Higher-Order Calculations of Anomalous Dimensions at Infrared Fixed Points in Gauge Theories and Studies of Renormalization-Group Behavior of Some Scalar Field Theories

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Outline

• Renormalization-group flow from UV to IR in asymptotically free gauge theory; types of IR behavior; role of IR fixed point

• Calculations of anomalous dimensions of fermion bilinear operators via series expansions in gauge coupling and via scheme-independent series expansions; application to theories with fermions in a single representation and in multiple representations of the gauge group

• Comparison of results with recent lattice measurements for SU(3) with $N_F = 10$ and SU(4) for $N_F = 4$ and $N_{A_2} = 4$ fermions, where $F$ and $A_2$ are the fundamental and antisymmetric tensor representations

• Higher-loop studies of the beta functions of $O(N)$ $|\vec{\phi}|_4^4$ and $|\vec{\phi}|_3^6$ theories

• Conclusions

This talk contains new results from Rytov and RS, 2307.12426 and from RS, PRD 107, 056018 (2023) [2301.01830] and RS, PRD 107, 096009 (2023) [2302.05422]
RG Flow from UV to IR; Types of IR Behavior and Role of IR Fixed Point

Consider an asymptotically free, vectorial gauge theory with gauge group $G$ and a set of massless fermions, either (i) $N_f$ fermions $f$ in a single representation $R$ of $G$, or (ii) $N_f$ fermions $f$ and $N_{f'}$ fermions $f'$ in different reps. $R$ and $R'$.

Asymptotic freedom $\Rightarrow$ theory is weakly coupled, properties are perturbatively calculable for large Euclidean momentum scale $\mu$ in deep ultraviolet (UV).

One can analyze the renormalization-group (RG) flow from large $\mu$ in the UV to small $\mu$ in the infrared (IR).

If a fermion had mass $m_0$, it would be integrated out in the effective low-energy field theory for $\mu < m_0$, and hence would not affect the IR limit of interest here, so no loss of generality in taking massless fermions (mass-split models can also be of interest).

Denote running gauge coupling at scale $\mu$ as $g = g(\mu)$, and let $\alpha(\mu) = g(\mu)^2/(4\pi)$ and $\alpha(\mu) = g(\mu)^2/(16\pi^2)$. 
The dependence of $\alpha(\mu)$ on $\mu$ is described by the $\beta$ function

$$\beta \equiv \frac{d\alpha}{d\ln \mu} = -2\alpha \sum_{\ell=1}^{\infty} b_{\ell} a^\ell$$

where $\ell =$ loop order of the coeff.

Coefficients $b_1$ (Gross and Wilczek, Politzer, 1973) and $b_2$ (Caswell, Jones, 1974) in $\beta$ are independent of regularization/renormalization scheme, while $b_{\ell}$ for $\ell \geq 3$ are scheme-dependent.

Asymptotic freedom means $b_1 > 0$, so $\beta < 0$ for small $\alpha(\mu)$, in neighborhood of UV fixed point (UVFP) at $\alpha = 0$.

As the scale $\mu$ decreases from large values, $\alpha(\mu)$ increases. For a sufficiently large fermion content satisfying asymptotic freedom, $\beta$ has an infrared (IR) zero, denoted $\alpha_{IR}$, which is an IR fixed point (IRFP) of the renormalization group.

At this IRFP, the theory is scale-invariant and is inferred to be conformal-invariant, hence the term “conformal window” (CW) for this regime.
The properties of the theory at such an IRFP are of fundamental interest. These include anomalous dimensions of (gauge-invariant) operators. Denoting the dimension of an operator $\mathcal{O}$ as $D_\mathcal{O}$, the anomalous dimension $\gamma_\mathcal{O}$ is given by $D_\mathcal{O} = D_\mathcal{O},\text{ free} - \gamma_\mathcal{O}$.

Besides the intrinsic field-theoretic interest in anomalous dimensions of operators in the conformal window, theories slightly below the lower end of the CW exhibit quasi-conformal behavior, with slow variation of the gauge coupling over an extended interval of Euclidean momentum, $\mu$, owing to the small $\beta$.

This is of interest for particle physics because as this theory flows into the IR and eventually undergoes spontaneous chiral symmetry breaking ($S\chi\text{SB}$), the dynamical breaking of scale invariance yields a light approx. Nambu-Goldstone boson, the dilaton. Insofar as the Higgs boson can be modelled as at least partially dilatonic, this could protect its mass from large radiative corrections. Lattice simulations (LSD, LatKMI, Lat-HC groups) have verified the appearance of a light $0^{++}$ scalar in these theories.

In the chirally broken phase, just as the IR zero of $\beta$ is only an approx. IRFP, so also, the $\gamma_{\bar{\psi}\psi,IR}$ is only approx., describing the running of $\bar{\psi}\psi$ over an extended interval of energies.
The asymptotic freedom condition is \( b_1 > 0 \), i.e. \( 11C_A - 4 \sum_f N_f T_f > 0 \). For a theory with fermions in a single rep., this sets an upper bound on \( N_f \):

\[
N_{f,u} = \frac{11C_A}{4T_f},
\]

where with \( T^a_R \) are generators of the Lie algebra of \( G \) in the representation \( R \), and \( d_R = \text{dim}(R) \), and group invariants include

\[
T^a_R T^a_R = C_2(R) I_{d_R \times d_R}, \quad \text{Tr}_R(T^a_R T^b_R) = T(R) \delta^{ab}
\]

We use the notation \( C_A \equiv C_2(G) \) and, for \( f \) in \( R \), \( T_f \equiv T(R) \) and \( C_f \equiv C_2(R) \).

The condition that the 2-loop \( \beta \) fn. should have an IR zero (IRZ) is \( b_2 < 0 \), i.e.,

\[
34C_A^2 - 4 \sum_f (5C_A + 3C_f) N_f T_f < 0
\]

which sets a lower bound on \( N_f \). The region in which \( b_1 > 0 \) and \( b_2 < 0 \) is denoted \( I_{IRZ} \). The upper (\( u \)) and lower (\( \ell \)) boundaries \( \mathcal{B}_{IRZ,u} \) and \( \mathcal{B}_{IRZ,\ell} \) of the IRZ regions are the lines \( b_1 = 0 \) and \( b_2 = 0 \).

The upper and lower boundaries of the conformal window are denoted \( \mathcal{B}_{CW,u} = \mathcal{B}_{IRZ,u} \) and \( \mathcal{B}_{CW,\ell} \). We first discuss theories with fermions in a single representation, and then theories with fermions in multiple different reps.
Calculations of Anomalous Dimensions

An operator of particular interest is the fermion bilinear, $\bar{\psi}\psi = \sum_{j=1}^{N_f} \bar{\psi}_j \psi_j$ with anom. dim. $\gamma_{\bar{\psi}\psi}$ and its value at the IRFP, $\gamma_{\bar{\psi}\psi,IR}$.

One way to calculate $\gamma_{\bar{\psi}\psi,IR}$ is via a power series expansion in the coupling, $a = \alpha/(4\pi)$:

$$\gamma_{\bar{\psi}\psi} = \sum_{\ell=1}^{\infty} c_\ell a^\ell$$

where $c_\ell$ is $\ell$-loop coefficient. To calculate the $n$-loop result for the anom. dim., $\gamma_{\bar{\psi}\psi,n\ell,IR}$, one first calculates $\alpha_{IR,n\ell,IR}$ from the IR zero in the $n$-loop beta function and then sets $\alpha = \alpha_{IR,n\ell,IR}$ in above eq.

