Higher-Order Calculations of Anomalous Dimensions at Infrared Fixed Points in Gauge Theories and Studies of Renormalization-Group Behavior of Some Scalar Field Theories

Robert Shrock

C. N. Yang Institute for Theoretical Physics and Dept. of Physics and Astronomy, Stony Brook University, email: robert.shrock@stonybrook.edu

Lattice-2023, Fermilab, July 31-Aug. 4, 2023

Outline

- Renormalization-group flow from UV to IR in asymptotically free gauge theory; types of IR behavior; role of IR fixed point

- Calculations of anomalous dimensions of fermion bilinear operators via series expansions in gauge coupling and via scheme-independent series expansions; application to theories with fermions in a single representation and in multiple representations of the gauge group

- Comparison of results with recent lattice measurements for SU(3) with $N_F = 10$ and SU(4) for $N_F = 4$ and $N_{A_2} = 4$ fermions, where F and A_2 are the fundamental and antisymmetric tensor representations

- Higher-loop studies of the beta functions of O(N) $|\vec{\phi}|_4^4$ and $|\vec{\phi}|_3^6$ theories

- Conclusions

This talk contains new results from Ryttov and RS, 2307.12426 and from RS, PRD 107, 056018 (2023) [2301.01830] and RS, PRD 107, 096009 (2023) [2302.05422]

RG Flow from UV to IR; Types of IR Behavior and Role of IR Fixed Point

Consider an asymptotically free, vectorial gauge theory with gauge group G and a set of massless fermions, either (i) N_f fermions f in a single representation R of G, or (ii) N_f fermions f and $N_{f'}$ fermions f' in different reps. R and R'.

Asymptotic freedom $\Rightarrow$ theory is weakly coupled, properties are perturbatively calculable for large Euclidean momentum scale μ in deep ultraviolet (UV).

One can analyze the renormalization-group (RG) flow from large μ in the UV to small μ in the infrared (IR).

If a fermion had mass m_0, it would be integrated out in the effective low-energy field theory for $\mu < m_0$, and hence would not affect the IR limit of interest here, so no loss of generality in taking massless fermions (mass-split models can also be of interest).

Denote running gauge coupling at scale μ as $g = g(\mu)$, and let $\alpha(\mu) = g(\mu)^2/(4\pi)$ and $a(\mu) = g(\mu)^2/(16\pi^2)$.

The dependence of $\alpha(\mu)$ on μ is described by the β function

$$\beta \equiv \frac{d\alpha}{d\ln\mu} = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \, a^\ell$$

where $\ell =$ loop order of the coeff.

Coefficients b_1 (Gross and Wilzcek, Politzer, 1973) and b_2 (Caswell, Jones, 1974) in β are independent of regularization/renormalization scheme, while b_ℓ for $\ell \geq 3$ are scheme-dependent.

Asymptotic freedom means $b_1 > 0$, so $\beta < 0$ for small $\alpha(\mu)$, in neighborhood of UV fixed point (UVFP) at $\alpha = 0$.

As the scale μ decreases from large values, $\alpha(\mu)$ increases. For a sufficiently large fermion content satisfying asymptotic freedom, β has an infrared (IR) zero, denoted α_{IR}, which is an IR fixed point (IRFP) of the renormalization group.

At this IRFP, the theory is scale-invariant and is inferred to be conformal-invariant, hence the term “conformal window” (CW) for this regime.

The properties of the theory at such an IRFP are of fundamental interest. These include anomalous dimensions of (gauge-invariant) operators. Denoting the dimension of an operator $\mathcal{O}$ as $D_{\mathcal{O}}$, the anomalous dimension $\gamma_{\mathcal{O}}$ is given by $D_{\mathcal{O}} = D_{\mathcal{O},\ free} - \gamma_{\mathcal{O}}$.

Besides the intrinsic field-theoretic interest in anomalous dimensions of operators in the conformal window, theories slightly below the lower end of the CW exhibit quasi-conformal behavior, with slow variation of the gauge coupling over an extended interval of Euclidean momentum, μ, owing to the small β.

This is of interest for particle physics because as this theory flows into the IR and eventually undergoes spontaneous chiral symmetry breaking (SχSB), the dynamical breaking of scale invariance yields a light approx. Nambu-Goldstone boson, the dilaton. Insofar as the Higgs boson can be modelled as at least partially dilatonic, this could protect its mass from large radiative corrections. Lattice simulations (LSD, LatKMI, Lat-HC groups) have verified the appearance of a light 0^{++} scalar in these theories.

In the chirally broken phase, just as the IR zero of β is only an approx. IRFP, so also, the $\gamma_{\bar{\psi}\psi,IR}$ is only approx., describing the running of $\bar{\psi}\psi$ over an extended interval of energies.

The asymptotic freedom condition is $b_1 > 0$, i.e. $11C_A - 4\sum_f N_f T_f > 0$. For a theory with fermions in a single rep., this sets an upper bound on N_f: $N_{f,u} = 11C_A/(4T_f)$, where with T_R^a are generators of the Lie algebra of G in the representation R, and $d_R = \dim(R)$, and group invariants include

$$T_R^a T_R^a = C_2(R) I_{d_R \times d_R} , \qquad \mathrm{Tr}_R(T_R^a T_R^b) = T(R)\delta^{ab}$$

We use the notation $C_A \equiv C_2(G)$ and, for f in R, $T_f \equiv T(R)$ and $C_f \equiv C_2(R)$.

The condition that the 2-loop β fn. should have an IR zero (IRZ) is $b_2 < 0$, i.e.,

$$34C_A^2 - 4\sum_f (5C_A + 3C_f) N_f T_f < 0$$

which sets a lower bound on N_f. The region in which $b_1 > 0$ and $b_2 < 0$ is denoted I_{IRZ}. The upper (u) and lower (ℓ) boundaries $\mathcal{B}_{IRZ,u}$ and $\mathcal{B}_{IRZ,\ell}$ of the IRZ regions are the lines $b_1 = 0$ and $b_2 = 0$.

The upper and lower boundaries of the conformal window are denoted $\mathcal{B}_{CW,u} = \mathcal{B}_{IRZ,u}$ and $\mathcal{B}_{CW,\ell}$. We first discuss theories with fermions in a single representation, and then theories with fermions in multiple different reps.

Calculations of Anomalous Dimensions

An operator of particular interest is the fermion bilinear, $\bar\psi\psi = \sum_{j=1}^{N_f} \bar\psi_j \psi_j$ with anom. dim. $\gamma_{\bar\psi\psi}$ and its value at the IRFP, $\gamma_{\bar\psi\psi,IR}$.

One way to calculate $\gamma_{\bar\psi\psi,IR}$ is via a power series expansion in the coupling, $a = \alpha/(4\pi)$:

$$\gamma_{\bar\psi\psi} = \sum_{\ell=1}^{\infty} c_\ell \, a^\ell$$

where c_ℓ is ℓ-loop coefficient. To calculate the n-loop result for the anom. dim., $\gamma_{\bar\psi\psi,n\ell,IR}$, one first calculates $\alpha_{IR,n\ell,IR}$ from the IR zero in the n-loop beta function and then sets $\alpha = \alpha_{IR,n\ell,IR}$ in above eq.

For a given G and R, as N_f decreases below $N_{f,u}$, $\alpha_{IR,2\ell}$ increases. This motivates calculation of the IR zero in β to higher-loop order. With T. Ryttov, we calculated $\gamma_{\bar\psi\psi,IR}$ in this way up to 5-loop order in Ryttov and RS, PRD 94, 105014 (2016), PRD 94, 105015 (2016).

The anom. dim. $\gamma_{\bar\psi\psi,IR}$ is a physical quantity and is independent of the scheme used for regularization and renormalization.

The conventional expansion of $\gamma_{\bar\psi\psi,IR}$ as a power series in α, calculated to finite order, does not maintain this scheme independence beyond the lowest order, since the b_ℓ are scheme-dependent for $\ell \geq 3$ and the c_ℓ are scheme-dependent for $\ell \geq 2$.

This scheme-dependence of higher-order calculations is well-known in QCD and uncertainties due to it are routinely taken into account in comparing higher-order QCD calculations with data, e.g., from the Tevatron and LHC.

We studied the effects of scheme dependence by applying scheme transformations in a series of papers, incl. Ryttov and RS, PRD 86, 065032, 085005 (2012); RS, PRD 88, 036003 (2013); RS, PRD 90, 045011 (2014); Choi and RS, PRD 90 125029 (2014); PRD 94, 065038 (2016); Ryttov, PRD 89, 016013 (2014); PRD 89, 056001 (2014); PRD 90, 056007 (2014); also J. Gracey, Simms, PRD 91, 085037 (2015); Gracey et al., 2306.09056.

It is valuable to calculate and analyze series expansions for physical quantities such as anomalous dimensions that are scheme-independent at each order.

Since $\alpha_{IR} \to 0$ as $N_f \to N_{f,u}$ and also $\Delta_f = N_{f,u} - N_f$ has the property that $\Delta_f \to 0$ as $N_f \to N_{f,u}$, one can alternatively express these quantities as power series in Δ_f rather than α (Banks-Zaks). Note that $\Delta_f = 3b_1/(4T_f)$.

Because Δ_f depends only on the group G, the rep. R, and the number N_f, these power series are obviously scheme-independent (at each order).

This scheme-independent series expansion is

$$\gamma_{\bar{\psi}\psi,IR} = \sum_{j=1}^{\infty} \kappa_j \Delta_f^j$$

We denote the truncation of the above series to maximal order (power) j as $\gamma_{\bar{\psi}\psi,IR,\Delta_f^j}$.

The calculation of κ_j requires, as inputs, the values of the b_ℓ for $1 \le \ell \le j+1$ and the c_ℓ for $1 \le \ell \le j$. It It may also provide a rough guide for anomalous dimensions in quasi-conformal theories that are close to the lower edge of the conformal window.

Define a denominator factor $D = 7C_A + 11C_f$. The first two κ_j are

$$\kappa_1 = \frac{8C_f T_f}{C_A D} \, ,$$

$$\kappa_2 = \frac{4C_f T_f^2 (5C_A + 88C_f)(7C_A + 4C_f)}{3C_A^2 D^3} \, ,$$

and similarly for κ_3 (Ryttov, PRL 117, 071601 (2016). In Ryttov and RS, PRD 94, 105014 (2016) we calculated κ_4 and hence $\gamma_{\bar{\psi}\psi,IR}$ to $O(\Delta_f^4)$ for SU(3).

