multilevel performance, i.e. $\Sigma_{\text{rel}} \gg \mathcal{O}(1\ /\ N_2)$

Performance of multilevel at fixed $\Delta t = 8$

$\Gamma = E^{++}, T_2^{++}$

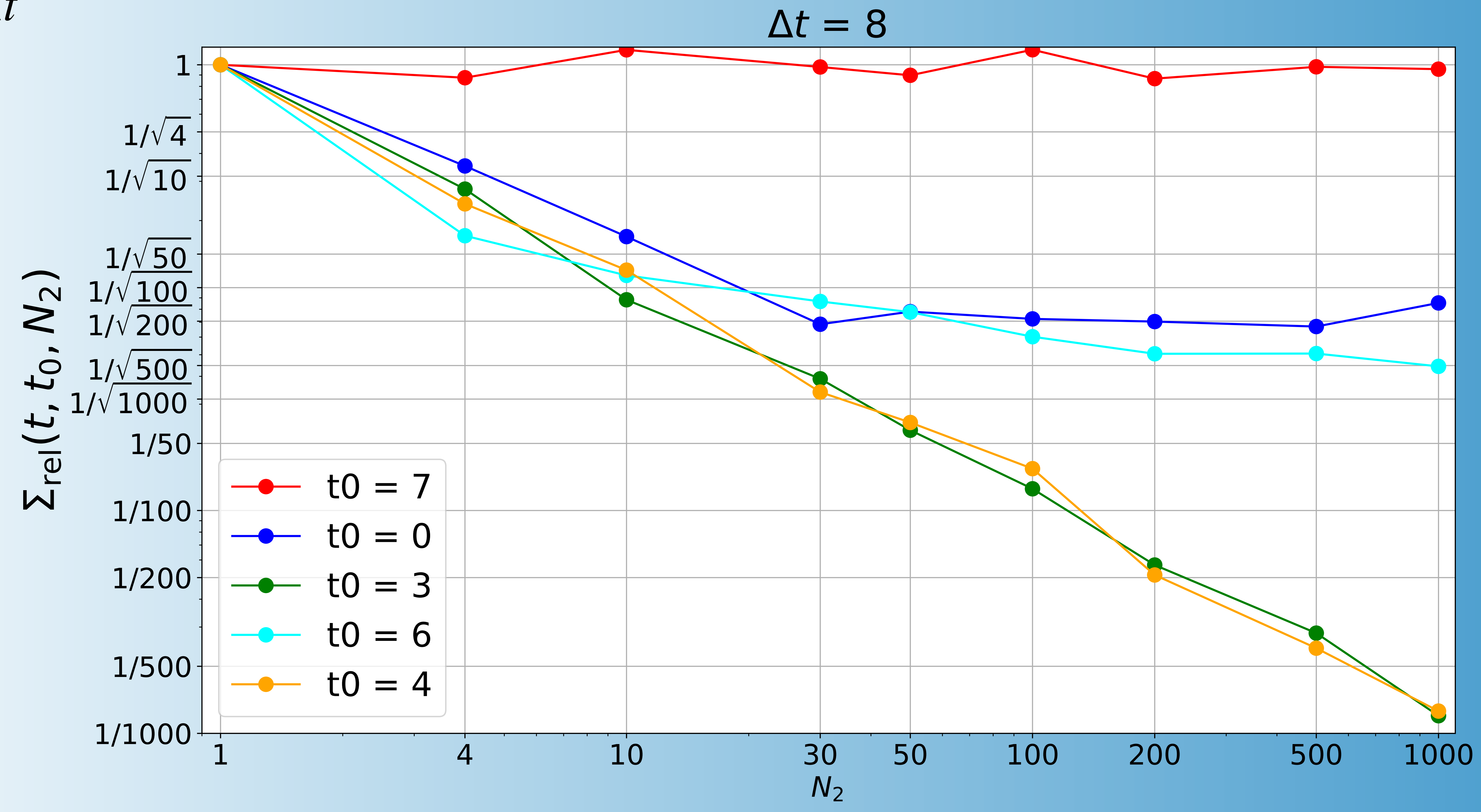
- $\widetilde{\Lambda}_7$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23