For a given $G$ and $R$, as $N_f$ decreases below $N_{f,u}$, $\alpha_{IR,2\ell}$ increases. This motivates calculation of the IR zero in $\beta$ to higher-loop order. With T. Ryttov, we calculated $\gamma_{\bar{\psi}\psi,IR}$ in this way up to 5-loop order in Ryttov and RS, PRD 94, 105014 (2016), PRD 94, 105015 (2016).

The anom. dim. $\gamma_{\bar{\psi}\psi,IR}$ is a physical quantity and is independent of the scheme used for regularization and renormalization.
The conventional expansion of $\gamma_{\bar{\psi}\psi,IR}$ as a power series in $\alpha$, calculated to finite order, does not maintain this scheme independence beyond the lowest order, since the $b_\ell$ are scheme-dependent for $\ell \geq 3$ and the $c_\ell$ are scheme-dependent for $\ell \geq 2$.

This scheme-dependence of higher-order calculations is well-known in QCD and uncertainties due to it are routinely taken into account in comparing higher-order QCD calculations with data, e.g., from the Tevatron and LHC.

We studied the effects of scheme dependence by applying scheme transformations in a series of papers, incl. Rytov and RS, PRD 86, 065032, 085005 (2012); RS, PRD 88, 036003 (2013); RS, PRD 90, 045011 (2014); Choi and RS, PRD 90 125029 (2014); PRD 94, 065038 (2016); Rytov, PRD 89, 016013 (2014); PRD 89, 056001 (2014); PRD 90, 056007 (2014); also J. Gracey, Simms, PRD 91, 085037 (2015); Gracey et al., 2306.09056.

It is valuable to calculate and analyze series expansions for physical quantities such as anomalous dimensions that are scheme-independent at each order.
Since $\alpha_{IR} \to 0$ as $N_f \to N_{f,u}$ and also $\Delta_f = N_{f,u} - N_f$ has the property that $\Delta_f \to 0$ as $N_f \to N_{f,u}$, one can alternatively express these quantities as power series in $\Delta_f$ rather than $\alpha$ (Banks-Zaks). Note that $\Delta_f = 3b_1/(4T_f)$.

Because $\Delta_f$ depends only on the group $G$, the rep. $R$, and the number $N_f$, these power series are obviously scheme-independent (at each order).

This scheme-independent series expansion is

$$\gamma_{\bar{\psi}\psi,IR} = \sum_{j=1}^{\infty} \kappa_j \Delta_f^j$$

We denote the truncation of the above series to maximal order (power) $j$ as $\gamma_{\bar{\psi}\psi,IR,\Delta_f^j}$.

The calculation of $\kappa_j$ requires, as inputs, the values of the $b_\ell$ for $1 \leq \ell \leq j + 1$ and the $c_\ell$ for $1 \leq \ell \leq j$. It may also provide a rough guide for anomalous dimensions in quasi-conformal theories that are close to the lower edge of the conformal window.
Define a denominator factor $D = 7C_A + 11C_f$. The first two $\kappa_j$ are

$$\kappa_1 = \frac{8C_fT_f}{C_AD},$$

$$\kappa_2 = \frac{4C_fT_f^2(5C_A + 88C_f)(7C_A + 4C_f)}{3C_A^2D^3},$$

and similarly for $\kappa_3$ (Rytov, PRL 117, 071601 (2016). In Rytov and RS, PRD 94, 105014 (2016) we calculated $\kappa_4$ and hence $\gamma_{\bar{\psi}\psi,IR}$ to $O(\Delta_4^f)$ for SU(3).

In Rytov and RS, PRD 95, 085012 (2017); PRD 95, 105004 (2017); PRD 96, 105015 (2017) we carried out these scheme-independent series expansions of $\gamma_{\bar{\psi}\psi,IR}$ for an arbitrary gauge group $G$ and fermion representation $R$ up to $O(\Delta_4^f)$ and analyzed them in detail for specific groups and reps. We used $b_5$ from Baikov, Chetyrkin, and Kühn, PRL 118, 082002 (2017); JHEP 04 (2017) 119 and the Vermaseren group, Herzog et al., JHEP 02 (2017) 090, together with earlier calculations of $c_4$ by Chetyrkin, and by Vermaseren, Larin, and van Ritbergen.

Our result for $\kappa_4$: 
\[
\kappa_A = \frac{T_f^2}{3^5 C_A^5 D^7} \left[ C_A C_f T_f^2 \left( 19515671 C_A^6 - 131455044 C_A^5 C_f + 1289299872 C_A^4 C_f^2 + 2660221312 C_A^3 C_f^3 \\
+ 1058481072 C_A^2 C_f^4 + 6953709312 C_A C_f^5 + 1275715584 C_f^6 \right) \right] \\
+ 2^{10} C_f T_f^2 D \left( 5789 C_A^2 - 4168 C_A C_f - 6820 C_f^2 \right) \frac{d_{abcd} d_{abcd}}{d_A} \\
- 2^{10} C_A C_f T_f D \left( 41671 C_A^2 - 125477 C_A C_f - 53240 C_f^2 \right) \frac{d_{abcd} d_{abcd}}{d_A} \\
- 2^8 \cdot 11^2 C_A^2 C_f D \left( 2569 C_A^2 + 18604 C_A C_f - 7964 C_f^2 \right) \frac{d_{abcd} d_{abcd}}{d_A} \\
- 2^{14} \cdot 3 C_A T_f D^3 \frac{d_{abcd} d_{abcd}}{d_R} + 2^{13} \cdot 33 C_A^2 T_f D^3 \frac{d_{abcd} d_{abcd}}{d_R} \\
+ 2^8 D \left[ - 3 C_A C_f T_f^2 D \left( 4991 C_A^4 - 17606 C_A^3 C_f + 33240 C_A^2 C_f^2 - 30672 C_A C_f^3 + 9504 C_f^4 \right) \right] \\
- 2^4 C_f T_f^2 \frac{d_{abcd} d_{abcd}}{d_A} \left( 17206 C_A^2 - 60511 C_A C_f - 45012 C_f^2 \right) \\
+ 40 C_A C_f T_f \frac{d_{abcd} d_{abcd}}{d_A} \left( 35168 C_A^2 - 154253 C_A C_f - 88572 C_f^2 \right) \\
- 88 C_A^2 C_f \frac{d_{abcd} d_{abcd}}{d_A} \left( 973 C_A^2 - 93412 C_A C_f - 56628 C_f^2 \right) \\
+ 1440 C_A T_f^2 D^2 \frac{d_{abcd} d_{abcd}}{d_R} - 7920 C_A T_f D^2 \frac{d_{abcd} d_{abcd}}{d_R} \right] \zeta_3
\]
\[ + \frac{4505600C_A C_f D^2}{d_A} \left[ -4T_f^2 d_A^{abcd} d_A^{abcd} + 2T_f d_R^{abcd} d_A^{abcd} (10C_A + 3C_f) + 11C_A d_R^{abcd} d_R^{abcd} (C_A - 3C_f) \right] \zeta_5 \]

where \((a, b, c, d)\) are group indices

\[ d_R^{abcd} = \frac{1}{3!} \text{Tr}_R \left[ T_R^a (T_R^b T_R^c T_R^d + \text{cycl.}) \right] \]