In Ryttov and RS, PRD 95, 085012 (2017); PRD 95, 105004 (2017); PRD 96, 105015 (2017) we carried out these scheme-independent series expansions of $\gamma_{\bar{\psi}\psi,IR}$ for an arbitrary gauge group G and fermion representation R up to $O(\Delta_f^4)$ and analyzed them in detail for specific groups and reps. We used b_5 from Baikov, Chetyrkin, and Kühn, PRL 118, 082002 (2017); JHEP 04 (2017) 119 and the Vermaseren group, Herzog et al., JHEP 02 (2017) 090, together with earlier calculations of c_4 by Chetyrkin, and by Vermaseren, Larin, and van Ritbergen.

Our result for κ_4:

$$
\begin{aligned}
\kappa_4 = \frac{T_f^2}{3^5 C_A^5 D^7}\Bigg[& C_A C_f T_f^2 \Big(19515671 C_A^6 - 131455044 C_A^5 C_f + 1289299872 C_A^4 C_f^2 + 2660221312 C_A^3 C_f^3 \\
& + 1058481072 C_A^2 C_f^4 + 6953709312 C_A C_f^5 + 1275715584 C_f^6\Big) \\
& + 2^{10} C_f T_f^2 D \Big(5789 C_A^2 - 4168 C_A C_f - 6820 C_f^2\Big)\frac{d_A^{abcd} d_A^{abcd}}{d_A} \\
& - 2^{10} C_A C_f T_f D \Big(41671 C_A^2 - 125477 C_A C_f - 53240 C_f^2\Big)\frac{d_R^{abcd} d_A^{abcd}}{d_A} \\
& - 2^8 \cdot 11^2 C_A^2 C_f D (2569 C_A^2 + 18604 C_A C_f - 7964 C_f^2\Big)\frac{d_R^{abcd} d_R^{abcd}}{d_A} \\
& - 2^{14} \cdot 3 C_A T_f^2 D^3 \frac{d_R^{abcd} d_A^{abcd}}{d_R} + 2^{13} \cdot 33 C_A^2 T_f D^3 \frac{d_R^{abcd} d_R^{abcd}}{d_R} \\
& + 2^8 D \Big[- 3 C_A C_f T_f^2 D \Big(4991 C_A^4 - 17606 C_A^3 C_f + 33240 C_A^2 C_f^2 - 30672 C_A C_f^3 + 9504 C_f^4\Big) \\
& - 2^4 C_f T_f^2 \frac{d_A^{abcd} d_A^{abcd}}{d_A}\Big(17206 C_A^2 - 60511 C_A C_f - 45012 C_f^2\Big) \\
& + 40 C_A C_f T_f \frac{d_R^{abcd} d_A^{abcd}}{d_A}\Big(35168 C_A^2 - 154253 C_A C_f - 88572 C_f^2\Big) \\
& - 88 C_A^2 C_f \frac{d_R^{abcd} d_R^{abcd}}{d_A}\Big(973 C_A^2 - 93412 C_A C_f - 56628 C_f^2\Big) \\
& + 1440 C_A T_f^2 D^2 \frac{d_R^{abcd} d_A^{abcd}}{d_R} - 7920 C_A^2 T_f D^2 \frac{d_R^{abcd} d_R^{abcd}}{d_R} \Big]\zeta_3
\end{aligned}
$$

$$+\frac{4505600 C_A C_f D^2}{d_A}\Big[-4T_f^2 d_A^{abcd} d_A^{abcd} + 2T_f d_R^{abcd} d_A^{abcd}(10C_A + 3C_f) + 11C_A d_R^{abcd} d_R^{abcd}(C_A - 3C_f)\Big]\zeta_5\Bigg]$$

where $(a,\ b,\ c,\ d$ are group indices)

$$d_R^{abcd} = \frac{1}{3!}\,\mathrm{Tr}_R\Big[T^a_{(R)}\Big(T^b_{(R)}T^c_{(R)}T^d_{(R)} + cycl.\Big)\Big]$$

$d_A^{abcd} = d_R^{abcd}$ for $R = adj$, $d_R = \dim(R)$, and $\zeta_s = \sum_{n=1}^{\infty}\frac{1}{n^s}$ is the Riemann zeta function.

For $G = \mathrm{SU}(N_c)$ and $R = F$, our results for general G and R reduce to

$$\kappa_{1,F} = \frac{4(N_c^2-1)}{N_c(25N_c^2-11)}, \qquad \kappa_{2,F} = \frac{4(N_c^2-1)(9N_c^2-2)(49N_c^2-44)}{3N_c^2(25N_c^2-11)^3}$$

$$\begin{aligned}\kappa_{3,F} \;=\; & \frac{8(N_c^2-1)}{3^3N_c^3(25N_c^2-11)^5}\Big[\Big(274243N_c^8 - 455426N_c^6 - 114080N_c^4 + 47344N_c^2 + 35574\Big) \\ & - \;4224N_c^2(4N_c^2-11)(25N_c^2-11)\zeta_3\;\Big]\end{aligned}$$

$$\begin{aligned}\kappa_{4,F} \;=\; & \frac{4(N_c^2-1)}{3^4N_c^4(25N_c^2-11)^7}\Big[\Big(263345440N_c^{12} - 673169750N_c^{10} + 256923326N_c^8 \\ & - \;290027700N_c^6 + 557945201N_c^4 - 208345544N_c^2 + 6644352\Big) \\ & + \;384(25N_c^2-11)\Big(4400N_c^{10} - 123201N_c^8 + 480349N_c^6 \\ & - \;486126N_c^4 + 84051N_c^2 + 1089\Big)\zeta_3 \\ & + \;211200N_c^2(25N_c^2-11)^2(N_c^6+3N_c^4-16N_c^2+22)\zeta_5\;\Big]\end{aligned}$$

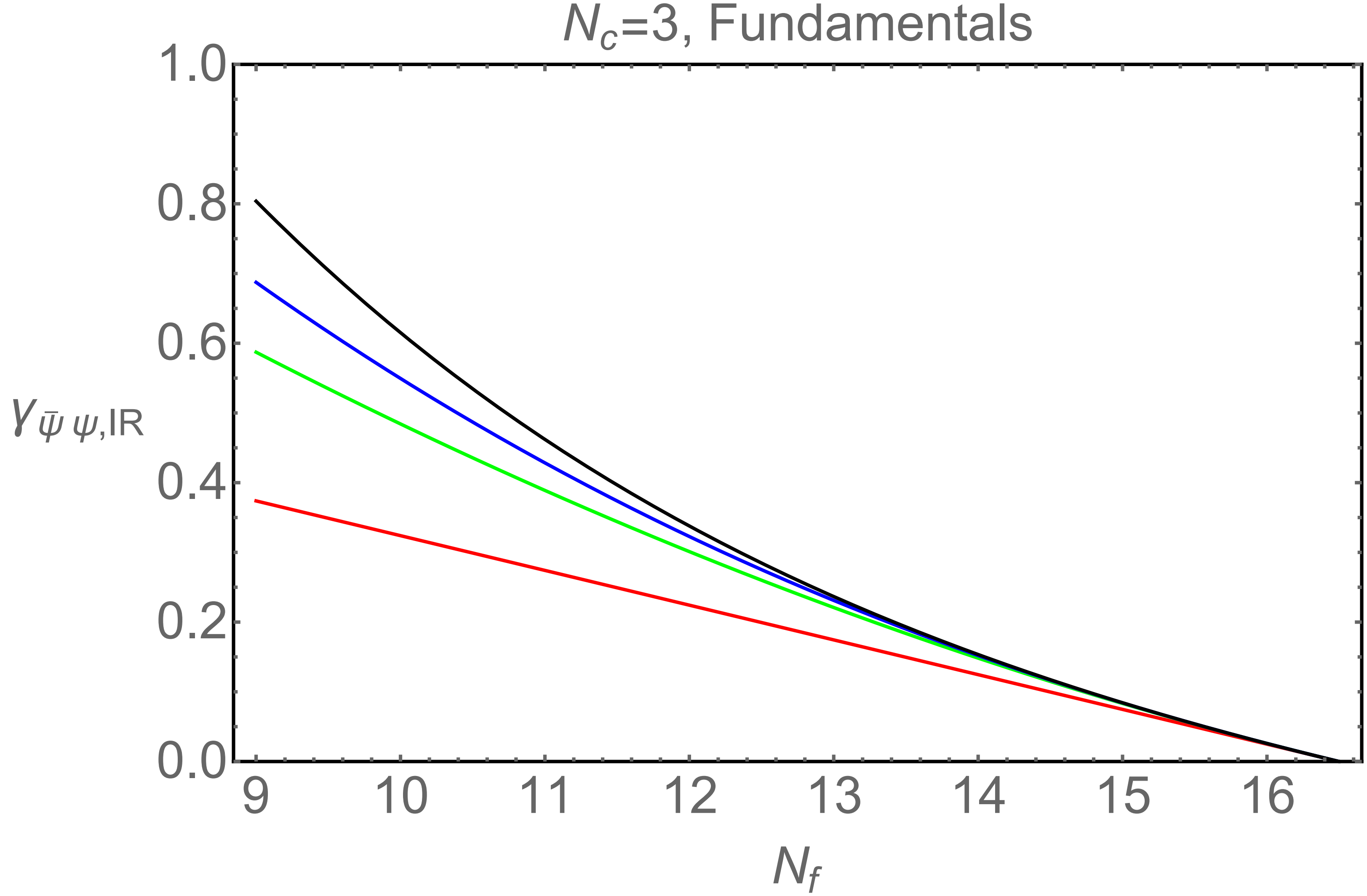
Plot of $\gamma_{\bar\psi\psi,IR,\Delta_f^j}$ with $1 \le j \le 4$ for SU(3) and fermion rep. $R = F$, as functions of $N_f \in I$ from Ryttov and RS, PRD 94, 105014 (2016) [1608.00068]. Curves: $\gamma_{\bar\psi\psi,IR,F,\Delta_f}$ (red), $\gamma_{\bar\psi\psi,IR,F,\Delta_f^2}$ (green), $\gamma_{\bar\psi\psi,IR,F,\Delta_f^3}$ (blue), $\gamma_{\bar\psi\psi,IR,F,\Delta_f^4}$ (black).

As one moves down from the upper end of the conformal window, $\gamma_{\bar\psi\psi,IR}$ increases. Approximate analysis of Schwinger-Dyson eq. for fermion propagator suggests SχSB occurs at $\gamma_{\bar\psi\psi,IR} = 1$, which thus determines the lower boundary $\mathcal{B}_{CW,\ell}$ of the conformal window (Appelquist et al (1988); Cohen and Georgi (1989))

A rigorous bound in the conformal window is $\gamma_{\bar\psi\psi,IR} < 2$ (Mack, 1977), but this need not be saturated.

