QCD at nonzero isospin chemical potential

6144 pions in a box

Based on 2307.15014 with
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[NPLQCD collaboration]

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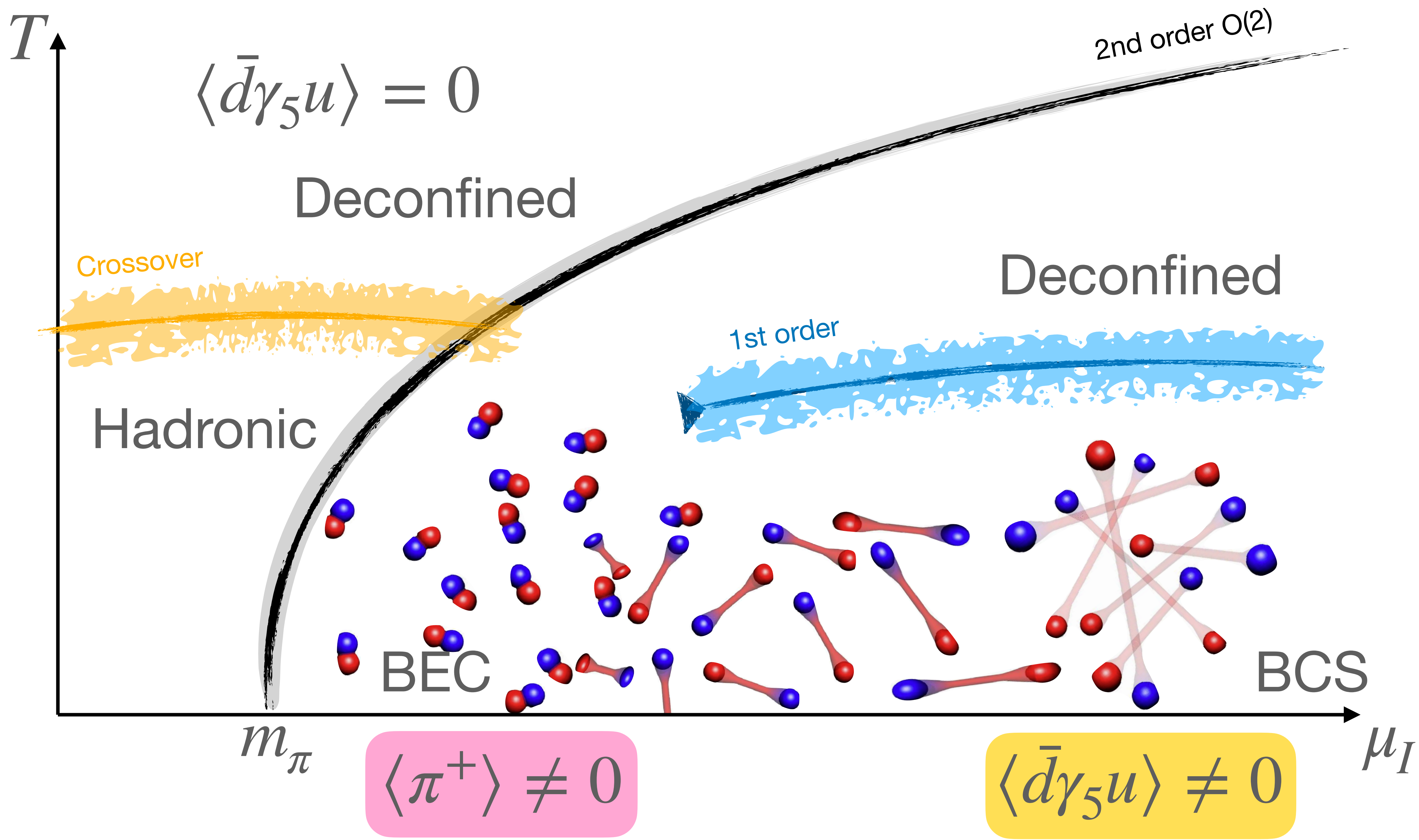
QCD at $\mu_I \neq 0$