$\langle [W(t)]_{\widetilde{\Lambda}_7} \; [W(t_0)]_{\widetilde{\Lambda}_7} \rangle$

$$\Sigma_{\text{rel}}(t, t_0, N_2) = \frac{\sigma(t, t_0, N_2)}{\sigma(t, t_0, N_2 = 1)}$$

$t = t_0 + \Delta t$



$\langle [W(14)]_{\widetilde{\Lambda}_7} \; [W(6)]_{\widetilde{\Lambda}_7} \rangle$

$\langle [W(12)]_{\widetilde{\Lambda}_7} \; [W(4)]_{\widetilde{\Lambda}_7} \rangle$

$\langle [W(11)]_{\widetilde{\Lambda}_7} \; [W(3)]_{\widetilde{\Lambda}_7} \rangle$

$\Sigma_{\text{rel}} \sim 1/1000$

Performance of multilevel at fixed $t_0 = 3$

- $\widetilde{\Lambda}_7$

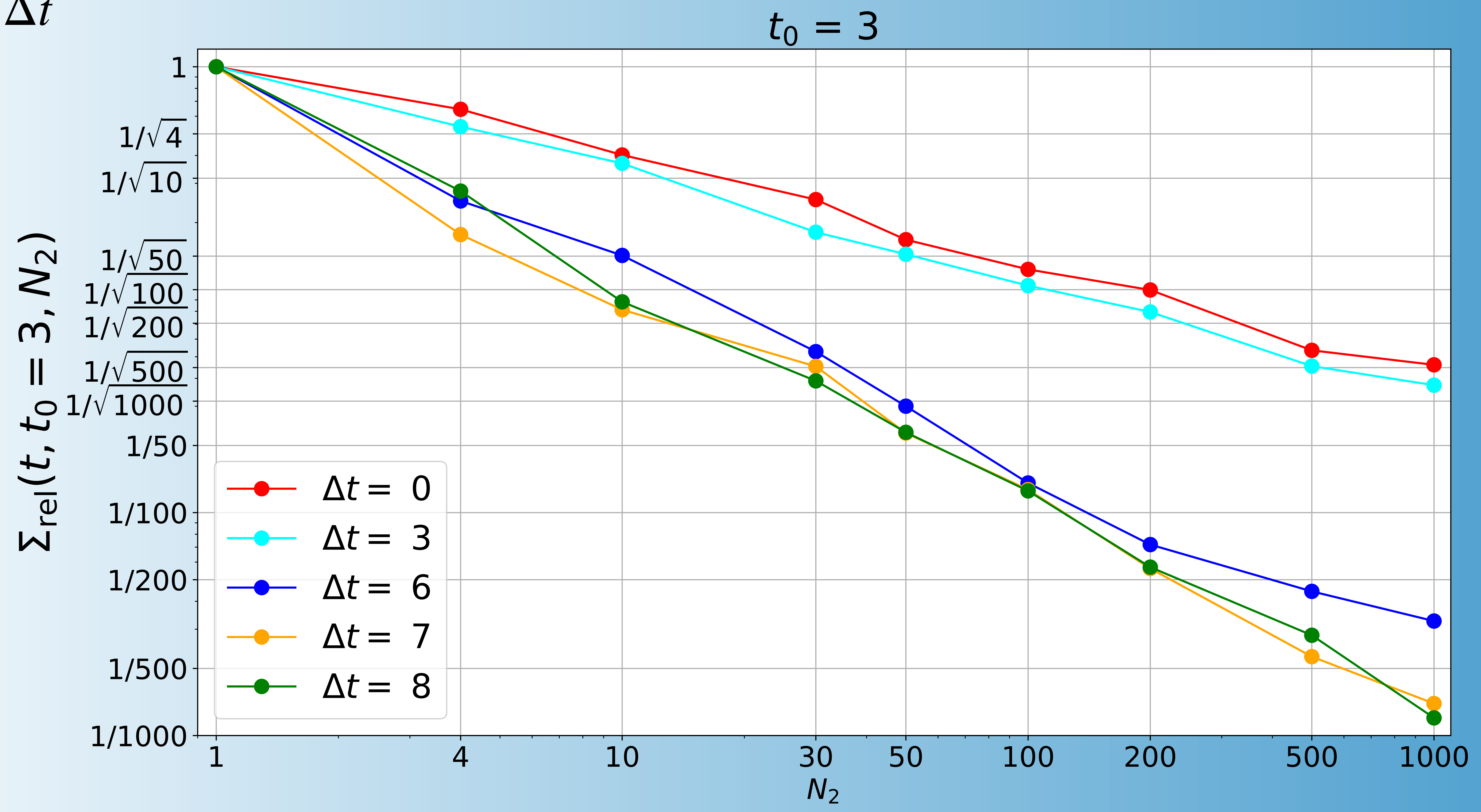
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23