\[ d_A^{abcd} = d_R^{abcd} \text{ for } R = \text{adj}, \quad d_R = \dim(R), \text{ and } \zeta_s = \sum_{n=1}^{\infty} \frac{1}{n^s} \text{ is the Riemann zeta function.} \]
For $G = \text{SU}(N_c)$ and $R = F$, our results for general $G$ and $R$ reduce to

$$\kappa_{1,F} = \frac{4(N^2_c - 1)}{N_c(25N^2_c - 11)}, \quad \kappa_{2,F} = \frac{4(N^2_c - 1)(9N^2_c - 2)(49N^2_c - 44)}{3N^2_c(25N^2_c - 11)^3}$$

$$\kappa_{3,F} = \frac{8(N^2_c - 1)}{3^3N^3_c(25N^2_c - 11)^5} \left[ \left(274243N^8_c - 455426N^6_c - 114080N^4_c + 47344N^2_c + 35574\right) - 4224N^2_c(4N^2_c - 11)(25N^2_c - 11)\zeta_3 \right]$$

$$\kappa_{4,F} = \frac{4(N^2_c - 1)}{3^4N^4_c(25N^2_c - 11)^7} \left[ \left(263345440N^{12}_c - 673169750N^{10}_c + 256923326N^8_c - 290027700N^6_c + 557945201N^4_c - 208345544N^2_c + 6644352\right) + 384(25N^2_c - 11)\left(4400N^{10}_c - 123201N^8_c + 480349N^6_c - 486126N^4_c + 84051N^2_c + 1089\right)\zeta_3 \right] + 211200N^2_c(25N^2_c - 11)^2(N^6_c + 3N^4_c - 16N^2_c + 22)\zeta_5$$
Plot of $\gamma_{\bar{\psi}\psi,IR,\Delta_f^j}$ with $1 \leq j \leq 4$ for SU(3) and fermion rep. $R = F$, as functions of $N_f \in I$ from Rytov and RS, PRD 94, 105014 (2016) [1608.00068]. Curves: $\gamma_{\bar{\psi}\psi,IR,F,\Delta_f}$ (red), $\gamma_{\bar{\psi}\psi,IR,F,\Delta_f^2}$ (green), $\gamma_{\bar{\psi}\psi,IR,F,\Delta_f^3}$ (blue), $\gamma_{\bar{\psi}\psi,IR,F,\Delta_f^4}$ (black).

As one moves down from the upper end of the conformal window, $\gamma_{\bar{\psi}\psi,IR}$ increases. Approximate analysis of Schwinger-Dyson eq. for fermion propagator suggests $S_{\chi SB}$ occurs at $\gamma_{\bar{\psi}\psi,IR} = 1$, which thus determines the lower boundary $B_{CW,\ell}$ of the conformal window (Appelquist et al (1988); Cohen and Georgi (1989))

A rigorous bound in the conformal window is $\gamma_{\bar{\psi}\psi,IR} < 2$ (Mack, 1977), but this need not be saturated.

Extrapolations of these $O(\Delta_f^4)$ results to $O(\Delta_f^j)$ with $\lim_{j \to \infty}$ given in Rytov-RS, PRD 94, 105014 (2016). Combining this extrapolation with the $\gamma_{IR} = 1$ condition yields $9 < N_{f,cr} < 10$.

This agrees with extensive lattice simulations of the SU(3), $N_F = 8$ theory (LSD, LatKMI, LatHC...), which indicate that it is slightly below $B_{CW,\ell}$ and with Hasenfratz et al., 2306.07236, who find that the SU(3) $N_f = 10$ theory is in the CW.
<table>
<thead>
<tr>
<th>$N_c$</th>
<th>$N_f$</th>
<th>$\gamma_{IR,F,\Delta f}$</th>
<th>$\gamma_{IR,F,\Delta^2_f}$</th>
<th>$\gamma_{IR,F,\Delta^3_f}$</th>
<th>$\gamma_{IR,F,\Delta^4_f}$</th>
<th>$\gamma_{IR,F,ext}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>8</td>
<td>0.424</td>
<td>0.698</td>
<td>0.844</td>
<td>1.036</td>
<td>—</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>0.374</td>
<td>0.587</td>
<td>0.687</td>
<td>0.804</td>
<td>1.4(2)</td>
</tr>
<tr>
<td>3</td>
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<td>0.324</td>
<td>0.484</td>
<td>0.549</td>
<td>0.615</td>
<td>0.95(6)</td>
</tr>
<tr>
<td>3</td>
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<td>0.389</td>
<td>0.428</td>
<td>0.462</td>
<td>0.62(2)</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>0.224</td>
<td>0.301</td>
<td>0.323</td>
<td>0.338</td>
<td>0.400(5)</td>
</tr>
<tr>
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<td>0.221</td>
<td>0.231</td>
<td>0.237</td>
<td>0.257(5)</td>
</tr>
<tr>
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<td>14</td>
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<td>0.148</td>
<td>0.152</td>
<td>0.153</td>
<td>0.154(4)</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>0.0748</td>
<td>0.0833</td>
<td>0.0841</td>
<td>0.0843</td>
<td>0.0841(2)</td>
</tr>
<tr>
<td>3</td>
<td>16</td>
<td>0.0249</td>
<td>0.0259</td>
<td>0.0259</td>
<td>0.0259</td>
<td>0.0259(1)</td>
</tr>
</tbody>
</table>

Values of $\gamma_{\bar{\psi}\psi,IR,\Delta^j_f} = \gamma_{IR,\Delta^j_f}$ with $1 \leq j \leq 4$ for SU(2), SU(3), and $R = F$. Last column shows extrapolations to $j \rightarrow \infty$, denoted $\gamma_{IR,F,ext}$. 
In our papers we discussed the accuracy of these finite order calculations and resultant $\gamma_{IR,\Delta_f}$ values. A rough estimate can be obtained from the figures.

Where the curves for the $\gamma_{IR,\Delta_f}$ with different $j$ are close to each other, higher-order terms are expected to be small. As $N_f$ decreases, these curves deviate progressively more from each other, and higher-order terms are more important.

Additional estimates of effects of higher-order terms were obtained via calculation and analysis of Padé approximants, e.g., in Rytov-RS, PRD 97, 025004 (2018).

The approximate analysis of the Schwinger-Dyson eq. (Appelquist, Lane, Mahanta, 1988) also suggested a quadratic criticality condition $\gamma_{CC, \gamma_{\bar{f}f,IR}}(2 - \gamma_{\bar{f}f,IR}) = 1$ for the onset of $S\chi$SB. This eq. has a double root at $\gamma_{\bar{f}f,IR} = 1$ and hence is formally equivalent to the linear $\gamma_{CC, \gamma_{\bar{f}f,IR}} = 1$.

When applied in the context of a truncated series expansions, the quadratic $\gamma_{CC}$ yields a slightly larger value of $N_{f,cr}$ at $B_{CW,\ell}$ than the linear $\gamma_{CC}$ (B. S. Kim, D. K. Hong, and J.-W. Lee, PRD 101, 056008 (2020); J.-W. Lee, PRD 103, 076006 (2021); J.-W. Lee, talk at this conf.) This difference decreases as the order $O(\Delta_f^j)$ increases.
For SU(3), with our \( O(\Delta^4_f) \) order scheme-independent inputs, the quadratic \( \gamma_{CC} \) condition gives \( 9 < N_{f,c} < 10 \), in agreement with our extrapolation in Rytov-RS, PRD 94, 105014 (2016). The CFT bound \( \gamma < 2 \) would give \( 8 < N_{f,c} < 9 \), also in agreement with lattice simulations.