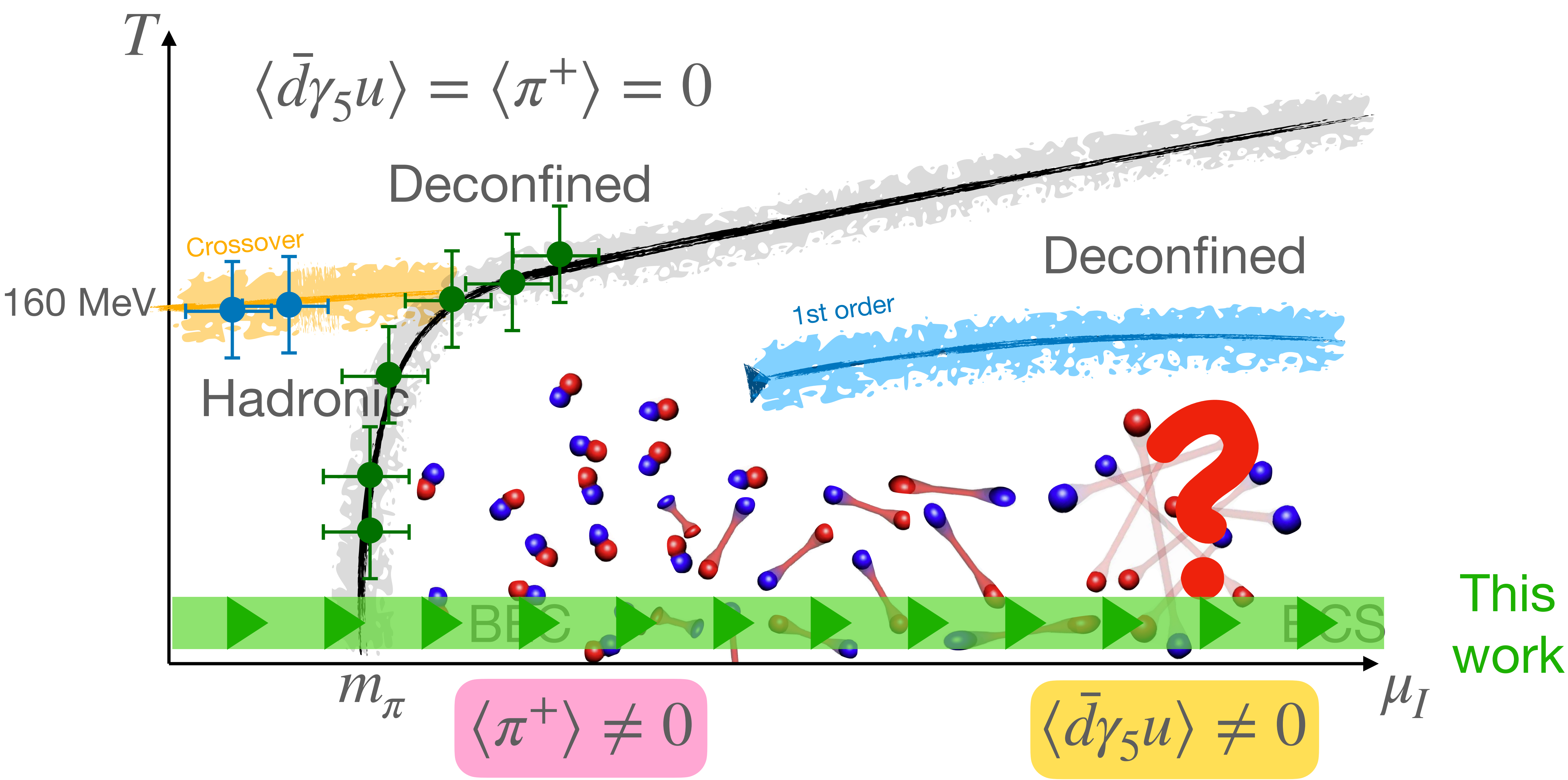
Conjectured phase diagram

- Son & Stephanov PRL 2001



QCD at $\mu_I \neq 0$
Conjectured phase diagram

- Status in 2023 - much recent work from Brandt, Cuteri & Endrodi



(Grand) Canonical approach

Isospin chemical potential

- Isospin chemical potential

$$S \longrightarrow S + \mu_I \int dx \left[\bar{u}(x)\gamma_0 u(x) - \bar{d}(x)\gamma_0 d(x)\right]$$

- Canonical approach: thermodynamic relation

$$\mu_I = \frac{dE}{dn_I}$$

- Study energy of system as isospin charge changes

- Correlation functions with quantum numbers of many charged pions

$$C_n(t) = \left\langle \left(\sum_x \pi^-(\mathbf{x}, 0) \right)^n \prod_{i=1}^{n} \pi^+(\mathbf{y}_i, t) \right\rangle$$

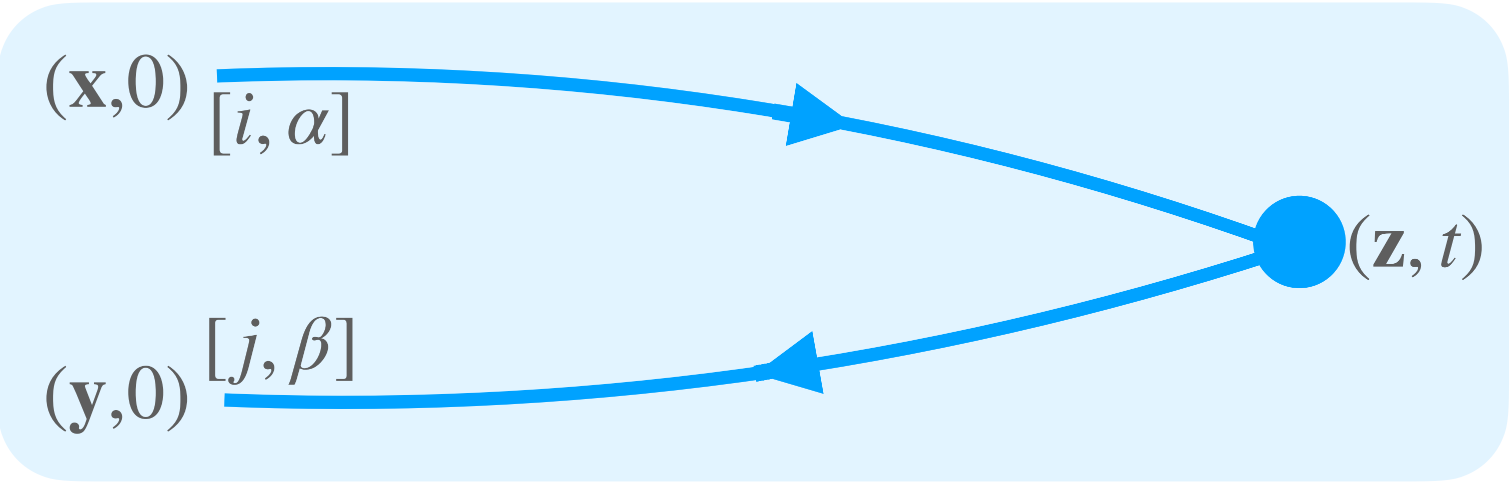
but large number of Wick contractions: $\sim 10^{40,000}$ for $n = 6144$

Many pion correlation functions

Pion blocks

- Previous studies used
  - Traces, Recursion relations, Vandermonde matrices & FFTs
  - Limited in n by cost (best algorithm $\sim \mathcal{O}(n^4)$) and numerical precision demands
- Made use of zero-momentum pion block ($12L^3 \times 12L^3$ matrix)

$$\Pi_{(i,\alpha)(j,\beta)}(\mathbf{x},\mathbf{y};t) = \sum_{k,\gamma,\mathbf{z}} S_{(i,\alpha)(k,\gamma)}(\mathbf{x},0;\mathbf{z},t) S^{\dagger}_{(k,\gamma)(j,\beta)}(\mathbf{y},0;\mathbf{z},t)$$



Many pion correlation functions

Symmetric polynomial algorithm

- New algorithm based on symmetric polynomials over eigenvalues of Π (denoted $\overrightarrow{x} = \{x_1, \ldots x_N\}$ with $N = 12L^3$)

$$C_n(t) = n!\, E_n(\vec{x})$$

  where for $1 \leq n \leq N$

$$E_n(\vec{x}) \equiv E_n(\{x_1, \ldots, x_N\}) \equiv \sum_{i_1 < \cdots < i_n}^{N} x_{i_1} \ldots x_{i_n}$$

- Recurrence relation for

$$E_k(\{x_1, \ldots, x_M\}) = x_M E_{k-1}(\{x_1, \ldots x_{M-1}\}) + E_k(\{x_1, \ldots, x_{M-1}\}),$$

  (numerically stable and cost in $\mathcal{O}(N^2)$ for all $n \in \{1, \ldots, N\}$)