Extrapolations of these $O(\Delta_f^4)$ results to $O(\Delta_f^j)$ with $\lim_{j\to\infty}$ given in Ryttov-RS, PRD 94, 105014 (2016). Combining this extrapolation with the $\gamma_{IR} = 1$ condition yields $9 < N_{f,cr} < 10$.

This agrees with extensive lattice simulations of the SU(3), $N_F = 8$ theory (LSD, LatKMI, LatHC...), which indicate that it is slightly below $\mathcal{B}_{CW,\ell}$ and with Hasenfratz et al., 2306.07236, who find that the SU(3) $N_f = 10$ theory is in the CW.



N_c=3, Fundamentals
1.0
0.8
0.6
0.4
0.2
0.0
$\gamma_{\bar{\psi}\psi,\mathrm{IR}}$
9
10
11
12
13
14
15
16
N_f

N_c	N_f	γ_{IR,F,Δ_f}	γ_{IR,F,Δ_f^2}	γ_{IR,F,Δ_f^3}	γ_{IR,F,Δ_f^4}	$\gamma_{IR,F,ext}$
3	8	0.424	0.698	0.844	1.036	—
3	9	0.374	0.587	0.687	0.804	1.4(2)
3	10	0.324	0.484	0.549	0.615	0.95(6)
3	11	0.274	0.389	0.428	0.462	0.62(2)
3	12	0.224	0.301	0.323	0.338	0.400(5)
3	13	0.174	0.221	0.231	0.237	0.257(5)
3	14	0.125	0.148	0.152	0.153	0.154(4)
3	15	0.0748	0.0833	0.0841	0.0843	0.0841(2)
3	16	0.0249	0.0259	0.0259	0.0259	0.0259(1)

Values of $\gamma_{\bar{\psi}\psi,IR,\Delta_f^j} = \gamma_{IR,\Delta_f^j}$ with $1 \le j \le 4$ for SU(2), SU(3), and $R = F$. Last column shows extrapolations to $j \to \infty$, denoted $\gamma_{IR,F,ext}$.

In our papers we discussed the accuracy of these finite order calculations and resultant γ_{IR,Δ_f^j} values. A rough estimate can be obtained from the figures.

Where the curves for the γ_{IR,Δ_f^j} with different j are close to each other, higher-order terms are expected to be small. As N_f decreases, these curves deviate progressively more from each other, and higher-order terms are more important.

Additional estimates of effects of higher-order terms were obtained via calculation and analysis of Padé approximants, e.g., in Ryttov-RS, PRD 97, 025004 (2018).

The approximate analysis of the Schwinger-Dyson eq. (Appelquist, Lane, Mahanta, 1988) also suggested a quadratic criticality condition γCC, $\gamma_{\bar{f}f,IR}(2-\gamma_{\bar{f}f,IR})=1$ for the onset of SχSB. This eq. has a double root at $\gamma_{\bar{f}f,IR}=1$ and hence is formally equivalent to the linear γCC, $\gamma_{\bar{f}f,IR}=1$.

When applied in the context of a truncated series expansions, the quadratic γCC yields a slightly larger value of $N_{f,cr}$ at $\mathcal{B}_{CW,\ell}$ than the linear γCC (B. S. Kim, D. K. Hong, and J.-W. Lee, PRD 101, 056008 (2020); J.-W. Lee, PRD 103, 076006 (2021); J.-W. Lee, talk at this conf.) This difference decreases as the order $O(\Delta_f^j)$ increases.

For SU(3), with our $O(\Delta_f^4)$ order scheme-independent inputs, the quadratic γCC condition gives $9 < N_{f,c} < 10$, in agreement with our extrapolation in Ryttov-RS, PRD 94, 105014 (2016). The CFT bound $\gamma < 2$ would give $8 < N_{f,c} < 9$, also in agreement with lattice simulations.

We also calculated anomalous dimensions for other operators, including higher-spin fermion bilinears.

One such operator is the Lorentz tensor bilinear $\mathcal{O}_{T,\mu\nu} = \bar{\psi}\sigma_{\mu\nu}\psi$, with anom. dim. $\gamma_{T,IR}$ at the IRFP and scheme-independent series expansion

$$\gamma_{T,IR} = \sum_{j=1}^{\infty} \kappa_{T,j}\, \Delta_f^j$$

with truncation to $O(\Delta_f^j)$ denoted γ_{T,IR,Δ_f^j}.

For a general G and R, using the highest-order inputs available, we calculated $\gamma_{T,IR}$ up to $O(\Delta_f^3)$ in Ryttov-RS, PRD 94, 125005 (2016).

For the coefficients $\kappa_{T,j}$ in the scheme-independent expansions of these anomalous dimensions for the Lorentz tensor fermion bilinear, we obtain

$$\kappa_{T,1} = -\frac{8C_f T_f}{3C_A D}$$

$$\kappa_{T,2} = -\frac{4C_f T_f^2(259C_A^2 + 428C_A C_f - 528C_f^2)}{9C_A^2 D^3}$$

$$\begin{aligned}
\kappa_{T,3} \;=\; & \frac{4C_f T_f}{3^5 C_A^4 D^5}\Big[3C_A T_f^2\Big\{C_A^4(-11319 + 188160\zeta_3) + C_A^3 C_f(-337204 + 64512\zeta_3) + C_A^2 C_f^2(83616 - 890112\zeta_3) \\
+ \; & C_A C_f^3(1385472 - 354816\zeta_3) + C_f^4(-212960 + 743424\zeta_3)\Big\} - 512T_f^2 D(-5 + 132\zeta_3)\frac{d_A^{abcd} d_A^{abcd}}{d_A} \\
- \; & 15488C_A^2 D(-11 + 24\zeta_3)\frac{d_R^{abcd} d_R^{abcd}}{d_A} + 11264C_A T_f D(-4 + 39\zeta_3)\frac{d_R^{abcd} d_A^{abcd}}{d_A}\Big] \,.
\end{aligned}$$

Note that in contrast to the κ_j for $\gamma_{\bar\psi\psi,IR}$, which are positive for $1 \le j \le 4$, here for SU(3), $R = F$, $\kappa_{F,1}$ and $\kappa_{F,2}$ are negative, while $\kappa_{F,3}$ is positive:

$$\kappa_{T,\mathrm{SU(3)},F,1} = -(1.6615 \times 10^{-2}) \,, \qquad \kappa_{T,\mathrm{SU(3)},F,2} = -(1.12625 \times 10^{-3}) \,,$$

$$\kappa_{T,\mathrm{SU(3)},F,3} = 2.480155 \times 10^{-5}$$

For SU(3) and $R = F$, fundamental rep., with $\gamma_{T,IR} \equiv \gamma^{(\sigma)}_{IR,F}$, these give the following anomalous dimensions as a function of N_F:

N_f	$\gamma^{(\sigma)}_{IR,F,\Delta_f}$	$\gamma^{(\sigma)}_{IR,F,\Delta_f^2}$	$\gamma^{(\sigma)}_{IR,F,\Delta_f^3}$
8	-0.141	-0.223	-0.207
9	-0.125	-0.188	-0.1775
10	-0.108	-0.156	-0.149
11	-0.0914	-0.125	-0.121
12	-0.0748	-0.0976	-0.0953
13	-0.05815	-0.07195	-0.0709
14	-0.0415	-0.0486	-0.0482
15	-0.0249	-0.0275	-0.0274
16	-0.00831	-0.00859	-0.00859

Further analysis for other higher-spin operators in Ryttov-RS, PRD 101, 076018 (2020).

Another operator of interest is $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$, whose anom. dim. at the IRFP is given by $\beta'_{IR} = (d\beta/d\alpha)_{IR}$. Calculations in RS, PRD 87, 105005 (2013); Ryttov-RS, PRD 94, 125005 (2017); PRD 05, 105004 (2017) to $O(\Delta_f^5)$. Here we focus on anom. dims. of fermion bilinears.

It is of interest to compare our perturbative calculations of anomalous dimensions with lattice measurements.

In previous work we have done this for several theories for which there have been extensive simulations, such as SU(3) with $N_F = 12$ fermions in the fundamental rep., SU(3) with 2 fermions in the symmetric tensor rep., SU(2) with various reps. R.

For various G, R, and N_f, there is not yet a complete consensus as to whether a given theory is inside or outside of the conformal window.

Here we focus on new results on SU(3) with $N_F = 10$ fermions in the fundamental rep. Previous studies include the following:

Appelquist et al. (LSD Collab.), arXiv:1204.6000 early study

Appelquist et al. (LSD Collab.), PRD 103, 014504 (2021) find that this theory has an IRFP and hence is in the conformal window, and measure $\gamma_{\bar{\psi}\psi,IR} = 0.47 \pm 0.05$

Z. Fodor, K. Holland, J. Kuti, D. Nogradi, Wong, PoS, Lattice 2018, [1812.03972]; PoS, Lattice 2019 [1912.07653]; Z. Fodor, K. Holland, J. Kuti, Wong, PoS, Lattice 2021 [2203.15847] find that this theory is in the chirally broken phase (Kuti, talk at this conf.)

See also T.-W. Chiu, 1603.08854 and PRD 99, 014507 for study of β function and β'_{IR}

recently: Hasenfratz, Neil, Shamir, Svetitsky, 2306.07236 (A. Hasenfratz, talk at this conf.) find that this theory has an IRFP and hence is in the conformal window, and measure $\gamma_{\bar\psi\psi,IR}$ and $\gamma_{T,IR}$ to be

$$\gamma_{\bar\psi\psi,IR} \simeq 0.6 \ , \quad \gamma_{T,IR} \simeq -0.2$$

To within the uncertainties in our scheme-independent perturbative calculations of $\gamma_{\bar\psi\psi,IR}$ to $O(\Delta_F^4)$ and $\gamma_{T,IR}$ to $O(\Delta_F^3)$, they agree with these measurements:

$$\gamma_{\bar\psi\psi,IR,\Delta_F} = 0.324 \ , \qquad \gamma_{\bar\psi\psi,IR,\Delta_F^2} = 0.484$$

$$\gamma_{\bar\psi\psi,IR,\Delta_F^3} = 0.549 \ , \qquad \gamma_{\bar\psi\psi,IR,\Delta_F^4} = 0.615$$

$$\gamma_{T,IR,\Delta_F} = -0.108 \ , \qquad \gamma_{T,IR,\Delta_F^2} = -0.156 \ , \qquad \gamma_{T,IR,\Delta_F^3} = -0.149$$

Theories with Fermions in Multiple Different Representations

We generalized our scheme-independent calculations of anomalous dimensions to asymptotically free theories with fermions in multiple different representations in Ryttov-RS, PRD 98, 096003 (2018), giving results for an an arbitrary nonabelian gauge group G with (massless) fermions f in rep. R and f' in rep. R' of G. A generalized 't Hooft-Veneziano limit was studied in Girmohanta, Ryttov, RS, PRD 99, 116022 (2019). Further studies in B. S. Kim, D. K. Hong, J.-W. Lee, PRD 101, 056008 (2020); J.-W. Lee, PRD 103, 076006 (2021).) Here we report new results from Ryttov and RS, 2307.12426.