$\Gamma = E^{++}, T_2^{++}$

$\langle [W(t)]_{\widetilde{\Lambda}_7} \; [W(t_0 = 3)]_{\widetilde{\Lambda}_7} \rangle$

$$\Sigma_{\rm rel}(t,3,N_2) = \frac{\sigma(t,\,3,\,N_2)}{\sigma(t,\,3,\,N_2 = 1)}$$

$t = t_0 + \Delta t$



Distance from boundaries important for achieving high performance

$\langle [W(6)]_{\widetilde{\Lambda}_7} \; [W(3)]_{\widetilde{\Lambda}_7} \rangle$

Error decreased by ~ 1000 with $N_2 = 1000$

$\langle [W(10)]_{\widetilde{\Lambda}_7} \; [W(3)]_{\widetilde{\Lambda}_7} \rangle$

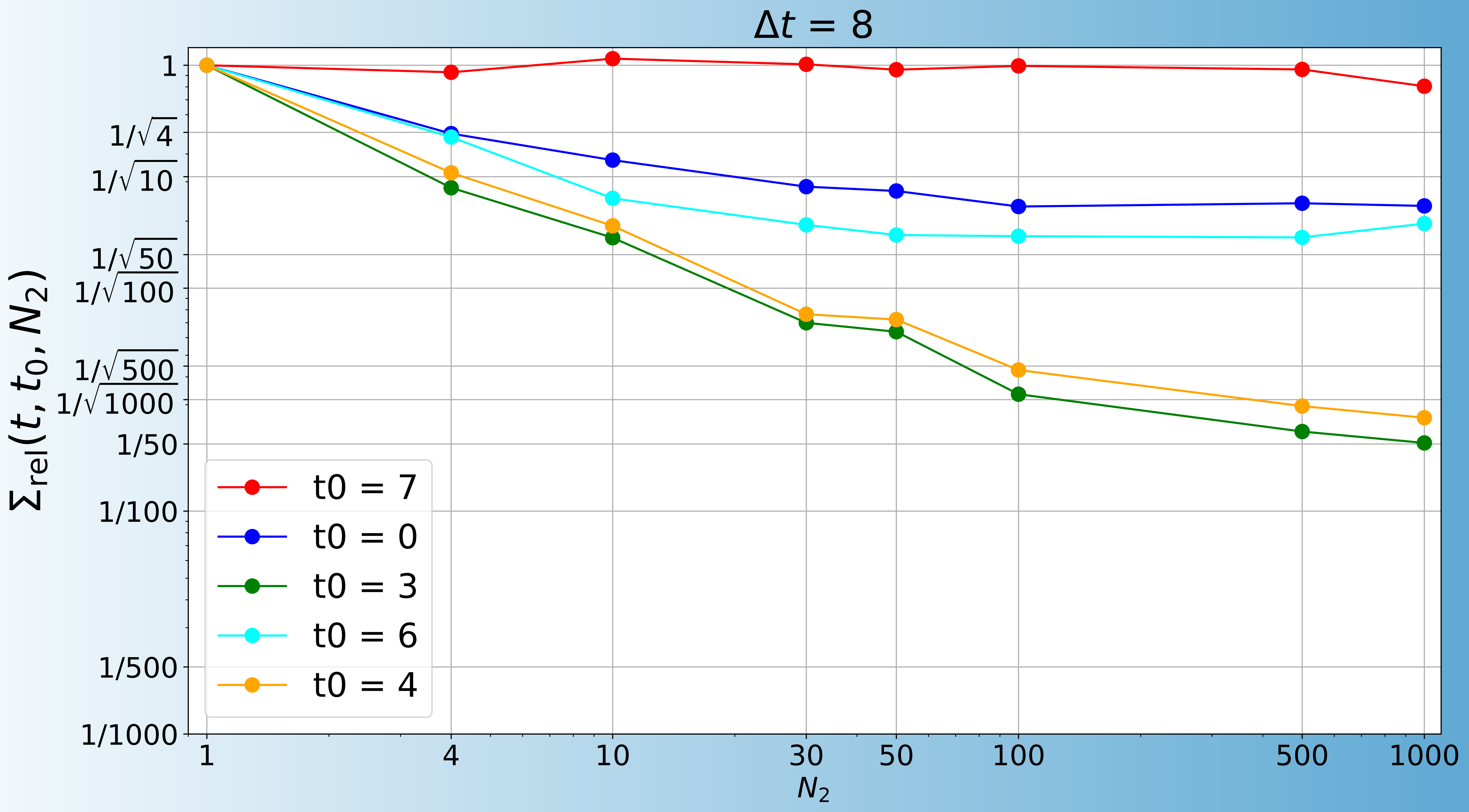
$\langle [W(11)]_{\widetilde{\Lambda}_7} \; [W(3)]_{\widetilde{\Lambda}_7} \rangle$

Performance of multilevel: A_1^{++} with $\langle W \rangle \neq 0$

- $\widetilde{\Lambda}_7$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23

$$\langle [\bar{W}(t)]_{\widetilde{\Lambda}_7} \; [\bar{W}(t_0)]_{\widetilde{\Lambda}_7} \rangle \qquad \bar{W}(t) = W(t) - \langle W(t) \rangle \qquad \Sigma_{\rm rel}(t, t_0, N_2) = \frac{\sigma(t, t_0, N_2)}{\sigma(t, t_0, N_2 = 1)}$$



Multilevel is not very effective for $\Gamma = A_1^{++}$

Boundary fluctuations (at t_B) of lightest state (vacuum) with $E = 0$

Covariance $\sigma(t, t_B) \approx \sigma(t_B) e^{-E(t-t_B)/2}$

[G. P. Lepage TASI (1989)]

Also observed in

[H. Meyer, JHEP 01 (2004)]