- Overall cost dominated by finding the eigenvalues: $\mathcal{O}(N^3)$

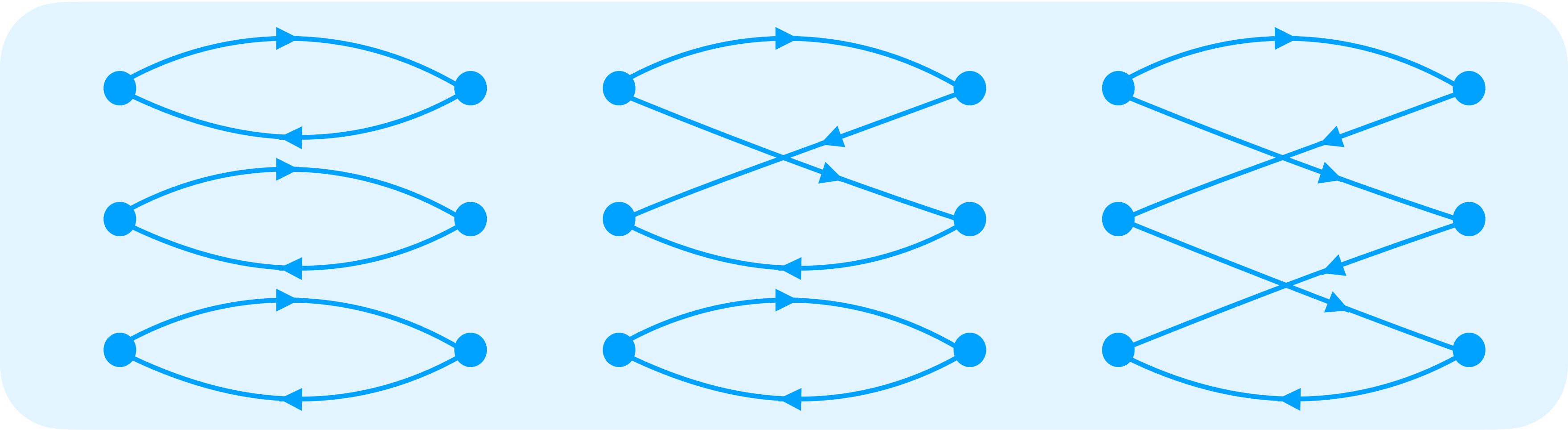
- See 2307.15014 for proof

Many pion correlation functions

Simple example (n=3 for N=4)

- $C_3(t)$ given by

$$C_3 = \mathrm{Tr}(\Pi)^3 - 3\mathrm{Tr}(\Pi^2)\mathrm{Tr}(\Pi) + 2\mathrm{Tr}(\Pi^3)$$



- Expand using trace as sum of powers of eigenvalues

$$\begin{aligned}
&= (x_1 + x_2 + x_3 + x_4)^3 \\
&\quad -3(x_1^2 + x_2^2 + x_3^2 + x_4^2)(x_1 + x_2 + x_3 + x_4) \\
&\quad +2(x_1^3 + x_2^3 + x_3^3 + x_4^3) \\
&= 6(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4)
\end{aligned}$$

Many pion correlation functions

Lattice QCD calculations

- Study on two ensembles of 2+1f clover gauge configurations with close to physical quark masses

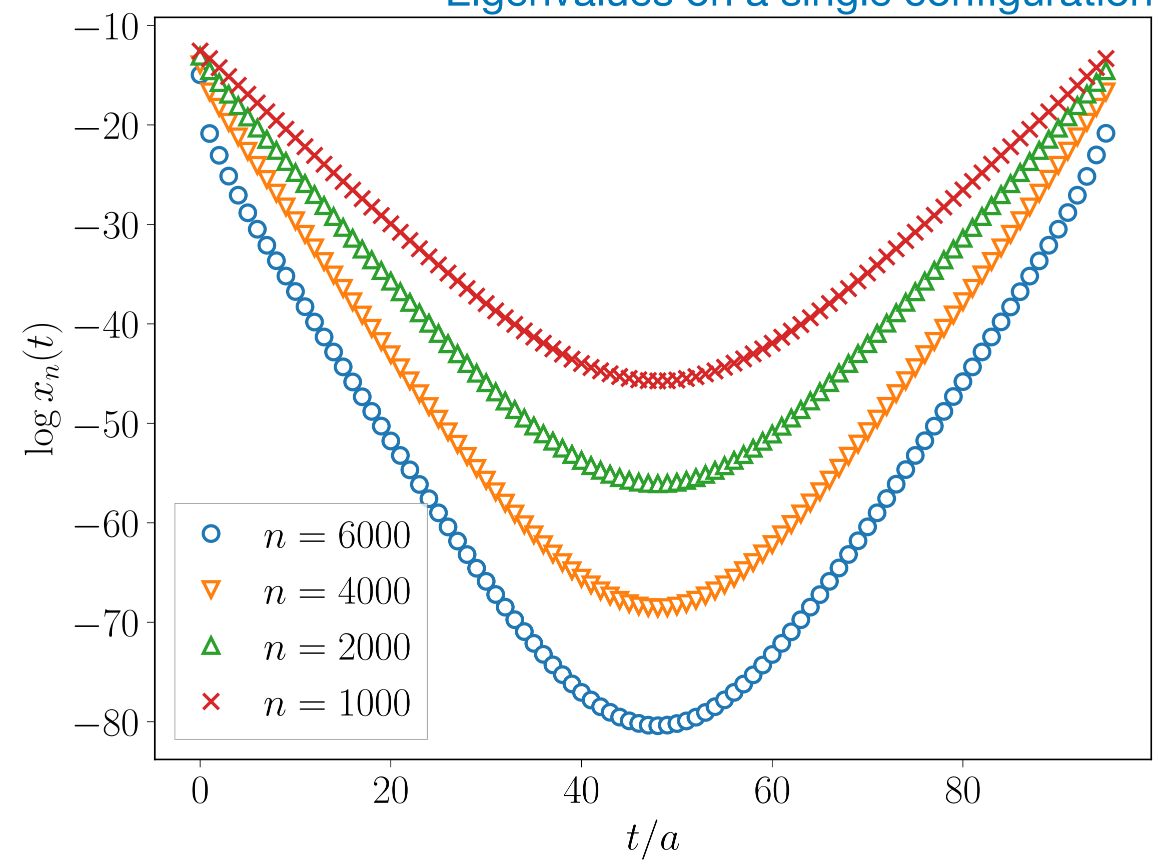
Label	$N_{\rm conf}$	β	C_{SW}	am_{ud}	am_s	$L^3 \times T$	a (fm)	M_π (MeV)	$M_\pi L$
A	201	6.3	1.20537	-0.2416	-0.2050	$48^3 \times 96$	0.091(1)	166(2)	3.7
B	322	6.3	1.20537	-0.2416	-0.2050	$64^3 \times 128$	0.091(1)	172(6)	5.08

- Sparsened quark propagators computed from grid of 8^3 sites on one timeslice: $N = 12 \times L^3 = 12 \times 8^3 = 6144$

- Eigenvalues computed by SVD of time sliced quark-propagator (since $\Pi = S^\dagger S$)