Here, asymptotic freedom (AF) condition is $b_1 > 0$ where $b_1 = (1/3)[11C_A - 4N_fT_f - 4N_{f'}T_{f'}]$. The eq. $b_1 = 0$ is the upper boundary $\mathcal{B}_{IRZ,u} = \mathcal{B}_{CW,u}$ of the IRZ region and conformal window. The resultant upper bounds on N_f and $N_{f'}$ from AF are $N_f < N_{f,u}$ and $N_{f'} < N_{f',u}$, where

$$N_{f,u} = \frac{11C_A - 4N_{f'}T_{f'}}{4T_f}\,, \qquad N_{f',u} = \frac{11C_A - 4N_fT_f}{4T_{f'}}$$

The scheme-independent expansion parameters are

$$\Delta_f = N_{f,u} - N_f = \frac{3b_1}{4T_f}, \qquad \Delta_{f'} = N_{f',u} - N_{f'} = \frac{3b_1}{4T_{f'}}$$

so $\Delta_{f'} = (T_f/T_{f'})\Delta_f$.

Scheme-independent series expansions of anom. dims at the IRFP are

$$\gamma_{\bar{f}f,IR} = \sum_{j=1}^{\infty} \kappa_j^{(f)} \, \Delta_f^j \, , \qquad \gamma_{\bar{f}'f',IR} = \sum_{j=1}^{\infty} \kappa_j^{(f')} \, \Delta_{f'}^j$$

Denote truncations of these series to order j as $\gamma_{\bar{f}f,IR,\Delta_f^j}$ and $\gamma_{\bar{f}'f',IR,\Delta_{f'}^j}$.

Define the denominator factor $\mathcal{D}_f = C_A(7C_A + 11C_f) + 4N_{f'}T_{f'}(C_{f'} - C_f)$.

For $\kappa_j^{(f)}$, $j = 1, 2, 3$ we obtained $\kappa_1^{(f)} = 8C_fT_f/\mathcal{D}_f$,

$$\kappa_2^{(f)} = \frac{4C_fT_f^2}{3\mathcal{D}_f^3}\bigg[C_A(7C_A + 4C_f)(5C_A + 88C_f) + 2^4N_{f'}T_{f'}(C_{f'} - C_f)\Big(10C_A + 8C_f + C_{f'}\Big)\bigg]$$

$$\kappa_3^{(f)} = \frac{4C_f T_f}{3^4 \mathcal{D}_f^5} \Big[A_0^{(f)} + A_1^{(f)} N_{f'} + A_2^{(f)} N_{f'}^2 + A_3^{(f)} N_{f'}^3 \Big]$$

where $A_0^{(f)}$, $A_1^{(f)}$, $A_2^{(f)}$, and $A_3^{(f)}$ are more complicated functions (given in our paper).

The $\kappa^{(f')}$ are obtained from these $\kappa_j^{(f)}$ by interchanging f and f' in all expressions.

A particular theory of interest has an SU(4) gauge group and (massless) Dirac fermion content consisting of N_F fermions in the fundamental (F) rep. and N_{A_2} fermions in the antisymmetric rank-2 tensor rep. (A_2). In this theory, the (6-dim.) A_2 rep. is self-conjugate, so N_{A_2} Dirac fermions are equivalent to $2N_{A_2}$ Majorana fermions. This theory is motivated as a model of dynamical electroweak symmetry breaking (EWSB) that addresses the issue of the large top quark mass.

There have been many earlier efforts at dynamical EWSB models, e.g. Weinberg, PRD 19, 1277 (1979); Susskind, PRD 20, 2619 (1979); Dimopoulos and Susskind, NPB 155, 23, (1979); Eichten and Lane PLB 90, 125 (1980). Early models for a heavy top quark include Hill, PLB 266, 419 (1991); Kaplan, NPB 365, 259 (1991); Appelquist and Terning, PRD 50, 2116 (1994); Lane and Eichten, PLB 352, 382 (1995); Chivukula, Dobrescu, and Terning, PLB 353, 289 (1995); Chivukula and Simmons, PRD 66, 015006 (2002) among others.

Reasonably UV-complete theories of this type involved sequential breaking of an asymptotically free chiral gauge theory in stages, leading to a vectorial gauge theory that becomes strongly coupled at the TeV scale. The sequential breaking provided a way of explaining the hierarchical structure of the SM fermion generations with a low-scale seesaw for neutrino masses: Appelquist and RS, PLB 548, 204 (2002); Appelquist and RS, PRL 90, 201801 (2003); Appelquist, Piai, and RS, PRD 69 015002 (2004); Christensen and RS, PRL 94, 241801 (2004) (challenge of getting t-b mass splitting and still satisfying precision EW constraints). See also Ferretti and Karateev, JHEP 03 (2014) 077. Another approach assumed a higher-dimensional spacetime, with SM fermions having wave functions in the extra dimensions that are strongly localized: Arkani-Hamed and Schmaltz, PRD 61, 033005 (2000); Nussinov and RS, PLB 526, 137 (2002).

In the 2018 Ryttov-RS paper we noted that lattice simulations had been performed of an SU(4) theory with Dirac fermion content $N_F = 2$ and $N_{A_2} = 2$ by Ayyar, DeGrand, et al., PRD 97, 074505, 114505 (2018), but this theory was found to be in the chirally broken phase where our calculations do not apply directly.

Recently, in PRD 107, 114504 (2023) [2304.11729], Hasenfratz, Neil, Shamir, Svetitsky, and Witzel have reported results from lattice simulations of the SU(4) theory with $N_F = 4$ and $N_{A_2} = 4$ Dirac fermions (c.f. talk by Y. Shamir at this conf.). These authors find that this theory has an IRFP and hence is in the conformal window, and measure

$$\gamma_m^{(4)} \simeq 0.75 \ , \qquad \gamma_m^{(6)} \simeq 1.0$$

An interesting question is whether our general higher-order perturbative calculations of anomalous dimensions of fermion bilinears, when specialized to this theory, yield results in agreement with the values measured in this recent lattice study by Hasenfratz et al.

We address and answer this question in Ryttov-RS 2307.12426. We find agreement.

It is instructive to give results for the more general case of an SU(N_c) theory with massless Dirac fermion content consisting of N_F fermions in the F rep. and N_{A_2} fermions in the A_2 rep. We label an SU(N_c) theory with N_F and N_{A_2} Dirac fermions as (N_c, N_F, N_{A_2})

Denote the F and A_2 fermion as ψ_i^a and $\chi_j^{ab} = -\chi_j^{ba}$, where a, b are SU(N_c) gauge indices and the flavor indices are $i = 1, ..., N_F$ and $j = 1, ..., N_{A_2}$. We calculate anomalous dimensions of the operators

$$\bar{\psi}\psi = \sum_{i=1}^{N_f} \bar{\psi}_{a,i}\psi_i^a \ , \qquad \bar{\chi}\chi = \sum_{j=1}^{N_{A_2}} \bar{\chi}_{ab,j}\chi_j^{ab}$$

The scheme-independent expansion variables are

$$\Delta_F = \frac{11}{2}N_c - N_F - (N_c - 2)N_{A_2}$$

$$\Delta_{A_2} = \frac{11N_c - 2N_F - 2(N_c - 2)N_{A_2}}{2(N_c - 2)}$$

N.B.: $\Delta_{A_2} = \frac{T_F}{T_{A_2}}\Delta_F = \frac{\Delta_F}{N_c - 2}$

The scheme-independent expansions of the anomalous dimensions are

$$\gamma_{\bar{\psi}\psi,IR} = \sum_{j=1}^{\infty} \kappa_j^{(F)} \Delta_F^j \ , \qquad \gamma_{\bar{\chi}\chi,IR} = \sum_{j=1}^{\infty} \kappa_j^{(A_2)} \Delta_{A_2}^j$$

Truncations of these series to order j are denoted $\gamma_{\bar{\psi}\psi,IR,\Delta_F^j}$ and $\gamma_{\bar{\chi}\chi,IR,\Delta_{A_2}^j}$.

Notational equivalence for SU(4): $\gamma_{\bar{\psi}\psi,IR} \equiv \gamma_m^{(4)}$ and $\gamma_{\bar{\chi}\chi,IR} \equiv \gamma_m^{(6)}$.

Define denominator factors

$$\mathcal{D}_F = N_c(25N_c^2 - 11) + 2N_{A_2}(N_c - 2)(N_c + 1)(N_c - 3)$$

$$\mathcal{D}_{A_2} = N_c(18N_c^2 - 11N_c - 22) - N_F(N_c - 3)(N_c + 1) \ .$$

We find

$$\kappa_1^{(F)} = \frac{4(N_c^2 - 1)}{\mathcal{D}_F} \ , \qquad \kappa_1^{(A_2)} = \frac{4(N_c - 2)^2(N_c + 1)}{\mathcal{D}_{A_2}}$$

$$\kappa_2^{(F)} = \frac{4(N_c^2 - 1)}{3\mathcal{D}_F^3}\left[N_c(9N_c^2 - 2)(49N_c^2 - 44) + 4N_{A_2}(N_c - 2)(N_c + 1)(N_c - 3)(3N_c - 2)(5N_c + 3)\right]$$

$$\kappa_2^{(A_2)} = \frac{(N_c - 2)^3(N_c + 1)}{3\mathcal{D}_{A_2}^3}\left[N_c(11N_c^2 - 4N_c - 8)(93N_c^2 - 88N_c - 176) - 2N_F(N_c - 3)(N_c + 1)(37N_c^2 - 16N_c - 33)\right]$$

$$\kappa_3^{(F)} = \frac{8(N_c^2 - 1)}{27\mathcal{D}_F^5}\left[A_0^{(F)} + A_1^{(F)}N_{A_2} + A_2^{(F)}N_{A_2}^2 + A_3^{(F)}N_{A_2}^3\right]$$