We also calculated anomalous dimensions for other operators, including higher-spin fermion bilinears.

One such operator is the Lorentz tensor bilinear \( \mathcal{O}_{T,\mu\nu} = \bar{\psi} \sigma_{\mu\nu} \psi \), with anom. dim. \( \gamma_{T,IR} \) at the IRFP and scheme-independent series expansion

\[
\gamma_{T,IR} = \sum_{j=1}^{\infty} \kappa_{T,j} \Delta_f^j
\]

with truncation to \( O(\Delta_f^j) \) denoted \( \gamma_{T,IR,\Delta_f^j} \).

For a general \( G \) and \( R \), using the highest-order inputs available, we calculated \( \gamma_{T,IR} \) up to \( O(\Delta_f^3) \) in Rytov-RS, PRD 94, 125005 (2016).
For the coefficients $\kappa_{T,j}$ in the scheme-independent expansions of these anomalous dimensions for the Lorentz tensor fermion bilinear, we obtain

$$\kappa_{T,1} = -\frac{8C_f T_f}{3C_A D}$$

$$\kappa_{T,2} = -\frac{4C_f T_f^2 (259C_A^2 + 428C_A C_f - 528C_f^2)}{9C_A^2 D^3}$$

$$\kappa_{T,3} = \frac{4C_f T_f}{3^5 C_A^4 D^5} \left[ 3C_A T_f^2 \left\{ C_A^3 (-11319 + 188160 \zeta_3) + C_A^3 C_f (-337204 + 64512 \zeta_3) + C_A^2 C_f^2 (83616 - 890112 \zeta_3) 
+ C_A C_f^3 (1385472 - 354816 \zeta_3) + C_f^4 (1385472 - 354816 \zeta_3) \right\} - 512 T_f^2 D (-5 + 132 \zeta_3) \frac{d_R^{abcd} d_R^{abcd}}{d_A} 
- 15488 C_A^2 D (-11 + 24 \zeta_3) \frac{d_R^{abcd} d_R^{abcd}}{d_A} + 11264 C_A T_f D (-4 + 39 \zeta_3) \frac{d_R^{abcd} d_R^{abcd}}{d_A} \right] .$$

Note that in contrast to the $\kappa_{j}$ for $\gamma \bar{\psi} \psi, IR$, which are positive for $1 \leq j \leq 4$, here for $SU(3)$, $R = F$, $\kappa_{F,1}$ and $\kappa_{F,2}$ are negative, while $\kappa_{F,3}$ is positive:

$$\kappa_{T,SU(3),F,1} = -(1.6615 \times 10^{-2})$$

$$\kappa_{T,SU(3),F,2} = -(1.12625 \times 10^{-3})$$

$$\kappa_{T,SU(3),F,3} = 2.480155 \times 10^{-5}$$
For SU(3) and $R = F$, fundamental rep., with $\gamma_{T,IR} \equiv \gamma_{IR,F}^{(\sigma)}$, these give the following anomalous dimensions as a function of $N_F$:

<table>
<thead>
<tr>
<th>$N_f$</th>
<th>$\gamma_{IR,F,\Delta_f}^{(\sigma)}$</th>
<th>$\gamma_{IR,F,\Delta_f^2}$</th>
<th>$\gamma_{IR,F,\Delta_f^3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>-0.141</td>
<td>-0.223</td>
<td>-0.207</td>
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<tr>
<td>9</td>
<td>-0.125</td>
<td>-0.188</td>
<td>-0.1775</td>
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<tr>
<td>10</td>
<td>-0.108</td>
<td>-0.156</td>
<td>-0.149</td>
</tr>
<tr>
<td>11</td>
<td>-0.0914</td>
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<tr>
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<td>16</td>
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<td>-0.00859</td>
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</tbody>
</table>

Further analysis for other higher-spin operators in Rytto-RS, PRD 101, 076018 (2020).

Another operator of interest is $\text{Tr}(F_{\mu\nu}F^{\mu\nu})$, whose anom. dim. at the IRFP is given by $\beta'_{IR} = (d\beta/d\alpha)_{IR}$. Calculations in RS, PRD 87, 105005 (2013); Rytto-RS, PRD 94, 125005 (2017); PRD 05, 105004 (2017) to $O(\Delta_f^5)$. Here we focus on anom. dims. of fermion bilinears.
It is of interest to compare our perturbative calculations of anomalous dimensions with lattice measurements.

In previous work we have done this for several theories for which there have been extensive simulations, such as SU(3) with $N_F = 12$ fermions in the fundamental rep., SU(3) with 2 fermions in the symmetric tensor rep., SU(2) with various reps. $R$.

For various $G$, $R$, and $N_f$, there is not yet a complete consensus as to whether a given theory is inside or outside of the conformal window.

Here we focus on new results on SU(3) with $N_F = 10$ fermions in the fundamental rep. Previous studies include the following:

Appelquist et al. (LSD Collab.), arXiv:1204.6000 early study

Appelquist et al. (LSD Collab.), PRD 103, 014504 (2021) find that this theory has an IRFP and hence is in the conformal window, and measure $\gamma_{\bar{\psi}\psi, IR} = 0.47 \pm 0.05$

See also T.-W. Chiu, 1603.08854 and PRD 99, 014507 for study of $\beta$ function and $\beta'_{IR}$ recently: Hasenfratz, Neil, Shamir, Svetitsky, 2306.07236 (A. Hasenfratz, talk at this conf.) find that this theory has an IRFP and hence is in the conformal window, and measure $\gamma_{\bar{\psi}\psi,IR}$ and $\gamma_{T,IR}$ to be

$$\gamma_{\bar{\psi}\psi,IR} \simeq 0.6 , \quad \gamma_{T,IR} \simeq -0.2$$

To within the uncertainties in our scheme-independent perturbative calculations of $\gamma_{\bar{\psi}\psi,IR}$ to $O(\Delta_F^4)$ and $\gamma_{T,IR}$ to $O(\Delta_F^3)$, they agree with these measurements:

$$\gamma_{\bar{\psi}\psi,IR,\Delta_F} = 0.324 , \quad \gamma_{\bar{\psi}\psi,IR,\Delta_F^2} = 0.484$$

$$\gamma_{\bar{\psi}\psi,IR,\Delta_F^3} = 0.549 , \quad \gamma_{\bar{\psi}\psi,IR,\Delta_F^4} = 0.615$$

$$\gamma_{T,IR,\Delta_F} = -0.108 , \quad \gamma_{T,IR,\Delta_F^2} = -0.156 , \quad \gamma_{T,IR,\Delta_F^3} = -0.149$$
Theories with Fermions in Multiple Different Representations


Here, asymptotic freedom (AF) condition is $b_1 > 0$ where $b_1 = (1/3)[11C_A − 4N_fT_f − 4N_{f'}T_{f'}]$. The eq. $b_1 = 0$ is the upper boundary $\mathcal{B}_{IRZ,u} = \mathcal{B}_{CW,u}$ of the IRZ region and conformal window. The resultant upper bounds on $N_f$ and $N_{f'}$ from AF are $N_f < N_{f,u}$ and $N_{f'} < N_{f',u}$, where

\[
N_{f,u} = \frac{11C_A - 4N_{f'}T_{f'}}{4T_f}, \quad N_{f',u} = \frac{11C_A - 4N_fT_f}{4T_{f'}}
\]
The scheme-independent expansion parameters are

\[ \Delta_f = N_{f,u} - N_f = \frac{3b_1}{4T_f}, \quad \Delta_{f'} = N_{f',u} - N_{f'} = \frac{3b_1}{4T_{f'}} \]

so \( \Delta_{f'} = \left(\frac{T_f}{T_{f'}}\right)\Delta_f \).