- Calculations performed in double, 2-double and 3-double

Eigenvalues on a single configuration

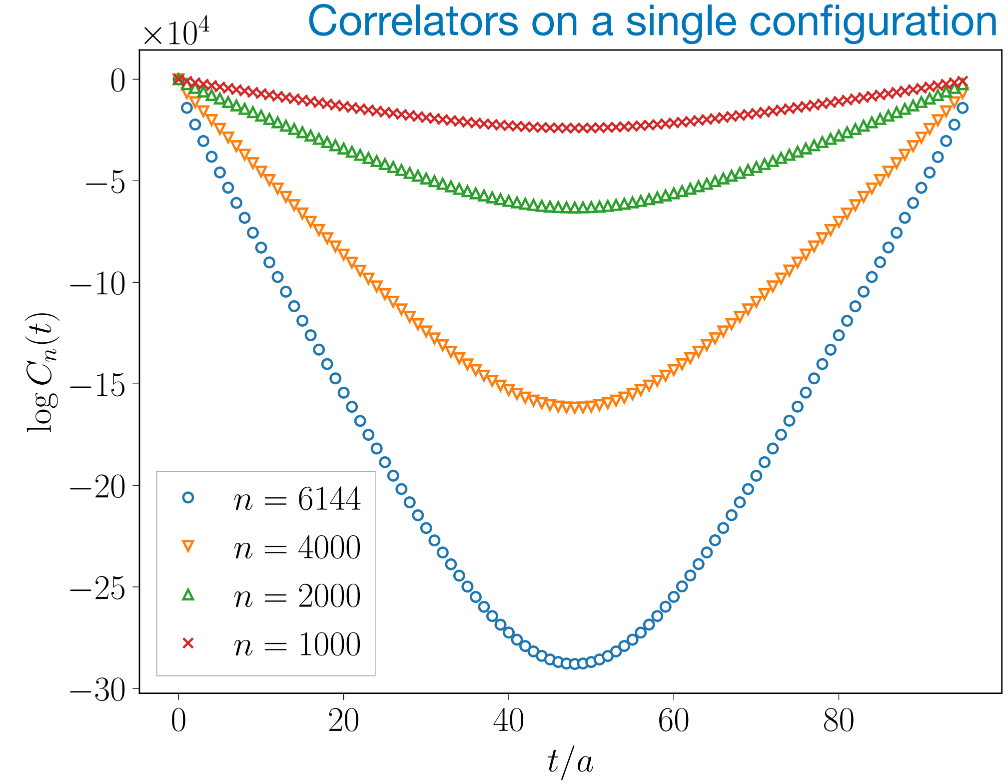


Many pion correlation functions

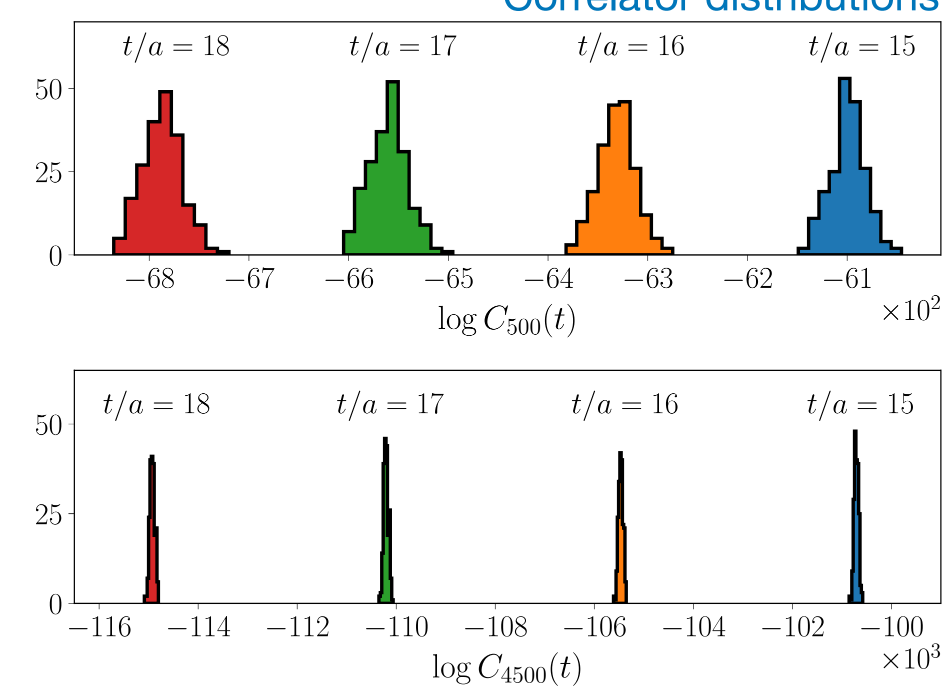
Lattice QCD calculations

- Correlation functions vary rapidly in Euclidean time
  - $C_{6144}(t)$ varies by $> 10^5$ orders of magnitude
- Correlation functions vary between samples by many orders of magnitude
  - Central Limit Theorem only valid at unachievable sample size 😒
  - Correlation function distributions are approximately log-normal 😌