$$\kappa_3^{(A_2)} = \frac{(N_c - 2)^3(N_c + 1)}{54\mathcal{D}_{A_2}^5}\left[A_0^{(A_2)} + A_1^{(A_2)}N_F + A_2^{(A_2)}N_F^2 + A_3^{(A_2)}N_F^3\right]$$

where

$$A_0^{(F)} = N_c^2\Big[\Big(274243N_c^8-455426N_c^6-114080N_c^4+47344N_c^2+35574\Big)-4224N_c^2(4N_c^2-11)(25N_c^2-11)\zeta_3\Big] ,$$

$$\begin{aligned} A_1^{(F)} &= 4N_c(N_c-2)(N_c-3)\Big[\Big(16981N_c^7+35460N_c^6+42927N_c^5+47342N_c^4+9432N_c^3-12849N_c^2 \\ &- 18843N_c-11616\Big)-576N_c^2\Big(25N_c^4+198N_c^3+187N_c^2-121N_c-121\Big)\zeta_3\Big] , \end{aligned}$$

$$\begin{aligned} A_2^{(F)} &= 8(N_c-2)(N_c-3)\Big[\Big(689N_c^8-1402N_c^7-9208N_c^6-15693N_c^5-9219N_c^4+16662N_c^3+19860N_c^2 \\ &+ 10617N_c+5598\Big)-192N_c^2\Big(3N_c^5-65N_c^4-238N_c^3-165N_c^2+231N_c+198\Big)\zeta_3\Big] , \end{aligned}$$

$$A_3^{(F)} = 128N_c(N_c-2)^2(N_c-3)^2(N_c+1)(3N_c^2+7N_c+6)(-11+24\zeta_3) ,$$

$$
\begin{aligned}
A_0^{(A_2)} \;=\;& N_c^2\Big[\Big(1670571N_c^9 - 7671402N_c^8 + 2181584N_c^7 + 25294256N_c^6 - 13413856N_c^5 \\
-\;& 17539136N_c^4 + 16707328N_c^3 + 3046912N_c^2 - 27320832N_c - 18213888\Big) \\
-\;& 8448N_c^2(N_c+2)(18N_c^2 - 11N_c - 22)(3N_c^3 - 28N_c^2 + 176)\zeta_3\Big]
\end{aligned}
$$

$$
\begin{aligned}
A_1^{(A_2)} \;=\;& -4N_c(N_c-3)\Big[\Big(60552N_c^8 - 150015N_c^7 - 373894N_c^6 + 138737N_c^5 + 300380N_c^4 \\
+\;& 421197N_c^3 + 768345N_c^2 + 858660N_c + 435468\Big) \\
-\;& 192N_c^2\Big(141N_c^5 - 2075N_c^4 - 6226N_c^3 + 1056N_c^2 + 17424N_c + 11616\Big)\zeta_3\Big]
\end{aligned}
$$

$$
\begin{aligned}
A_2^{(A_2)} \;=\;& 8(N_c-3)\Big[\Big(1148N_c^8 - 3919N_c^7 - 17365N_c^6 - 5724N_c^5 + 35724N_c^4 + 84915N_c^3 + 70641N_c^2 \\
+\;& 32928N_c + 15588\Big) - 192N_c^2\Big(3N_c^5 - 164N_c^4 - 271N_c^3 + 396N_c^2 + 1320N_c + 792\Big)\zeta_3\Big]
\end{aligned}
$$

$$
A_3^{(A_2)} = -128N_c(N_c+1)(N_c-3)^2(3N_c^2 + 7N_c + 6)(-11 + 24\zeta_3)\;.
$$

Specializing to $N_c = 4$, i.e., the SU(4) theory, we have

$$\kappa_1^{(F)} = \frac{15}{389 + 5N_{A_2}} \;, \qquad \kappa_2^{(F)} = \frac{25(5254 + 115N_{A_2})}{(389 + 5N_{A_2})^3}$$

$$\kappa_1^{(A_2)} = \frac{80}{888 - 5N_F} \;, \qquad \kappa_2^{(A_2)} = \frac{400(19456 - 165N_F)}{(888 - 5N_F)^3}$$

$$\begin{aligned}
\kappa_3^{(F)} &= \frac{5}{36(389 + 5N_{A_2})^5}\Big[(8039476475 - 696689664\zeta_3) + (479848740 - 197766144\zeta_3)N_{A_2} \\
&+ (-16264767 + 46568448\zeta_3)N_{A_2}^2 + (-288640 + 629760\zeta_3)N_{A_2}^3\Big]
\end{aligned}$$

$$\begin{aligned}
\kappa_3^{(A_2)} &= \frac{640}{27(888 - 5N_F)^5}\Big[(28645111296 + 7201751040\zeta_3) - (120552246 + 1055342592\zeta_3)N_F \\
&+ (-12526131 + 33675264\zeta_3)N_F^2 + (72160 - 157440\zeta_3)N_F^3\Big]
\end{aligned}$$

Substituting these into our scheme-independent expansions, we obtain

$$\gamma_{\bar\psi\psi,IR,\Delta_F} = 0.367 \ , \qquad \gamma_{\bar\psi\psi,IR,\Delta_F^2} = 0.576 \ , \qquad \gamma_{\bar\psi\psi,IR,\Delta_F^3} = 0.683$$

$$\gamma_{\bar\chi\chi,IR,\Delta_{A_2}} = 0.461 \ , \qquad \gamma_{\bar\chi\chi,IR,\Delta_{A_2}^2} = 0.748 \ , \qquad \gamma_{\bar\chi\chi,IR,\Delta_{A_2}^3} = 0.942$$

Because $\kappa_j^{(F)}$ and $\kappa_j^{(A_2)}$ are positive for all of the orders $j = 1, 2, 3$ for which we have calculated them, several monotonicity relations follow for these orders:

With fixed $\Delta_F = 2\Delta_{A_2}$, the anom. dims. $\gamma_{\bar\psi\psi,IR,\Delta_F^j}$ and $\gamma_{\bar\chi\chi,IR,\Delta_{A_2}^j}$ are monotonically increasing functions of j.

Second, for a fixed j, $\gamma_{\bar\psi\psi,IR,\Delta_F^j}$ is a monotonically increasing function of Δ_F and $\gamma_{\bar\chi\chi,IR,\Delta_{A_2}^j}$ is a monotonically increasing function of Δ_{A_2}.

Since finite-order perturbative calculations of this type become progressively less accurate as one approaches the lower boundary $\mathcal{B}_{CW,\ell}$ of the conformal window, one should assess the effect of higher-order corrections. From a rough extrapolation (ex) of our results for $j = 1, 2, 3$ to large j, we estimate that these higher orders would yield

$$\gamma_{\bar\psi\psi,IR,ex} \simeq 0.7 - 0.8 \qquad \gamma_{\bar\chi\chi,IR,ex} \simeq 1.0 - 1.1$$

To within the uncertainties in our extrapolation and in the lattice measurements, these results are in agreement with the values $\gamma_m^{(4)} \simeq 0.75$ and $\gamma_m^{(6)} \simeq 1.0$ obtained in Hasenfratz et al., PRD 107, 114504 (2023) [2304.11729]. (Recall notational equivalences in this SU(4) theory: $\gamma_m^{(4)} \equiv \gamma_{\bar\psi\psi,IR}$ and $\gamma_m^{(6)} \equiv \gamma_{\bar\chi\chi,IR}$.) We have also used Padé approximants for estimates.

More generally, we have calculated $\gamma_{\bar\psi\psi,IR,\Delta_F^j}$ and $\gamma_{\bar\chi\chi,IR,\Delta_{A_2}^j}$ for $j = 1, 2, 3$ in the SU(4) theory as functions of N_F and N_{A_2}.

In the figures we show the results on two line segments in I_{IRZ} that intersect at $(N_F, N_{A_2}) = (4, 4)$: with $N_{A_2} = 4$, varying $2 < N_F < 14$ and with $N_F = 4$, varying $3 < N_{A_2} < 9$. Color coding for $f = F,\ A_2$: $\gamma_{\bar f f,IR,\Delta_f}$ (red), $\gamma_{\bar f f,IR,\Delta_f^2}$ (green), $\gamma_{\bar f f,IR,\Delta_f^3}$ (blue).