Scheme-independent series expansions of anom. dims at the IRFP are

\[ \bar{\gamma}_{ff,IR} = \sum_{j=1}^{\infty} \kappa_j^{(f)} \Delta_f^j , \quad \bar{\gamma}_{f'f',IR} = \sum_{j=1}^{\infty} \kappa_j^{(f')} \Delta_{f'}^j \]

Denote truncations of these series to order \( j \) as \( \bar{\gamma}_{ff,IR,\Delta_f^j} \) and \( \bar{\gamma}_{f'f',IR,\Delta_{f'}^j} \).

Define the denominator factor \( D_f = C_A(7C_A + 11C_f) + 4N_{f'}T_{f'}(C_{f'} - C_f) \).

For \( \kappa_j^{(f)} , j = 1, 2, 3 \) we obtained \( \kappa_1^{(f)} = 8C_fT_f/D_f \),

\[ \kappa_2^{(f)} = \frac{4C_fT_f^2}{3D_f^3} \left[ C_A(7C_A + 4C_f)(5C_A + 88C_f) \right. \]

\[ \left. + \ 2^4 N_{f'}T_{f'}(C_{f'} - C_f)\left(10C_A + 8C_f + C_{f'}\right) \right] \]
\[
\kappa_3^{(f)} = \frac{4C_f T_f}{3^4 \mathcal{D}_f^5} \left[ A_0^{(f)} + A_1^{(f)} N_{f'} + A_2^{(f)} N_{f'}^2 + A_3^{(f)} N_{f'}^3 \right]
\]

where \( A_0^{(f)}, A_1^{(f)}, A_2^{(f)}, \) and \( A_3^{(f)} \) are more complicated functions (given in our paper).

The \( \kappa^{(f')} \) are obtained from these \( \kappa_j^{(f)} \) by interchanging \( f \) and \( f' \) in all expressions.

A particular theory of interest has an SU(4) gauge group and (massless) Dirac fermion content consisting of \( N_F \) fermions in the fundamental \((F)\) rep. and \( N_{A_2} \) fermions in the antisymmetric rank-2 tensor rep. \((A_2)\). In this theory, the (6-dim.) \( A_2 \) rep. is self-conjugate, so \( N_{A_2} \) Dirac fermions are equivalent to \( 2N_{A_2} \) Majorana fermions. This theory is motivated as a model of dynamical electroweak symmetry breaking (EWSB) that addresses the issue of the large top quark mass.

There have been many earlier efforts at dynamical EWSB models, e.g. Weinberg, PRD 19, 1277 (1979); Susskind, PRD 20, 2619 (1979); Dimopoulos and Susskind, NPB 155, 23, (1979); Eichten and Lane PLB 90, 125 (1980). Early models for a heavy top quark include Hill, PLB 266, 419 (1991); Kaplan, NPB 365, 259 (1991); Appelquist and Terning, PRD 50, 2116 (1994); Lane and Eichten, PLB 352, 382 (1995); Chivukula, Dobrescu, and Terning, PLB 353, 289 (1995); Chivukula and Simmons, PRD 66, 015006 (2002) among others.
Reasonably UV-complete theories of this type involved sequential breaking of an asymptotically free chiral gauge theory in stages, leading to a vectorial gauge theory that becomes strongly coupled at the TeV scale. The sequential breaking provided a way of explaining the hierarchical structure of the SM fermion generations with a low-scale seesaw for neutrino masses: Appelquist and RS, PLB 548, 204 (2002); Appelquist and RS, PRL 90, 201801 (2003); Appelquist, Piai, and RS, PRD 69 015002 (2004); Christensen and RS, PRL 94, 241801 (2004) (challenge of getting $t$-$b$ mass splitting and still satisfying precision EW constraints). See also Ferretti and Karateev, JHEP 03 (2014) 077. Another approach assumed a higher-dimensional spacetime, with SM fermions having wave functions in the extra dimensions that are strongly localized: Arkani-Hamed and Schmaltz, PRD 61, 033005 (2000); Nussinov and RS, PLB 526, 137 (2002).

In the 2018 Rytov-RS paper we noted that lattice simulations had been performed of an SU(4) theory with Dirac fermion content $N_F = 2$ and $N_{A_2} = 2$ by Ayyar, DeGrand, et al., PRD 97, 074505, 114505 (2018), but this theory was found to be in the chirally broken phase where our calculations do not apply directly.
Recently, in PRD 107, 114504 (2023) [2304.11729], Hasenfratz, Neil, Shamir, Svetitsky, and Witzel have reported results from lattice simulations of the SU(4) theory with $N_F = 4$ and $N_{A2} = 4$ Dirac fermions (c.f. talk by Y. Shamir at this conf.). These authors find that this theory has an IRFP and hence is in the conformal window, and measure

$$\gamma_m^{(4)} \approx 0.75 , \quad \gamma_m^{(6)} \approx 1.0$$

An interesting question is whether our general higher-order perturbative calculations of anomalous dimensions of fermion bilinears, when specialized to this theory, yield results in agreement with the values measured in this recent lattice study by Hasenfratz et al.

We address and answer this question in Rytov-RS 2307.12426. We find agreement.

It is instructive to give results for the more general case of an SU($N_c$) theory with massless Dirac fermion content consisting of $N_F$ fermions in the $F$ rep. and $N_{A2}$ fermions in the $A_2$ rep. We label an SU($N_c$) theory with $N_F$ and $N_{A2}$ Dirac fermions as $(N_c, N_F, N_{A2})$

Denote the $F$ and $A_2$ fermion as $\psi^a_i$ and $\chi^{ab}_j = -\chi^{ba}_j$, where $a, b$ are SU($N_c$) gauge indices and the flavor indices are $i = 1, \ldots, N_F$ and $j = 1, \ldots, N_{A2}$. We calculate anomalous dimensions of the operators
\[ \bar{\psi}\psi = \sum_{i=1}^{N_f} \bar{\psi}_{a,i} \psi^a_i , \quad \bar{\chi}\chi = \sum_{j=1}^{N_{A2}} \bar{\chi}_{ab,j} \chi^{ab}_j \]

The scheme-independent expansion variables are

\[ \Delta_F = \frac{11}{2} N_c - N_F - (N_c - 2)N_{A2} \]
\[ \Delta_{A2} = \frac{11 N_c - 2N_F - 2(N_c - 2)N_{A2}}{2(N_c - 2)} \]

N.B.: \( \Delta_{A2} = \frac{T_F}{T_{A2}} \Delta_F = \frac{\Delta_F}{N_c - 2} \)

The scheme-independent expansions of the anomalous dimensions are

\[ \gamma_{\bar{\psi}\psi,IR} = \sum_{j=1}^{\infty} \kappa_j^{(F)} \Delta_F^j , \quad \gamma_{\bar{\chi}\chi,IR} = \sum_{j=1}^{\infty} \kappa_j^{(A2)} \Delta_{A2}^j \]

Truncations of these series to order \( j \) are denoted \( \gamma_{\bar{\psi}\psi,IR,\Delta_F^j} \) and \( \gamma_{\bar{\chi}\chi,IR,\Delta_{A2}^j} \).