Correlators on a single configuration



Correlator distributions

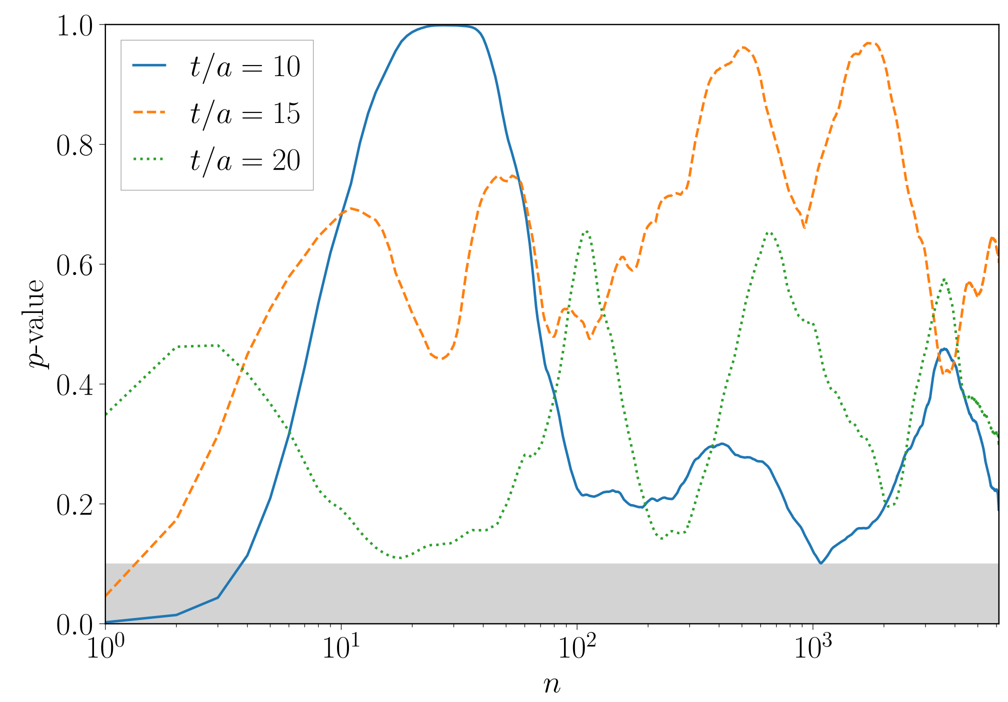


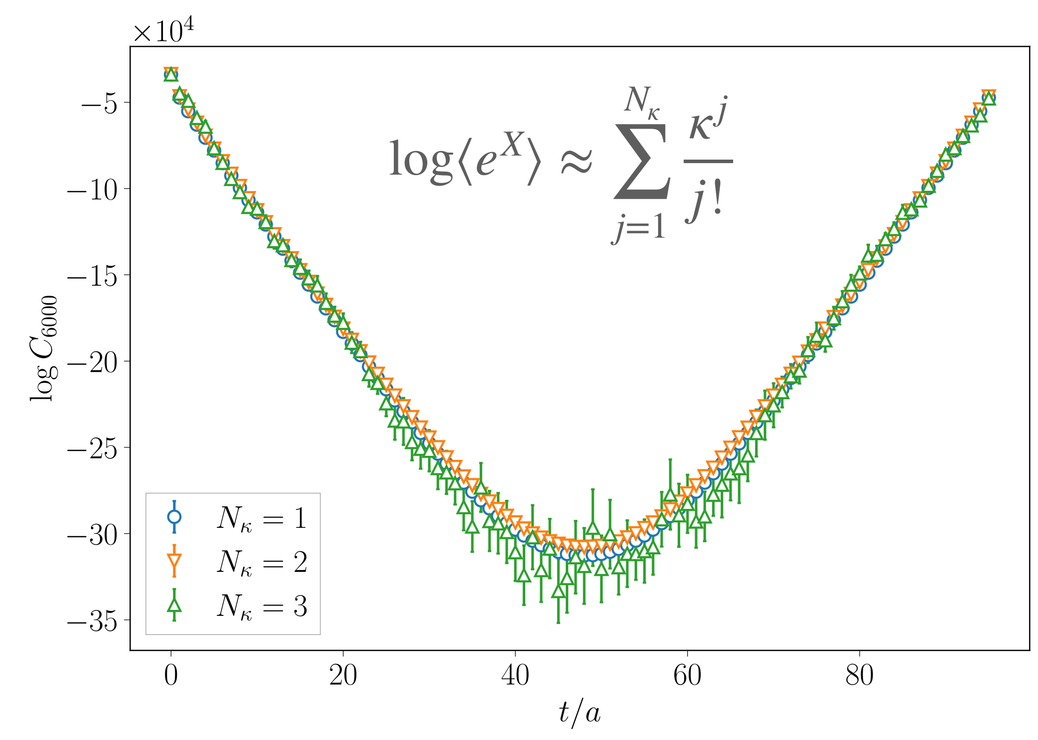
Many pion correlation functions

Log-normality tests and cumulants

- No statistically significant deviations from log-normal for $n > 4$
  - Shapiro-Wilk test $p > 0.1$
- Deviations can be incorporated through cumulants but only contribute noise
- Henceforth assume data are log-normal
  - i.e. $\log C_n(t)^{[U]} \sim \mathcal{N}(\mu_n(t), \sigma_n(t))$ where

$$\mu_n = \frac{1}{N_{\text{conf}}} \sum_{i=1}^{N_{\text{conf}}} \log C_n^{[U_i]}(t) \qquad \sigma_n^2 = \frac{1}{N_{\text{conf}} - 1} \sum_{i=1}^{N_{\text{conf}}} \left(\log C_n^{[U_i]}(t) - \mu_n\right)^2$$