Performance of multilevel: effective mass

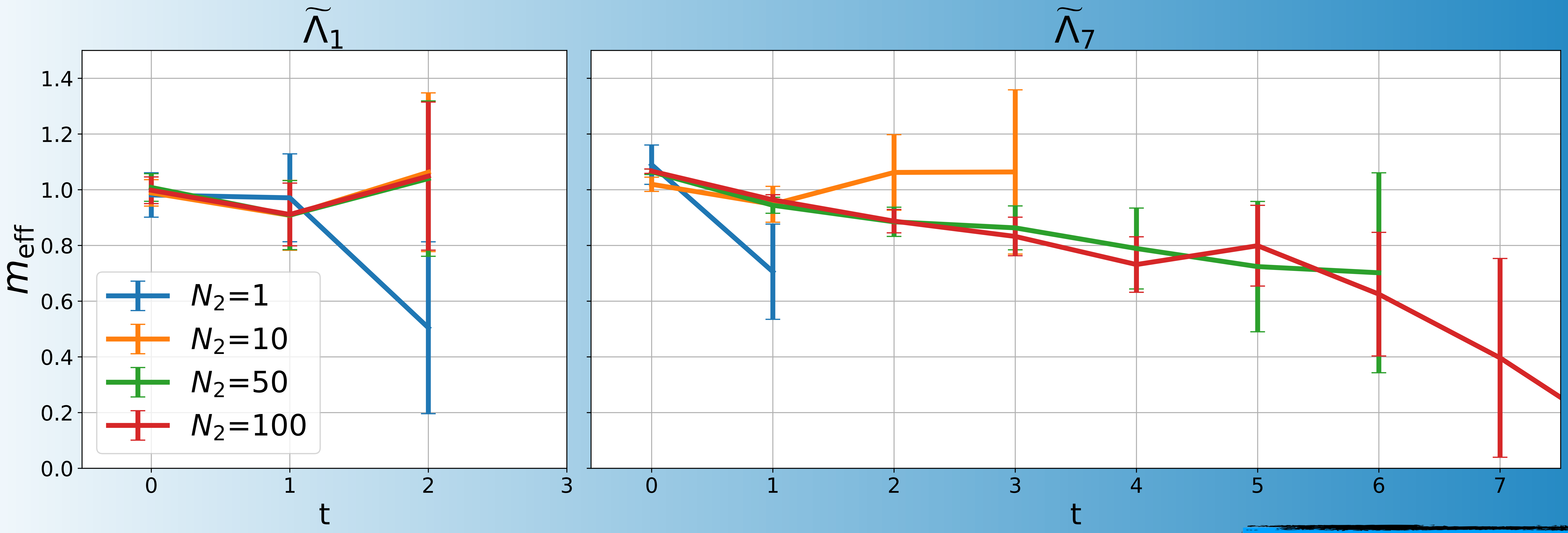
Weighted average

$$C(t, t_0) = \langle [W(t)]_{\widetilde{\Lambda}} [W(t_0)]_{\widetilde{\Lambda}} \rangle \quad \rightarrow \quad C(t) = \frac{\sum_{t_0} w(t, t_0)\, C(t, t_0)}{\sum_{t_0} w(t, t_0)}$$