Notational equivalence for SU(4): \( \gamma_{\bar{\psi}\psi,IR} \equiv \gamma_m^{(4)} \) and \( \gamma_{\bar{\chi}\chi,IR} \equiv \gamma_m^{(6)} \).
Define denominator factors

\[ D_F = N_c(25N_c^2 - 11) + 2N_{A_2}(N_c - 2)(N_c + 1)(N_c - 3) \]

\[ D_{A_2} = N_c(18N_c^2 - 11N_c - 22) - N_F(N_c - 3)(N_c + 1) . \]

We find

\[ \kappa_{(F)} = \frac{4(N_c^2 - 1)}{D_F}, \quad \kappa_{(A_2)} = \frac{4(N_c - 2)^2(N_c + 1)}{D_{A_2}} \]

\[ \kappa_{2}^{(F)} = \frac{4(N_c^2 - 1)}{3D_F^3} \left[ N_c(9N_c^2 - 2)(49N_c^2 - 44) + 4N_{A_2}(N_c - 2)(N_c + 1)(N_c - 3)(3N_c - 2)(5N_c + 3) \right] \]

\[ \kappa_{2}^{(A_2)} = \frac{(N_c - 2)^3(N_c + 1)}{3D_{A_2}^3} \left[ N_c(11N_c^2 - 4N_c - 8)(93N_c^2 - 88N_c - 176) - 2N_F(N_c - 3)(N_c + 1)(37N_c^2 - 16N_c - 33) \right] \]

\[ \kappa_{3}^{(F)} = \frac{8(N_c^2 - 1)}{27D_F^5} \left[ A_0^{(F)} + A_1^{(F)}N_{A_2} + A_2^{(F)}N_{A_2}^2 + A_3^{(F)}N_{A_2}^3 \right] \]

\[ \kappa_{3}^{(A_2)} = \frac{(N_c - 2)^3(N_c + 1)}{54D_{A_2}^5} \left[ A_0^{(A_2)} + A_1^{(A_2)}N_F + A_2^{(A_2)}N_F^2 + A_3^{(A_2)}N_F^3 \right] \]
where

\[
A_0^{(F)} = N_c^2 \left[ (274243N_c^8 - 455426N_c^6 - 114080N_c^4 + 47344N_c^2 + 35574) - 4224N_c^2(4N_c^2 - 11)(25N_c^2 - 11)\zeta_3 \right],
\]

\[
A_1^{(F)} = 4N_c(N_c - 2)(N_c - 3) \left[ (16981N_c^7 + 35460N_c^6 + 42927N_c^5 + 47342N_c^4 + 9432N_c^3 - 12849N_c^2 \\
- 18843N_c - 11616) - 576N_c^2(25N_c^4 + 198N_c^3 + 187N_c^2 - 121N_c - 121)\zeta_3 \right],
\]

\[
A_2^{(F)} = 8(N_c - 2)(N_c - 3) \left[ (689N_c^8 - 1402N_c^7 - 9208N_c^6 - 15693N_c^5 - 9219N_c^4 + 16662N_c^3 + 19860N_c^2 \\
+ 10617N_c + 5598) - 192N_c^2(3N_c^5 - 65N_c^4 - 238N_c^3 - 165N_c^2 + 231N_c + 198)\zeta_3 \right],
\]

\[
A_3^{(F)} = 128N_c(N_c - 2)^2(N_c - 3)^2(N_c + 1)(3N_c^2 + 7N_c + 6)(-11 + 24\zeta_3),
\]
\[ A_0^{(A_2)} = N_c^2 \left[ (1670571N_c^9 - 7671402N_c^8 + 2181584N_c^7 + 25294256N_c^6 - 13413856N_c^5 \\
- 17539136N_c^4 + 16707328N_c^3 + 3046912N_c^2 - 27320832N_c - 18213888) \\
- 8448N_c^2(N_c + 2)(18N_c^2 - 11N_c - 22)(3N_c^3 - 28N_c^2 + 176)\zeta_3 \right] \]

\[ A_1^{(A_2)} = -4N_c(N_c - 3) \left[ (60552N_c^8 - 150015N_c^7 - 373894N_c^6 + 138737N_c^5 + 300380N_c^4 \\
+ 421197N_c^3 + 768345N_c^2 + 858660N_c + 435468) \\
- 192N_c^2(141N_c^5 - 2075N_c^4 - 6226N_c^3 + 1056N_c^2 + 17424N_c + 11616)\zeta_3 \right] \]

\[ A_2^{(A_2)} = 8(N_c - 3) \left[ (1148N_c^8 - 3919N_c^7 - 17365N_c^6 - 5724N_c^5 + 35724N_c^4 + 84915N_c^3 + 70641N_c^2 \\
+ 32928N_c + 15588) - 192N_c^2(3N_c^5 - 164N_c^4 - 271N_c^3 + 396N_c^2 + 1320N_c + 792)\zeta_3 \right] \]

\[ A_3^{(A_2)} = -128N_c(N_c + 1)(N_c - 3)^2(3N_c^2 + 7N_c + 6)(-11 + 24\zeta_3) . \]
Specializing to $N_c = 4$, i.e., the SU(4) theory, we have

$$
\kappa_1^{(F)} = \frac{15}{389 + 5N_{A_2}}, \quad \kappa_2^{(F)} = \frac{25(5254 + 115N_{A_2})}{(389 + 5N_{A_2})^3}
$$

$$
\kappa_1^{(A_2)} = \frac{80}{888 - 5N_F}, \quad \kappa_2^{(A_2)} = \frac{400(19456 - 165N_F)}{(888 - 5N_F)^3}
$$

$$
\kappa_3^{(F)} = \frac{5}{36(389 + 5N_{A_2})^5} \left[ (8039476475 - 696689664\zeta_3) + (479848740 - 197766144\zeta_3)N_{A_2} \\
+ (-16264767 + 46568448\zeta_3)N_{A_2}^2 + (-288640 + 629760\zeta_3)N_{A_2}^3 \right]
$$

$$
\kappa_3^{(A_2)} = \frac{640}{27(888 - 5N_F)^5} \left[ (28645111296 + 7201751040\zeta_3) - (120552246 + 1055342592\zeta_3)N_F \\
+ (-12526131 + 33675264\zeta_3)N_F^2 + (72160 - 157440\zeta_3)N_F^3 \right]
$$
Substituting these into our scheme-independent expansions, we obtain

\[ \gamma_{\bar{\psi}\psi,IR,\Delta_F} = 0.367, \quad \gamma_{\bar{\psi}\psi,IR,\Delta_F^2} = 0.576, \quad \gamma_{\bar{\psi}\psi,IR,\Delta_F^3} = 0.683 \]

\[ \gamma_{\bar{\chi}\chi,IR,\Delta_{A_2}} = 0.461, \quad \gamma_{\bar{\chi}\chi,IR,\Delta_{A_2}^2} = 0.748, \quad \gamma_{\bar{\chi}\chi,IR,\Delta_{A_2}^3} = 0.942 \]

Because \( \kappa_j^{(F)} \) and \( \kappa_j^{(A_2)} \) are positive for all of the orders \( j = 1, 2, 3 \) for which we have calculated them, several monotonicity relations follow for these orders:

With fixed \( \Delta_F = 2\Delta_{A_2} \), the anom. dims. \( \gamma_{\bar{\psi}\psi,IR,\Delta_F^j} \) and \( \gamma_{\bar{\chi}\chi,IR,\Delta_{A_2}^j} \) are monotonically increasing functions of \( j \).