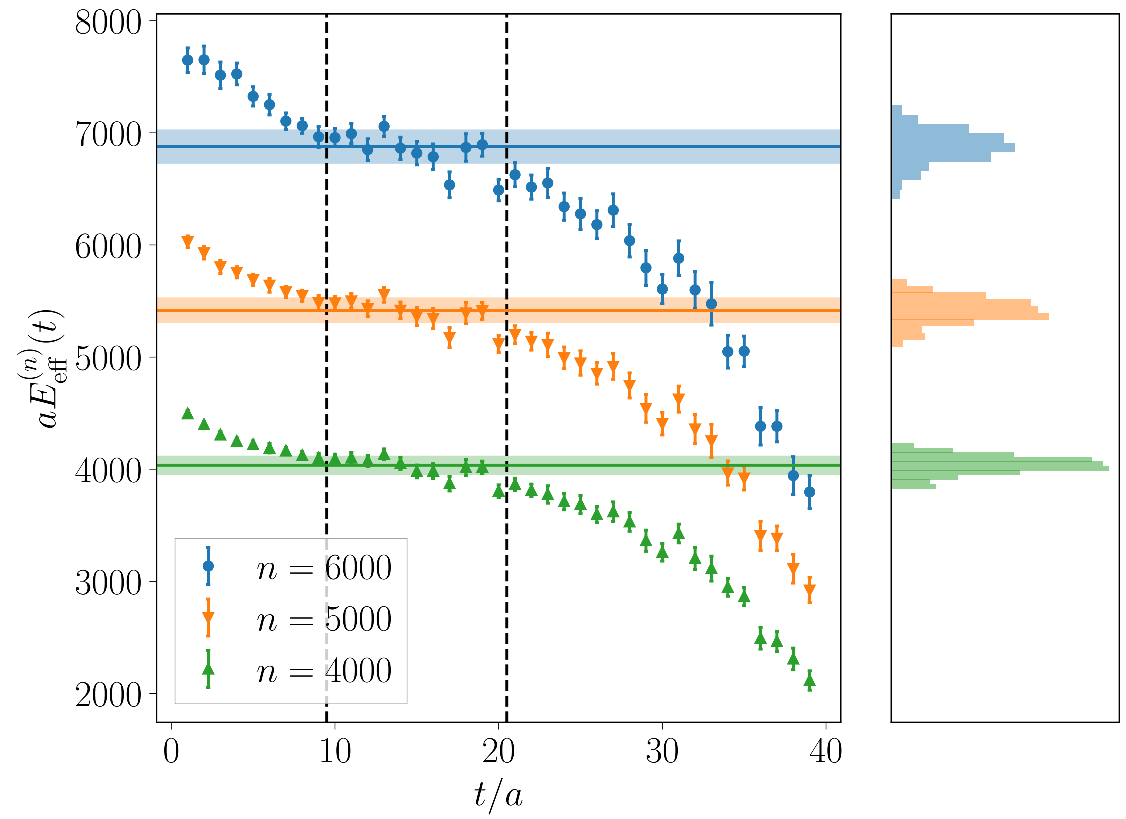
Many pion correlation functions

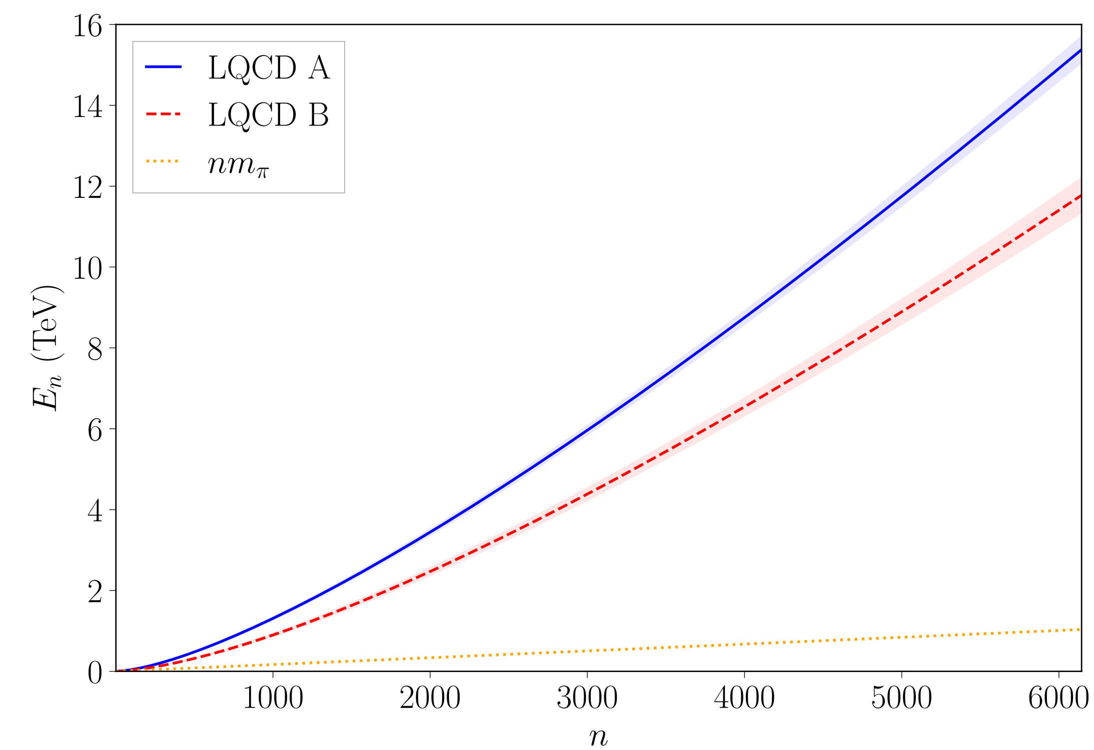
Many pion energies

- Effective energy from log-normality

$$E_{\text{eff}}^{(n)}(t) = \mu_n(t) - \mu_n(t-1) + \frac{\sigma_n^2(t)}{2} - \frac{\sigma_n^2(t-1)}{2}$$

- ~~CLT~~: χ^2-fitting makes no sense
- Bootstrap analysis takes value of $E_{\text{eff}}^{(n)}$ for random timeslice in plateau region
- Entire bootstrap histogram propagated into subsequent analysis
- Energy significantly larger than that of n free pions





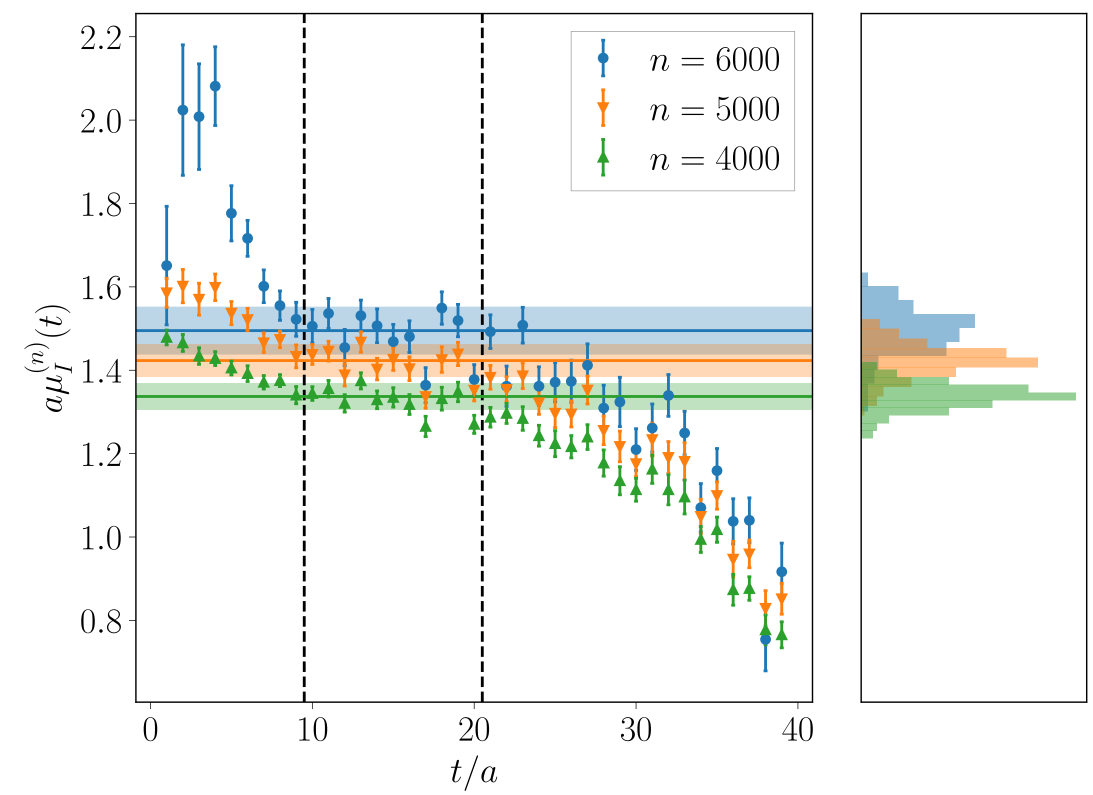
QCD at $\mu_I \neq 0$

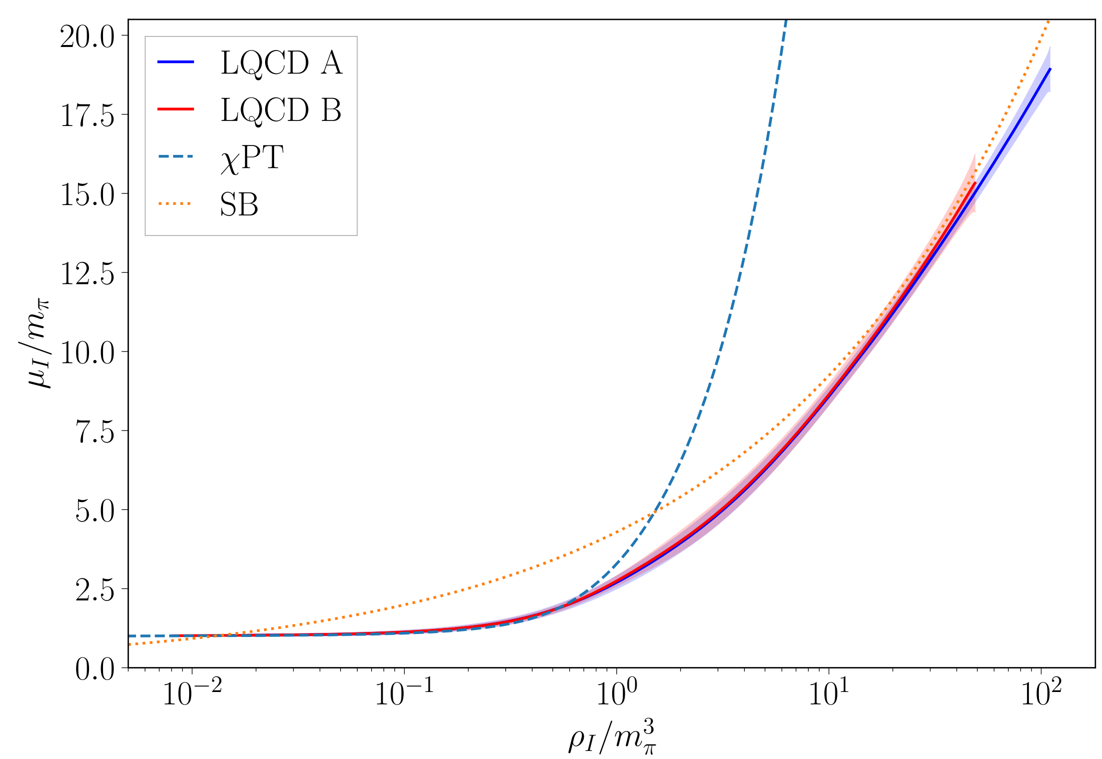
Isospin chemical potential

- Isospin chemical potential

$$\mu_I(n) = \frac{dE_n}{dn}\bigg|_{V\text{ const}} \approx \frac{E_{n+1} - E_{n-1}}{2}$$