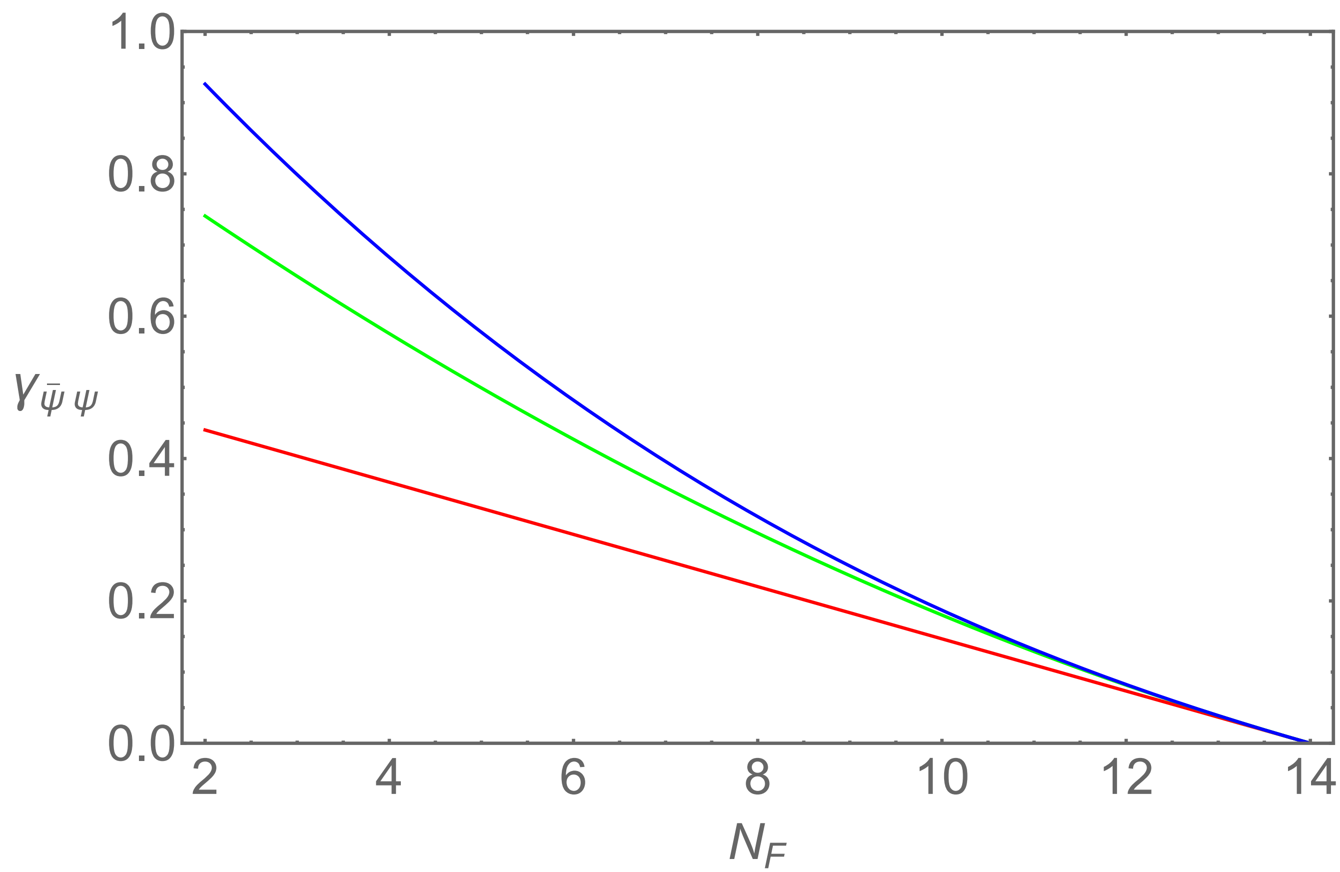


Figure 1: Plot of $\gamma_{\bar{\psi}\psi,IR,\Delta_F^j}$ calculated to order $j = 1, 2, 3$ for $G = \mathrm{SU}(4)$, and $N_{A_2} = 4$, as a function of $N_F \in I_{IRZ}$. From bottom to top, the curves refer to $\gamma_{\bar{\psi}\psi,IR,\Delta_F}$ (red), $\gamma_{\bar{\psi}\psi,IR,\Delta_F^2}$ (green), and $\gamma_{\bar{\psi}\psi,IR,\Delta_F^3}$ (blue).

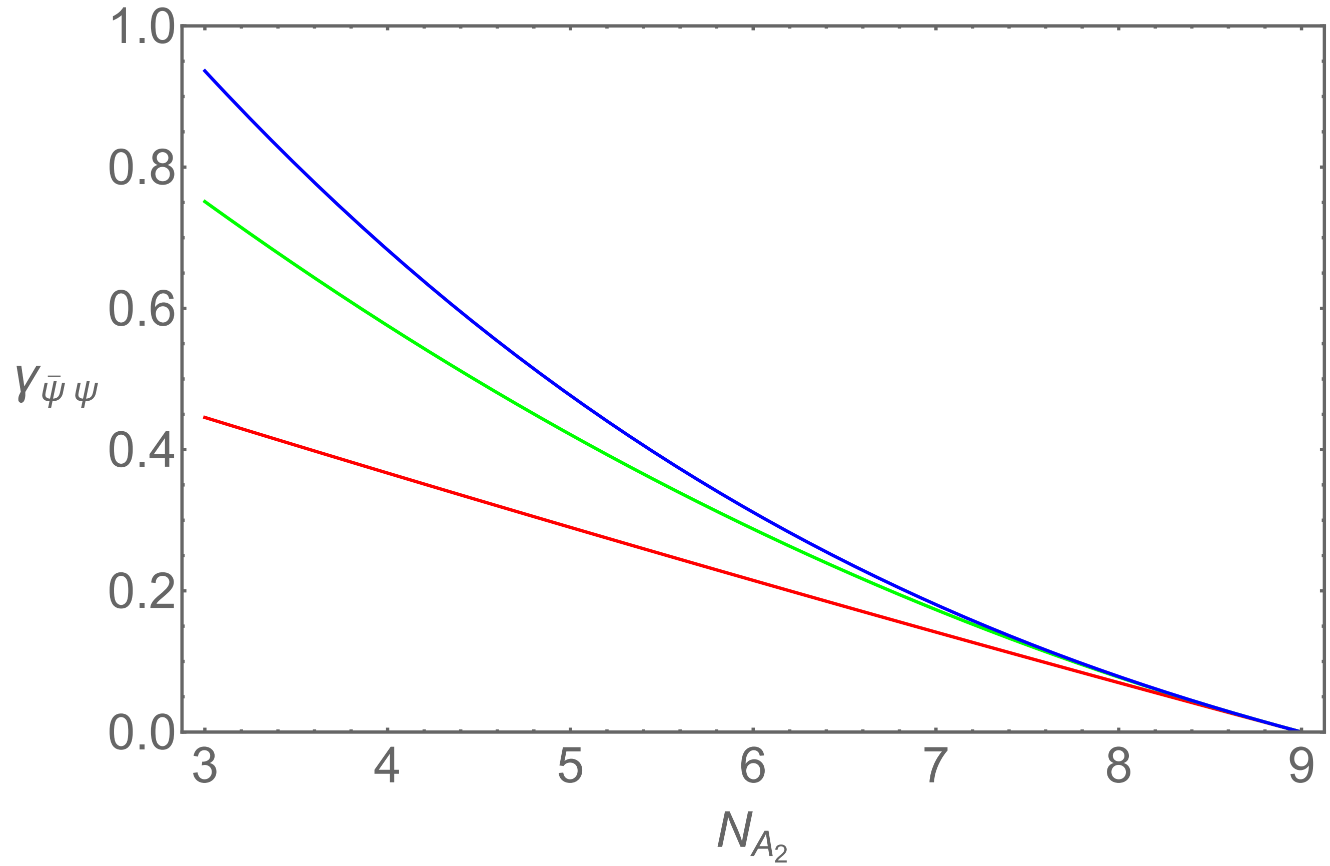


Figure 2: Plot of $\gamma_{\bar{\psi}\psi,IR,\Delta_F^j}$ calculated to order $j = 1, 2, 3$ for $G = \mathrm{SU}(4)$, and $N_F = 4$, as a function of $N_{A_2} \in I_{IRZ}$. From bottom to top, the curves refer to $\gamma_{\bar{\psi}\psi,IR,\Delta_F}$ (red), $\gamma_{\bar{\psi}\psi,IR,\Delta_F^2}$ (green), and $\gamma_{\bar{\psi}\psi,IR,\Delta_F^3}$ (blue).

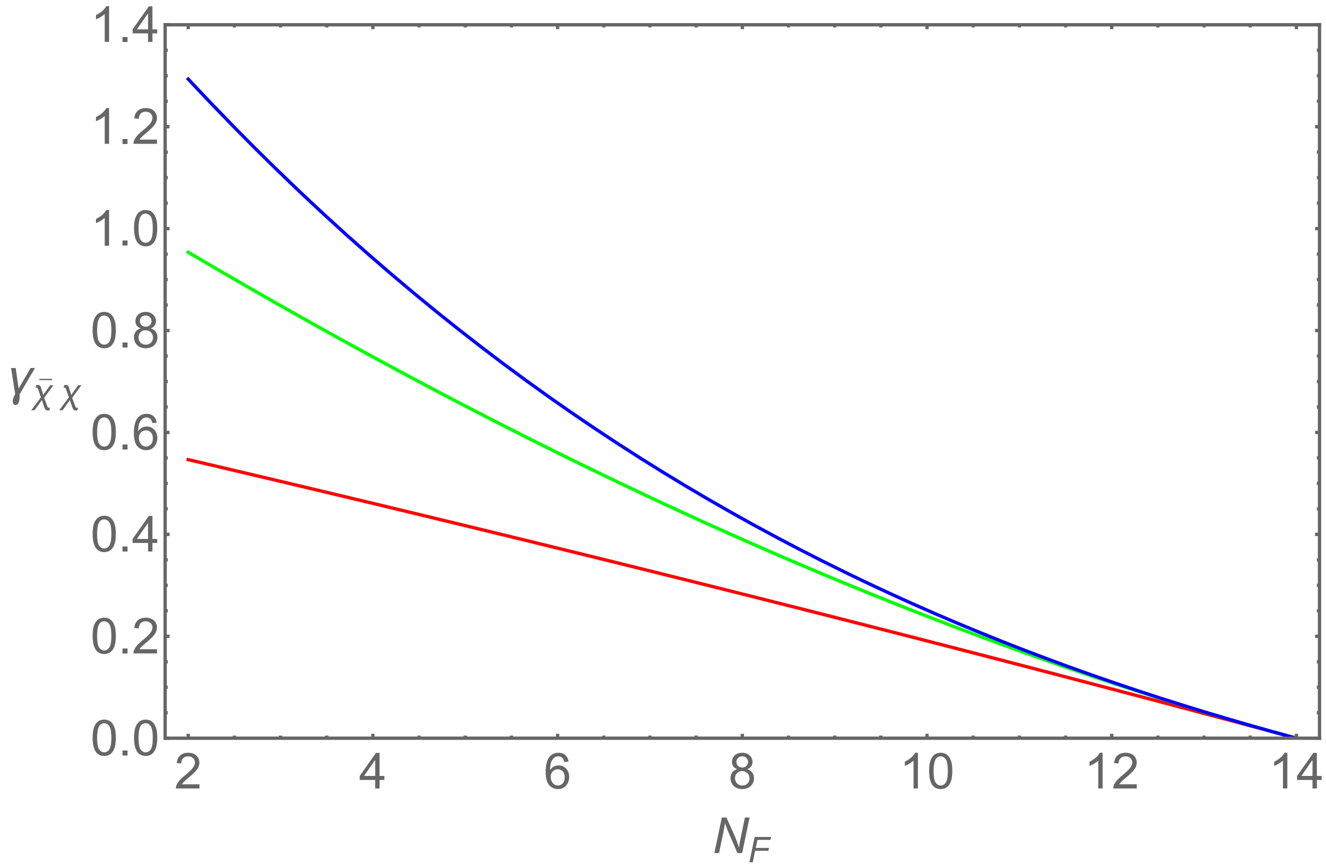


Figure 3: Plot of $\gamma_{\bar{\chi}\chi,IR,\Delta^j_{A_2}}$, calculated to order $j = 1, 2, 3$ for $G = \mathrm{SU}(4)$, and $N_{A_2} = 4$, as a function of $N_F \in I_{IRZ}$. From bottom to top, the curves refer to $\gamma_{\bar{\chi}\chi,IR,\Delta_{A_2}}$ (red), $\gamma_{\bar{\chi}\chi,IR,\Delta^2_{A_2}}$ (green), and $\gamma_{\bar{\chi}\chi,IR,\Delta^3_{A_2}}$ (blue).