Second, for a fixed \( j \), \( \gamma_{\bar{\psi}\psi,IR,\Delta_F^j} \) is a monotonically increasing function of \( \Delta_F \) and \( \gamma_{\bar{\chi}\chi,IR,\Delta_{A_2}^j} \) is a monotonically increasing function of \( \Delta_{A_2} \).

Since finite-order perturbative calculations of this type become progressively less accurate as one approaches the lower boundary \( \mathcal{B}_{CW,\ell} \) of the conformal window, one should assess the effect of higher-order corrections. From a rough extrapolation (ex) of our results for \( j = 1, 2, 3 \) to large \( j \), we estimate that these higher orders would yield
\[ \gamma_{\psi\psi,IR,ex} \simeq 0.7 - 0.8 \quad \gamma_{\chi\chi,IR,ex} \simeq 1.0 - 1.1 \]

To within the uncertainties in our extrapolation and in the lattice measurements, these results are in agreement with the values \[ \gamma^{(4)}_m \simeq 0.75 \] and \[ \gamma^{(6)}_m \simeq 1.0 \] obtained in Hasenfratz et al., PRD 107, 114504 (2023) [2304.11729]. (Recall notational equivalences in this SU(4) theory: \[ \gamma^{(4)}_m \equiv \gamma_{\psi\psi,IR} \] and \[ \gamma^{(6)}_m \equiv \gamma_{\chi\chi,IR}. \]) We have also used Padé approximants for estimates.

More generally, we have calculated \[ \gamma_{\psi\psi,IR,\Delta j_F} \] and \[ \gamma_{\chi\chi,IR,\Delta j_{A_2}} \] for \( j = 1, 2, 3 \) in the SU(4) theory as functions of \( N_F \) and \( N_{A_2} \).

In the figures we show the results on two line segments in \( I_{IRZ} \) that intersect at \( (N_F, N_{A_2}) = (4, 4) \): with \( N_{A_2} = 4 \), varying \( 2 < N_F < 14 \) and with \( N_F = 4 \), varying \( 3 < N_{A_2} < 9 \). Color coding for \( f = F, A_2 \): \( \gamma_{f\bar{f},IR,\Delta j_f} \) (red), \( \gamma_{f\bar{f},IR,\Delta^2 j_f} \) (green), \( \gamma_{f\bar{f},IR,\Delta^3 j_f} \) (blue).
Figure 1: Plot of $\gamma_{\bar{\psi}\psi,IR,jF}$ calculated to order $j = 1, 2, 3$ for $G = SU(4)$, and $N_{A_2} = 4$, as a function of $N_F \in I_{IRZ}$. From bottom to top, the curves refer to $\gamma_{\bar{\psi}\psi,IR,jF}$ (red), $\gamma_{\bar{\psi}\psi,IR,jF}$ (green), and $\gamma_{\bar{\psi}\psi,IR,jF}$ (blue).
Figure 2: Plot of $\gamma_{\bar{\psi}\psi,IR,\Delta^j_F}$ calculated to order $j = 1, 2, 3$ for $G = SU(4)$, and $N_F = 4$, as a function of $N_{A_2} \in IRZ$. From bottom to top, the curves refer to $\gamma_{\bar{\psi}\psi,IR,\Delta^1_F}$ (red), $\gamma_{\bar{\psi}\psi,IR,\Delta^2_F}$ (green), and $\gamma_{\bar{\psi}\psi,IR,\Delta^3_F}$ (blue).
Figure 3: Plot of $\bar{\gamma}_{XX,IR,\Delta A_2}$, calculated to order $j = 1, 2, 3$ for $G = SU(4)$, and $N_{A_2} = 4$, as a function of $N_F \in I_{IRZ}$. From bottom to top, the curves refer to $\bar{\gamma}_{XX,IR,\Delta A_2}$ (red), $\bar{\gamma}_{XX,IR,\Delta A_2}^2$ (green), and $\bar{\gamma}_{XX,IR,\Delta A_2}^3$ (blue).
Figure 4: Plot of $\gamma_{\bar{X}, IR, \Delta_{A_2}^j}$, calculated to order $j = 1, 2, 3$ for $G = SU(4)$, and $N_F = 4$, as a function of $N_F \in IRZ$. From bottom to top, the curves refer to $\gamma_{\bar{X}, IR, \Delta_{A_2}^1}$ (red), $\gamma_{\bar{X}, IR, \Delta_{A_2}^2}$ (green), and $\gamma_{\bar{X}, IR, \Delta_{A_2}^3}$ (blue).
Hasenfratz et al. find that the \((N_c, N_F, N_{A_2}) = (4, 4, 4)\) theory has an IFRP and hence is in the conformal window, close to the lower boundary, since \(\gamma_m^{(6)} \simeq 1\).

As one moves down from the upper boundary \(\mathcal{B}_{CW,u}\) of the conformal window toward the lower boundary, \(\mathcal{B}_{CW,\ell}\), the anomalous dimensions \(\gamma_{\bar{\psi}\psi, IR}\) and \(\gamma_{\bar{\chi}\chi, IR}\) increase.

The generalization of the condition in the single-rep. theory here is that the lower boundary \(\mathcal{B}_{CW,\ell}\) is reached when \(\max(\gamma_{\bar{\psi}\psi, IR}, \gamma_{\bar{\chi}\chi, IR}) = 1\). Since \(\gamma_{\bar{\chi}\chi, IR} > \gamma_{\bar{\psi}\psi, IR}\) here, this condition reduces to

\[
\gamma_{\bar{\chi}\chi, IR} = 1
\]

As in the single-rep. theory, the quadratic condition \(\gamma_{\bar{\chi}\chi, IR}(2 - \gamma_{\bar{\chi}\chi, IR}) = 1\), if solved for exactly, gives a double root at \(\gamma_{\bar{\chi}\chi, IR} = 1\) and hence is equivalent to the linear condition, but when applied in the context of a series expansion calculated to finite order, these yield different results.

See figure. The upper line (colored blue) is the upper boundary of the IRZ region and CW, \(\mathcal{B}_{IRZ,u} = \mathcal{B}_{CW,u}\). The locations of the lower CW boundary \(\mathcal{B}_{CW,\ell}\) from the quadratic \(\gamma_{CC}\) (green) from Kim, Hong, and Lee, PRD 101, 056008 (2020). For comparison, we have calculated \(\mathcal{B}_{CW,\ell}\) from the linear \(\gamma_{CC}\) (red). These are
approximately linear. The lower line (dotted) is the solution of $b_2$, the lower boundary of the IRZ region.

The boundary $B_{CW,\ell}$ as calculated with $\kappa_j^{(f)}$ up to $j = 3$ order from the linear condition includes the $(N_c, N_F, N_{A_2}) = (4, 4, 4)$ theory in the conformal window, while $B_{CW,\ell}$, as calculated with the same $j = 3$ inputs for the $\kappa_j^{(f)}$ coefficients excludes the $(4,4,4)$ theory from the conformal window.