- Two volumes: A=(4.4 fm)³, B=(5.8 fm)³ and two temporal extents: A=(9 fm), B=(12 fm)
- Curve collapse $\Longrightarrow$ thermodynamic limit ($T \sim 20$ MeV)
- Agreement with
  - Chiral perturbation theory for $\mu_I \to 0$
  - Stefan-Boltzmann/pQCD for $\mu_I \to \infty$

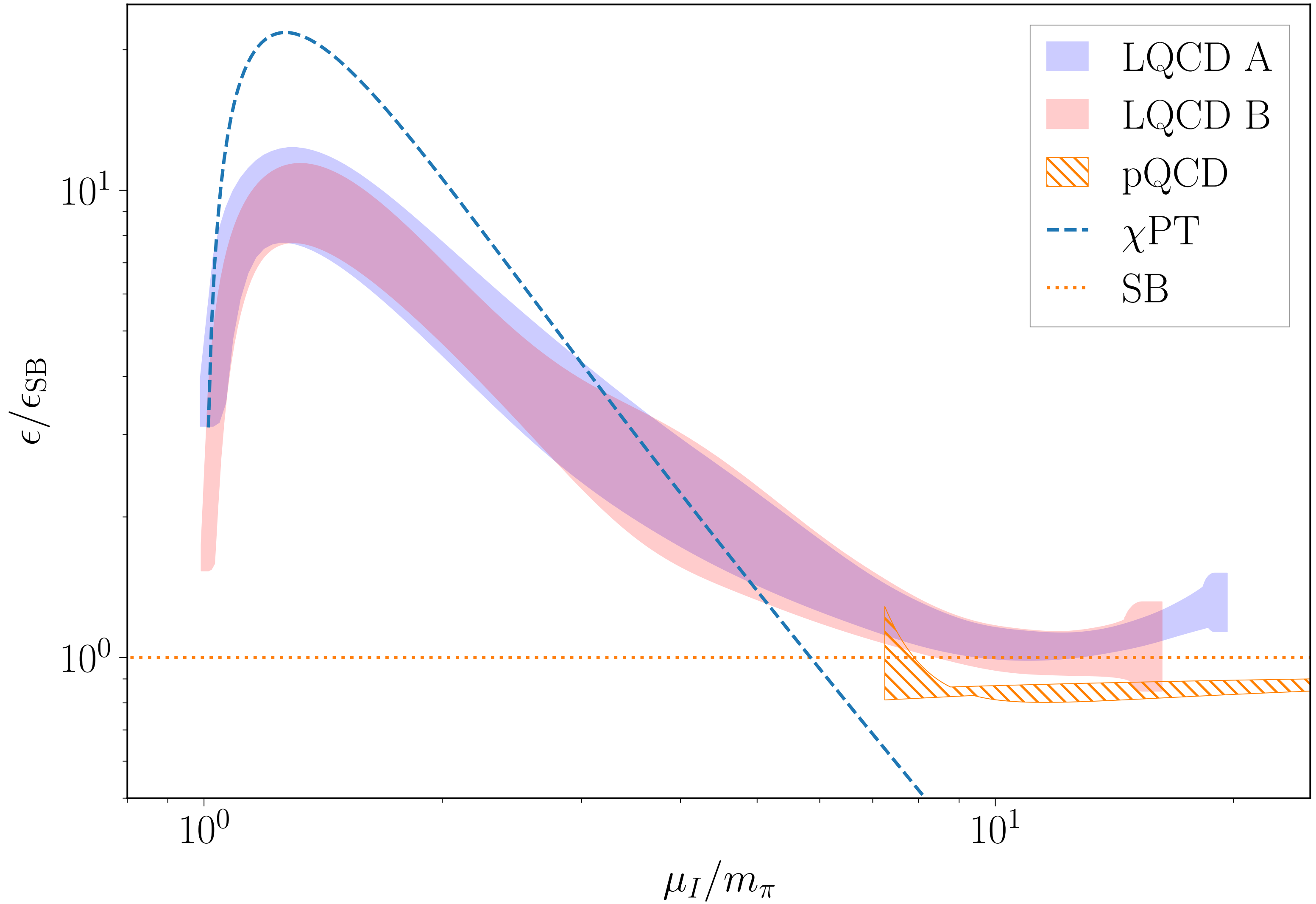




QCD at $\mu_I \neq 0$

Energy density

- Energy density ratio to SB expectation
  - Peak signals onset of pion BEC (in agreement with χPT)
  - Eventual approach to pQCD/ideal gas limit