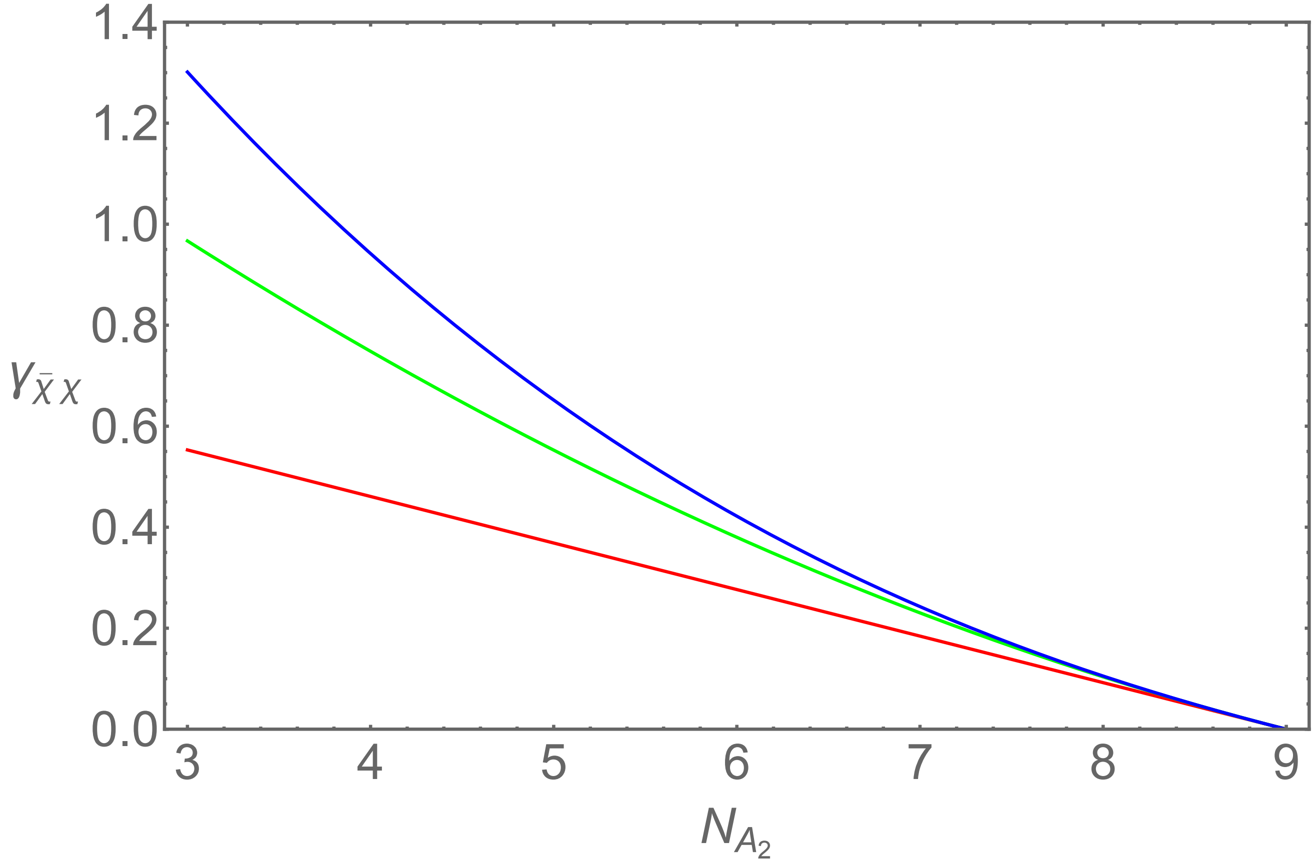


Figure 4: Plot of $\gamma_{\bar{\chi}\chi,IR,\Delta^j_{A_2}}$, calculated to order $j = 1, 2, 3$ for $G = \mathrm{SU}(4)$, and $N_F = 4$, as a function of $N_F \in I_{IRZ}$. From bottom to top, the curves refer to $\gamma_{\bar{\chi}\chi,IR,\Delta_{A_2}}$ (red), $\gamma_{\bar{\chi}\chi,IR,\Delta^2_{A_2}}$ (green), and $\gamma_{\bar{\chi}\chi,IR,\Delta^3_{A_2}}$ (blue).

Hasenfratz et al. find that the $(N_c, N_F, N_{A_2}) = (4, 4, 4)$ theory has an IFRP and hence is in the conformal window, close to the lower boundary, since $\gamma_m^{(6)} \simeq 1$.

As one moves down from the upper boundary $\mathcal{B}_{CW,u}$ of the conformal window toward the lower boundary, $\mathcal{B}_{CW,\ell}$, the anomalous dimensions $\gamma_{\bar\psi\psi,IR}$ and $\gamma_{\bar\chi\chi,IR}$ increase.

The generalization of the condition in the single-rep. theory here is that the lower boundary $\mathcal{B}_{CW,\ell}$ is reached when $\max(\gamma_{\bar\psi\psi,IR},\ \gamma_{\bar\chi\chi,IR}) = 1$. Since $\gamma_{\bar\chi\chi,IR} > \gamma_{\bar\psi\psi,IR}$ here, this condition reduces to

$$\gamma_{\bar\chi\chi,IR} = 1$$

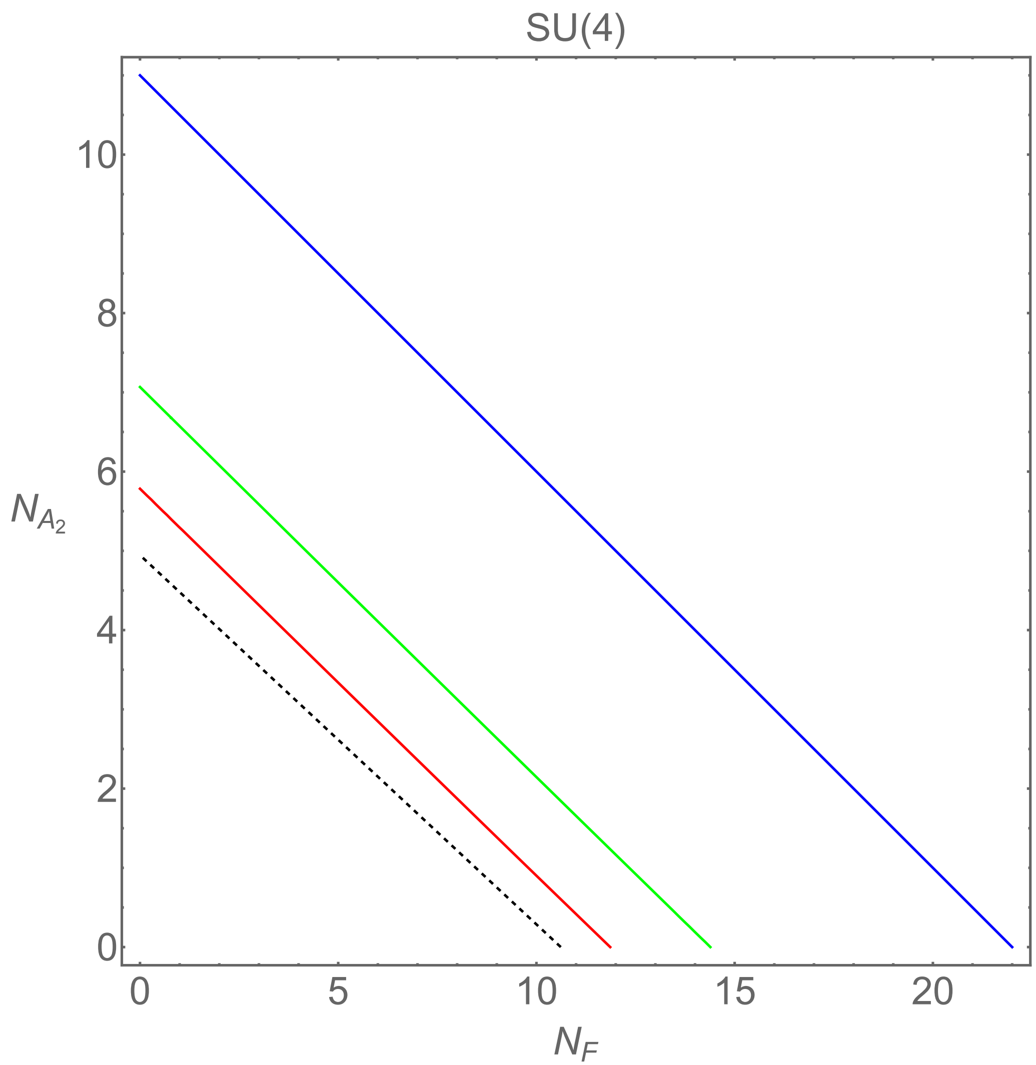
As in the single-rep. theory, the quadratic condition $\gamma_{\bar\chi\chi,IR}(2 - \gamma_{\bar\chi\chi,IR}) = 1$, if solved for exactly, gives a double root at $\gamma_{\bar\chi\chi,IR} = 1$ and hence is equivalent to the linear condition, but when applied in the context of a series expansion calculated to finite order, these yield different results.

See figure. The upper line (colored blue) is the upper boundary of the IRZ region and CW, $\mathcal{B}_{IRZ,u} = \mathcal{B}_{CW,u}$. The locations of the lower CW boundary $\mathcal{B}_{CW,\ell}$ from the quadratic γCC (green) from Kim, Hong, and Lee, PRD 101, 056008 (2020). For comparison, we have calculated $\mathcal{B}_{CW,\ell}$ from the linear γCC (red). These are approximately linear. The lower line (dotted) is the solution of b_2, the lower boundary of the IRZ region.

The boundary $\mathcal{B}_{CW,\ell}$ as calculated with $\kappa_j^{(f)}$ up to $j=3$ order from the linear condition includes the $(N_c, N_F, N_{A_2}) = (4,4,4)$ theory in the conformal window, while $\mathcal{B}_{CW,\ell}$, as calculated with the same $j=3$ inputs for the $\kappa_j^{(f)}$ coefficients excludes the (4,4,4) theory from the conformal window.

Of course, these are both finite-order perturbative calculations. The actual determination of the actual lower CW boundary $\mathcal{B}_{CW,\ell}$ requires a fully nonperturbative calculation, as provided by the lattice simulations.

Anomalous dimensions were also presented in the (4,4,4) theory by Hasenfratz et al. for several gauge-singlet composite fermion operators and were found to be $\lesssim 0.5$. The requisite inputs to compute these with our methods are not yet available, but this could be of interest for future work.