Of course, these are both finite-order perturbative calculations. The actual determination of the actual lower CW boundary $B_{CW,\ell}$ requires a fully nonperturbative calculation, as provided by the lattice simulations.

Anomalous dimensions were also presented in the $(4,4,4)$ theory by Hasenfratz et al. for several gauge-singlet composite fermion operators and were found to be $\lesssim 0.5$. The requisite inputs to compute these with our methods are not yet available, but this could be of interest for future work.
Higher-loop Studies of the Beta Functions of $|\vec{\phi}|_4^4$ and $|\vec{\phi}|_3^6$ Theories

Here we briefly mention some recent results on scalar field theories from RS, PRD 107, 056018 (2023) [2301.01830] and RS, PRD 107, 096009 (2023) [2302.05422]. These could be of interest to the lattice community.

If the $\beta$ function of a theory is positive near zero coupling, then this theory is IR-free. As the momentum scale $\mu$ increases from the IR toward the UV, the coupling grows. It is of interest to investigate whether an IR-free theory might have a UV zero in the $\beta$ function, which would be a UV fixed point (UVFP) of the renormalization group.

An example of an IR-free theory with a UVFP is the $O(N)$ nonlinear $\sigma$ model in $d = 2 + \epsilon$ dims. From an exact solution of this model in the large-$N$ limit we found

$$\beta(\lambda) = \frac{d\lambda}{d \ln \mu} = \epsilon \lambda \left(1 - \frac{\lambda}{\lambda_c}\right), \quad \text{i.e..,} \quad \beta(\bar{\lambda}) = \frac{d\bar{\lambda}}{d \ln \mu} = \epsilon \bar{\lambda} \left(1 - \frac{\bar{\lambda}}{\bar{\lambda}_c}\right)$$

where $\lambda$ is an effective coupling, $\bar{\lambda} = \lim_{N \to \infty} \lambda N$, and $\bar{\lambda}_c = 2\pi \epsilon$ with $\epsilon \ll 1$ (W. Bardeen, B. W. Lee, and RS, PRD 14, 985 (1976); also Brézin, Zinn-Justin, PRB 14, 3110 (1976)).
QED is also IR-free. In RS, PRD 89, 045019 (2014) we studied the beta function of a U(1) theory with $N_f$ fermions of charge $q$ up to the 5-loop level, finding evidence against a UVFP. Hence, in this theory, $\alpha(\mu)$ grows as $\mu$ increases, eventually exceeding the regime where perturbative calculations are applicable.

The $O(N) \lambda |\vec{\phi}|^4$ theory in $d = 4$ is IR-free. There has long been interest in whether this theory might have a UVFP (some early work: Wilson, 1971; Wilson and Kogut, 1974; Brézin, Le Guillou, Zinn-Justin, 1974; Aizenman, 1982; Freedman, Smolensky, Weingarten, 1982; Dashen and Neuberger, 1983; Lüscher and Weisz, 1987; Kuti, Lin, and Shen, 1988; Kleinert and Schulte-Frohlinde, 2001; Zinn-Justin, 2002).

Interaction term: \[ L_{\text{int.}} = -\frac{\lambda}{4!} |\vec{\phi}|^4, \text{ where } \vec{\phi} = (\phi_1, ..., \phi_N). \] Define $a = \lambda/(4\pi)^2$.

beta function: \[ \beta = \frac{da}{d \ln \mu} = a \sum_{\ell=1}^{\infty} b_\ell \, a^\ell, \] where here $b_\ell$ is the $\ell$-loop coefficient. Denote truncation to $n$-loop order as $\beta_{n\ell}$.

In RS, PRD 94, 125026 (2016); PRD 96, 056010 (2017), using Kompaniets and Panzer 6-loop calculation of $\beta$ in 1606.09210 (in $\overline{\text{MS}}$ scheme), we investigated whether this 6-loop beta function has a UV zero.
In the range of $\lambda$ where the perturbative calculation of the beta function is reliable, we found evidence against a UV zero. We used scheme transformations and Padé approximants to confirm our conclusions.

In RS, PRD 107, 056018 (2023) [2301.01830] we have carried this search for a UV zero to 7-loop order, using the calculation of the 7-loop $\beta$ fn. by Schnetz, PRD 97, 085018 (2021). Again, we used scheme transformations and Padé approximants as checks.

A necessary condition for there to be robust evidence for a zero in the beta function of a QFT is that the values calculated at successive loop orders should be close to each other. We find that this condition is not satisfied here. At $n = 3$, $n = 5$ and $n = 7$ loop order, $\beta_{n\ell}$ has no UV zero. Although $\beta_{2\ell}$ has a UV zero, it occurs at too large a value of $\lambda$ for the perturbative calculation to be reliable.

See figure for $N = 1$. color coding: $\beta_{2\ell}$ (red, solid); $\beta_{3\ell}$ (green, dashed); $\beta_{4\ell}$ (blue, dotted); $\beta_{5\ell}$ (black, dot-dashed); $\beta_{6\ell}$ (cyan, solid); $\beta_{7\ell}$ (brown, solid). Curves from bottom to top: $n = 6, 4, 2, 3, 5, 7$. 
Another IR-free scalar theory is the $O(N) |\phi|^6$ theory in $d = 3$, with Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_\nu \vec{\phi}) \cdot (\partial^\nu \vec{\phi}) - \frac{1}{2} m^2 |\vec{\phi}|^2 - \frac{\lambda}{4N} |\vec{\phi}|^4 - \frac{g}{6N^2} |\vec{\phi}|^6,$$

Since the coeffs. of the $|\vec{\phi}|^2$ and $|\vec{\phi}|^4$ terms in this $d = 3$ theory are dimensionful, and since $\lim_{\mu \to \infty} m^2/\mu^2 = 0$ and $\lim_{\mu \to \infty} \lambda/\mu = 0$, they are expected to be a negligible role in the UV limit.

This theory is known to have a UVFP in the large-$N$ limit (Townsend, 1977; Pisarski, 1982; Appelquist and Heinz, 1982).

An interesting question is: over what range of finite $N$ does this large-$N$ UVFP persist?

In RS, PRD 107, 096009 (2023) [2302.05422] we investigated this. As before, a necessary condition for a UVFP is that successive orders in perturbation theory should yield $g_{UVFP,n\ell}$ values that are close to each other.

Using a combination of direct analysis of the the 6-loop beta function from Hager, J. Phys. A 35, 2703 (2002), Padé approximants, and scheme transformations, we showed that there is robust evidence for a UVFP for $N \gtrsim 2 \times 10^3$. 
Conclusions

• Understanding the UV to IR evolution of an asymptotically free gauge theory and the nature of the IR behavior is of fundamental field-theoretic interest.

• Our higher-loop perturbative calculations of anomalous dimensions with T. A. Ryttov give information on properties at an IR fixed point for theories with fermions in a single representation and also theories with fermions in multiple different representations.

• Here have compared our higher-order scheme-independent calculations of anomalous dimensions with recent lattice measurements in an SU(3) gauge theory with $N_F = 10$ fermions in the fundamental representation and an SU(4) theory with $N_F = 4$ fermions in the fundamental rep. and $N_{A_2} = 4$ fermions in the antisymmetric rank-2 rep., finding agreement for both theories.

• We have also mentioned some results on UV behavior in a $|\phi|^4$ theory in $d = 4$ and a $|\phi|^6$ theory in $d = 3$.

THANK YOU