$$w(t, t_0) = 1 \,/\, \sigma^2(t, t_0)$$

$$m_{\mathrm{eff}} = \log \left(C(t) \,/\, C(t+1) \right)$$

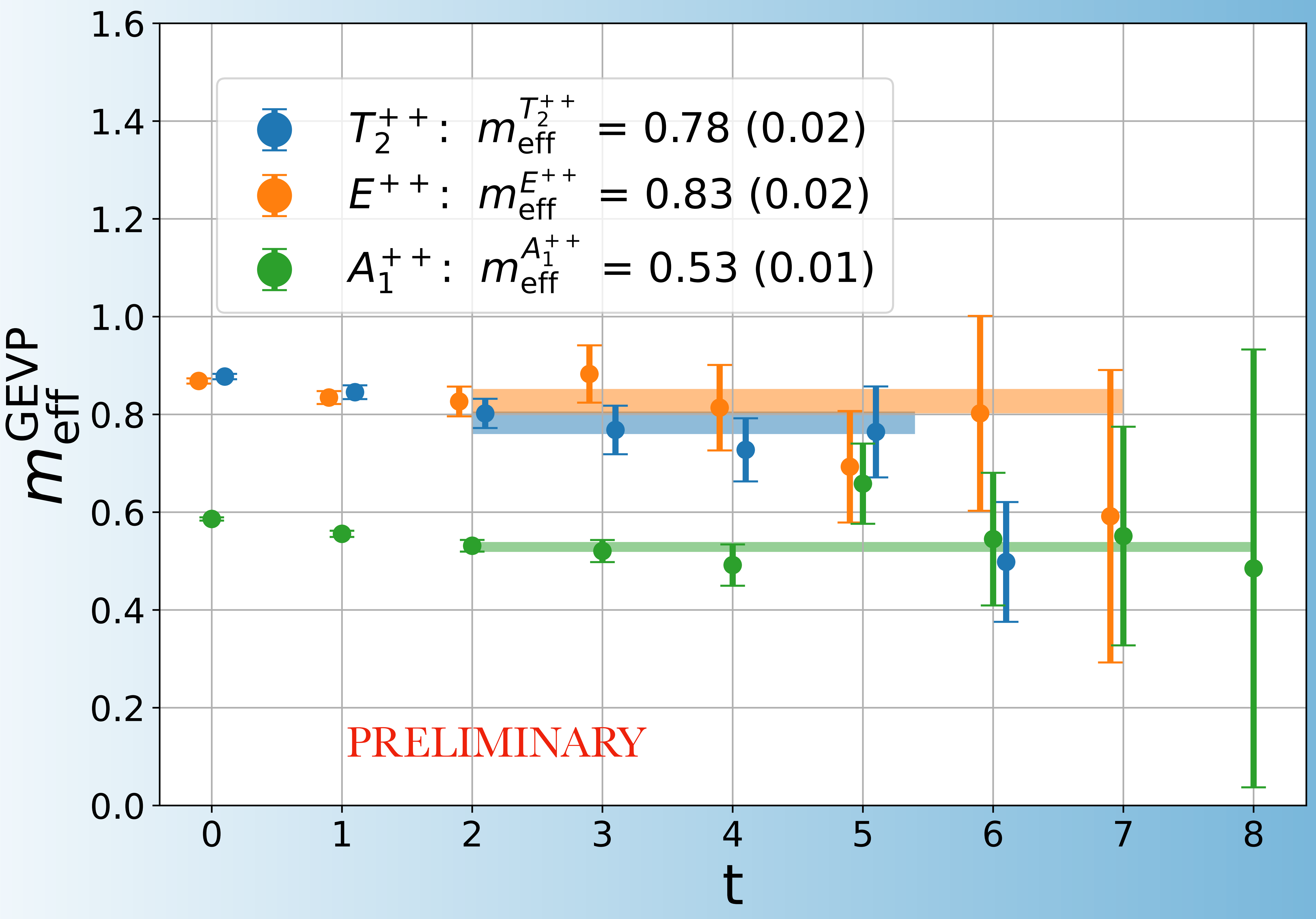
T_2^{++}



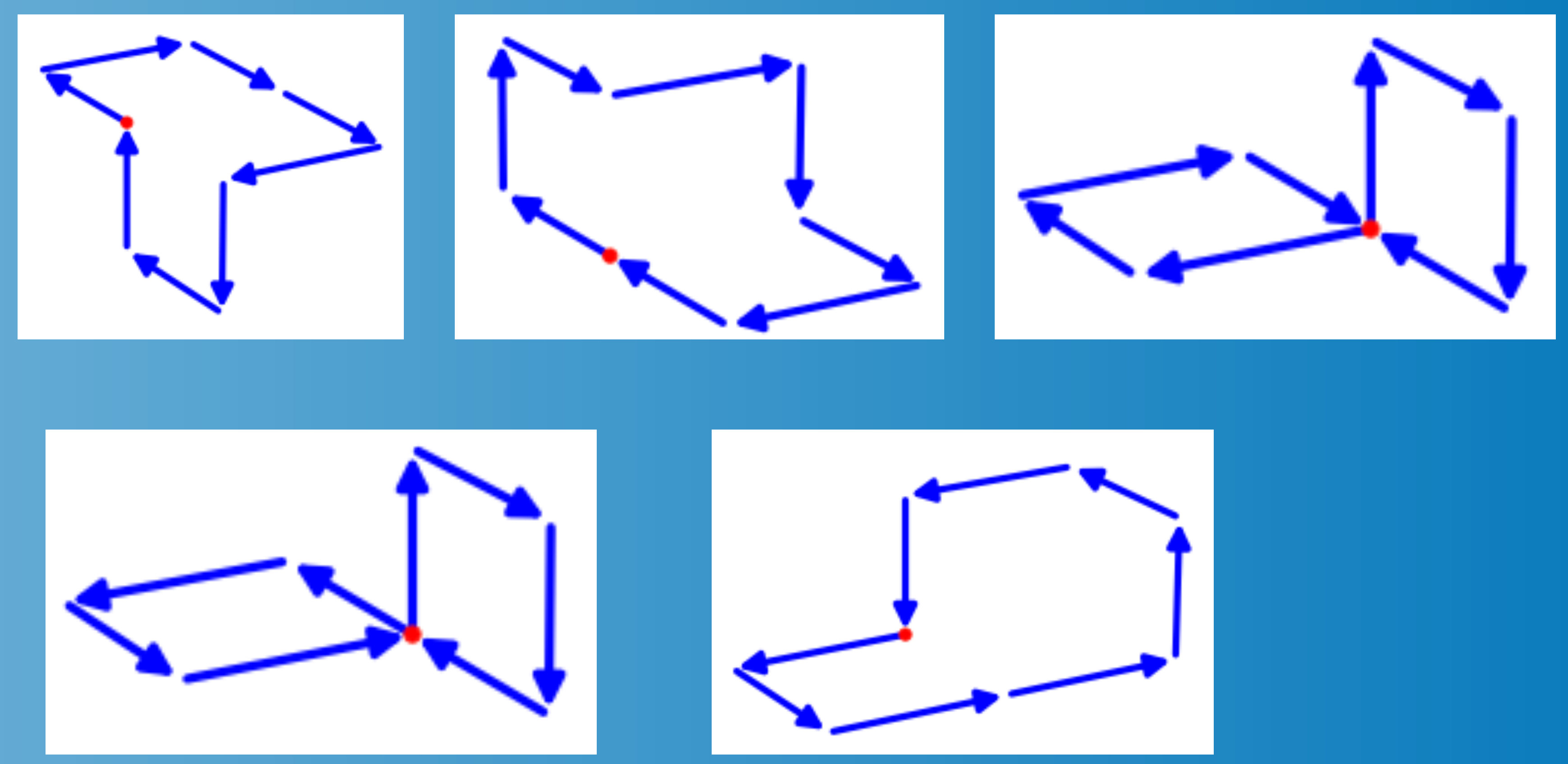
Larger sublattices achieve
Better error reduction

GEVP effective masses with 2-level algorithm

$N_1 = 101$ configurations $\times$ $N_2 = 1000$ submeasurements



5 length-8 operators x 4 smearing radii = 20 operators



Implemented by J. A. Urrea
(University of Wuppertal)

Simpler shapes (length-4 and -6)
will be soon implemented

Results consistent within errors with literature *[K. Sakai, S. Sasaki, PRD 107 (2023)]*

Conclusions

- Revisited SU(3) glueball spectrum in 4D in pure gauge theory with a 2-level algorithm
- Error reduction of $\sim N_2$ with $N_2 = 1, \ \ldots, \ 1000$ updates at fixed boundaries (except $\Gamma = A_1^{++}$)
- Observed dependence of the error reduction on the distance from boundaries
- GEVP with 5 length-8 operators x 4 APE smearing radii gives consistent results

Next

- Adopt less noisy operators (length-4, length-6)
- Investigate covariance $\sigma^2(t, t_0) \approx \langle W(t)W(0)W(t_0)W(0)\rangle - \langle W(t)W(0)\rangle\langle W(t_0)W(0)\rangle$
- Consider larger ensembles & continuum limit

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