SU(4)
N_{A_2}
N_F
0
2
4
6
8
10
0
5
10
15
20

Higher-loop Studies of the Beta Functions of $|\vec{\phi}|_4^4$ and $|\vec{\phi}|_3^6$ Theories

Here we briefly mention some recent results on scalar field theories from RS, PRD 107, 056018 (2023) [2301.01830] and RS, PRD 107, 096009 (2023) [2302.05422]. These could be of interest to the lattice community.

If the β function of a theory is positive near zero coupling, then this theory is IR-free. As the momentum scale μ increases from the IR toward the UV, the coupling grows. It is of interest to investigate whether an IR-free theory might have a UV zero in the β function, which would be a UV fixed point (UVFP) of the renormalization group.

An example of an IR-free theory with a UVFP is the O(N) nonlinear σ model in $d = 2 + \epsilon$ dims. From an exact solution of this model in the large-N limit we found

$$\beta(\lambda) = \frac{d\lambda}{d\ln\mu} = \epsilon\lambda\left(1 - \frac{\lambda}{\lambda_c}\right), \quad i.e., \quad \beta(\bar{\lambda}) = \frac{d\bar{\lambda}}{d\ln\mu} = \epsilon\bar{\lambda}\left(1 - \frac{\bar{\lambda}}{\bar{\lambda}_c}\right)$$

where λ is an effective coupling, $\bar{\lambda} = \lim_{N\to\infty} \lambda N$, and $\bar{\lambda}_c = 2\pi\epsilon$ with $\epsilon \ll 1$ (W. Bardeen, B. W. Lee, and RS, PRD 14, 985 (1976); also Brézin, Zinn-Justin, PRB 14, 3110 (1976)).

QED is also IR-free. In RS, PRD 89, 045019 (2014) we studied the beta function of a U(1) theory with N_f fermions of charge q up to the 5-loop level, finding evidence against a UVFP. Hence, in this theory, $\alpha(\mu)$ grows as μ increases, eventually exceeding the regime where perturbative calculations are applicable.

The O(N) $\lambda|\vec{\phi}|^4$ theory in $d = 4$ is IR-free. There has long been interest in whether this theory might have a UVFP (some early work: Wilson, 1971; Wilson and Kogut, 1974; Brézin, Le Guillou, Zinn-Justin, 1974; Aizenman, 1982; Freedman, Smolensky, Weingarten, 1982; Dashen and Neuberger, 1983; Lüscher and Weisz, 1987; Kuti, Lin, and Shen, 1988; Kleinert and Schulte-Frohlinde, 2001; Zinn-Justin, 2002).

Interaction term: $\mathcal{L}_{int.} = -\frac{\lambda}{4!}\,|\vec{\phi}|^4$, where $\vec{\phi} = (\phi_1, ..., \phi_N)$. Define $a = \lambda/(4\pi)^2$.

beta function: $\beta = \frac{da}{d\ln\mu} = a\sum_{\ell=1}^{\infty} b_\ell\, a^\ell$, where here b_ℓ is the ℓ-loop coefficient. Denote truncation to n-loop order as $\beta_{n\ell}$.

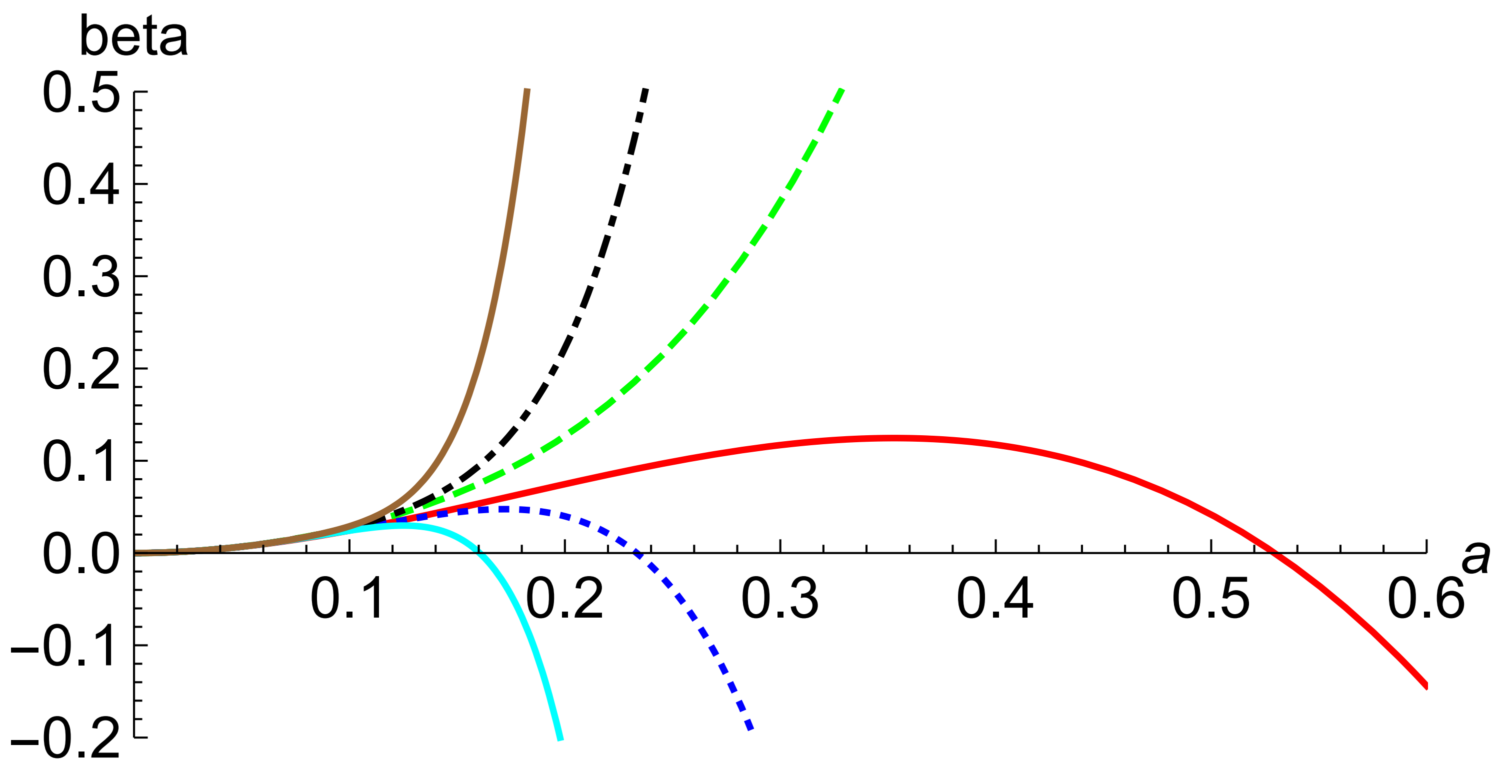
In RS, PRD 94, 125026 (2016); PRD 96, 056010 (2017), using Kompaniets and Panzer 6-loop calculation of β in 1606.09210 (in $\overline{\mathrm{MS}}$ scheme), we investigated whether this 6-loop beta function has a UV zero.

In the range of λ where the perturbative calculation of the beta function is reliable, we found evidence against a UV zero. We used scheme transformations and Padé approximants to confirm our conclusions.

In RS, PRD 107, 056018 (2023) [2301.01830] we have carried this search for a UV zero to 7-loop order, using the calculation of the 7-loop β fn. by Schnetz, PRD 97, 085018 (2021). Again, we used scheme transformations and Padé approximants as checks.

A necessary condition for there to be robust evidence for a zero in the beta function of a QFT is that the values calculated at successive loop orders should be close to each other. We find that this condition is not satisfied here. At $n = 3$, $n = 5$ and $n = 7$ loop order, $\beta_{n\ell}$ has no UV zero. Although $\beta_{2\ell}$ has a UV zero, it occurs at too large a value of λ for the perturbative calculation to be reliable.

See figure for $N = 1$. color coding: $\beta_{2\ell}$ (red, solid); $\beta_{3\ell}$ (green, dashed); $\beta_{4\ell}$ (blue, dotted); $\beta_{5\ell}$ (black, dot-dashed); $\beta_{6\ell}$ (cyan, solid); $\beta_{7\ell}$ (brown, solid). Curves from bottom to top: $n = 6,\ 4,\ 2,\ 3,\ 5,\ 7$.


beta
0.5
0.4
0.3
0.2
0.1
0.0
−0.1
−0.2
0.1
0.2
0.3
0.4
0.5
0.6
a

Another IR-free scalar theory is the O(N) $|\phi|^6$ theory in $d = 3$, with Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\nu \vec{\phi}) \cdot (\partial^\nu \vec{\phi}) - \frac{1}{2} m^2 |\vec{\phi}|^2 - \frac{\lambda}{4N} |\vec{\phi}|^4 - \frac{g}{6N^2} |\vec{\phi}|^6 ,$$

Since the coeffs. of the $|\vec{\phi}|^2$ and $|\vec{\phi}|^4$ terms in this $d = 3$ theory are dimensionful, and since $\lim_{\mu \to \infty} m^2/\mu^2 = 0$ and $\lim_{\mu \to \infty} \lambda/\mu = 0$, they are expected to be a negligible role in the UV limit.

This theory is known to have a UVFP in the large-N limit (Townsend, 1977; Pisarski, 1982; Appelquist and Heinz, 1982).

An interesting question is: over what range of finite N does this large-N UVFP persist?

In RS, PRD 107, 096009 (2023) [2302.05422] we investigated this. As before, a necessary condition for a UVFP is that successive orders in perturbation theory should yield $g_{UVFP,n\ell}$ values that are close to each other.

Using a combination of direct analysis of the the 6-loop beta function from Hager, J. Phys. A 35, 2703 (2002), Padé approximants, and scheme transformations, we showed that there is robust evidence for a UVFP for $N \gtrsim 2 \times 10^3$.

Conclusions

- Understanding the UV to IR evolution of an asymptotically free gauge theory and the nature of the IR behavior is of fundamental field-theoretic interest.

- Our higher-loop perturbative calculations of anomalous dimensions with T. A. Ryttov give information on properties at an IR fixed point for theories with fermions in a single representation and also theories with fermions in multiple different representations.

- Here have compared our higher-order scheme-independent calculations of anomalous dimensions with recent lattice measurements in an SU(3) gauge theory with $N_F = 10$ fermions in the fundamental representation and an SU(4) theory with $N_F = 4$ fermions in the fundamental rep. and $N_{A_2} = 4$ fermions in the antisymmetric rank-2 rep., finding agreement for both theories.

- We have also mentioned some results on UV behavior in a $|\vec{\phi}|^4$ theory in $d = 4$ and a $|\vec{\phi}|^6$ theory in $d = 3$.

THANK YOU