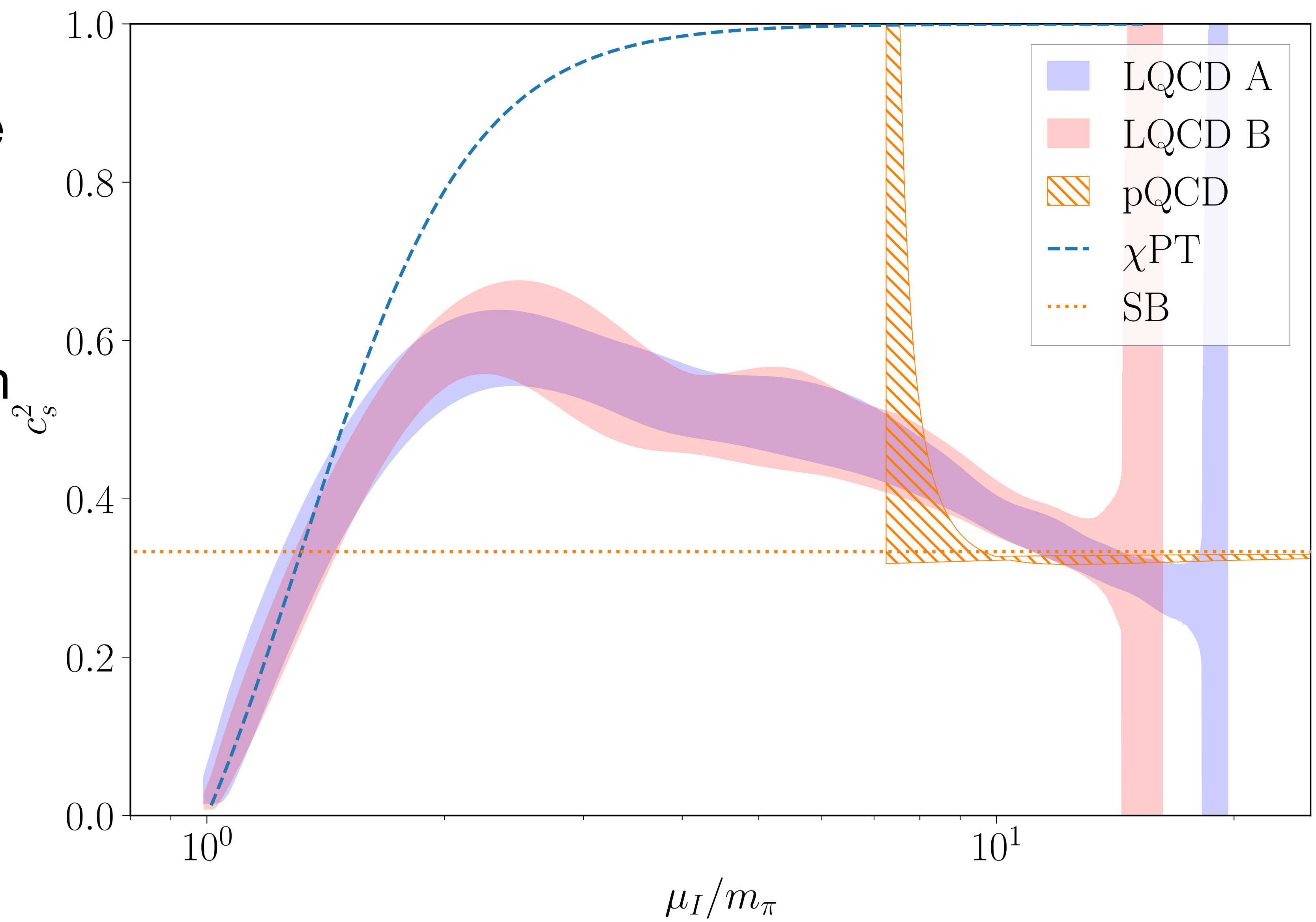
QCD at $\mu_I \neq 0$

Speed of sound

- Since temperature is $0 \sim T \leq 20$ MeV, isentropic speed-of-sound can be determined

$$c_s^2 = \frac{dp}{d\epsilon} = \frac{n}{\mu_I}\frac{d\mu_I}{dn} = \frac{n}{dE/dn}\frac{d^2E}{dn^2} \approx 2n\frac{E_{n+1} - 2E_n + E_{n-1}}{E_{n+1} - E_{n-1}},$$

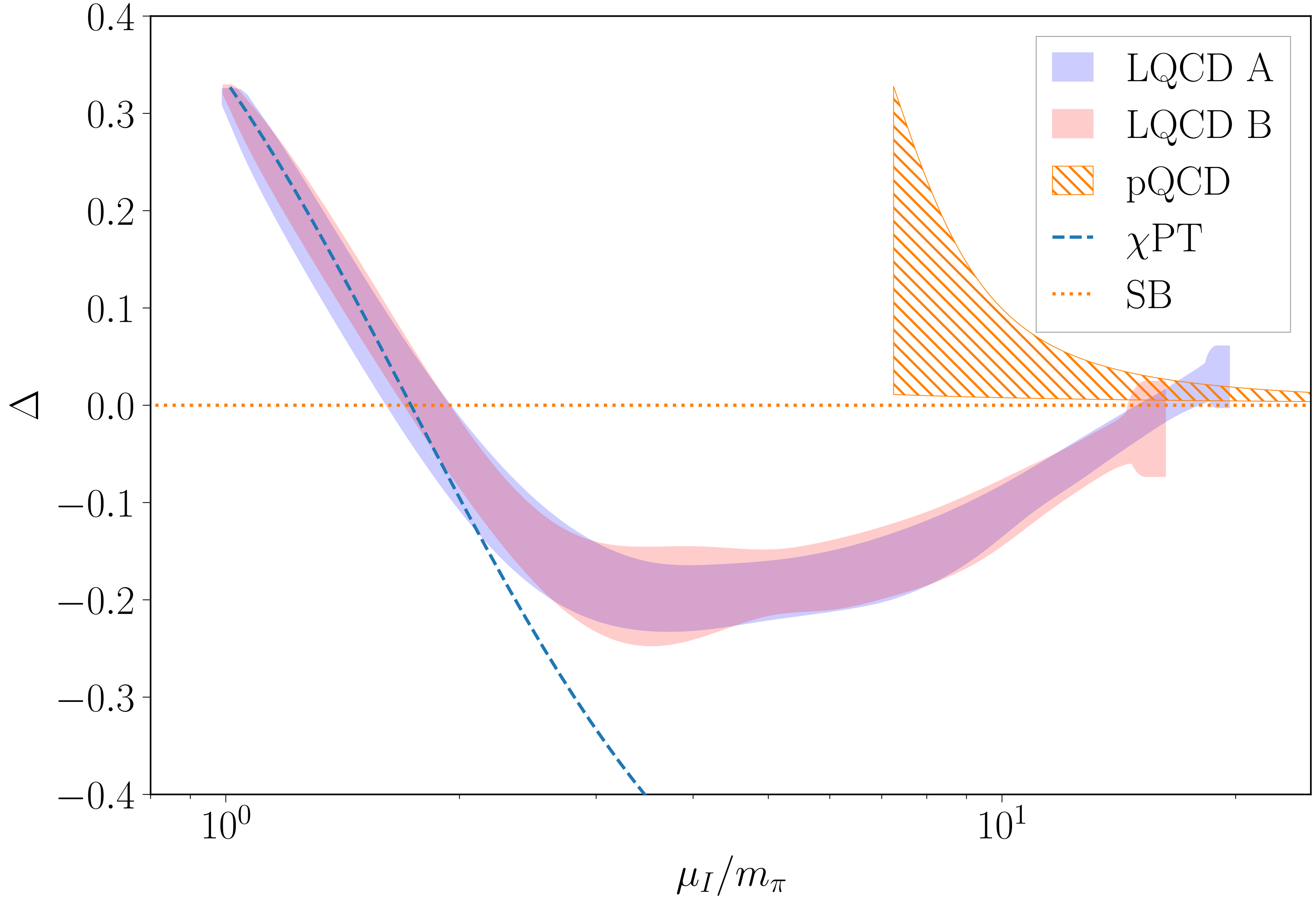
- Exceeds conformal bound $c_s^2 \leq 1/3$ over wide range of μ_I
- Similar behaviour seen in grand canonical approach [Brandt, Cuteri, Endrodi 2212.14016]
- Similar behaviour seen in $N_c = 2$ QCD [E. Itou, Friday]
- Eventually relaxes to pQCD/ideal gas



QCD at $\mu_I \neq 0$

Trace anomaly

- Trace anomaly $\Delta = 1/3 - p/\epsilon$ provides a measure of interactions
  - Also shows large chemical potential needed to reach pQCD



QCD at $\mu_I \neq 0$

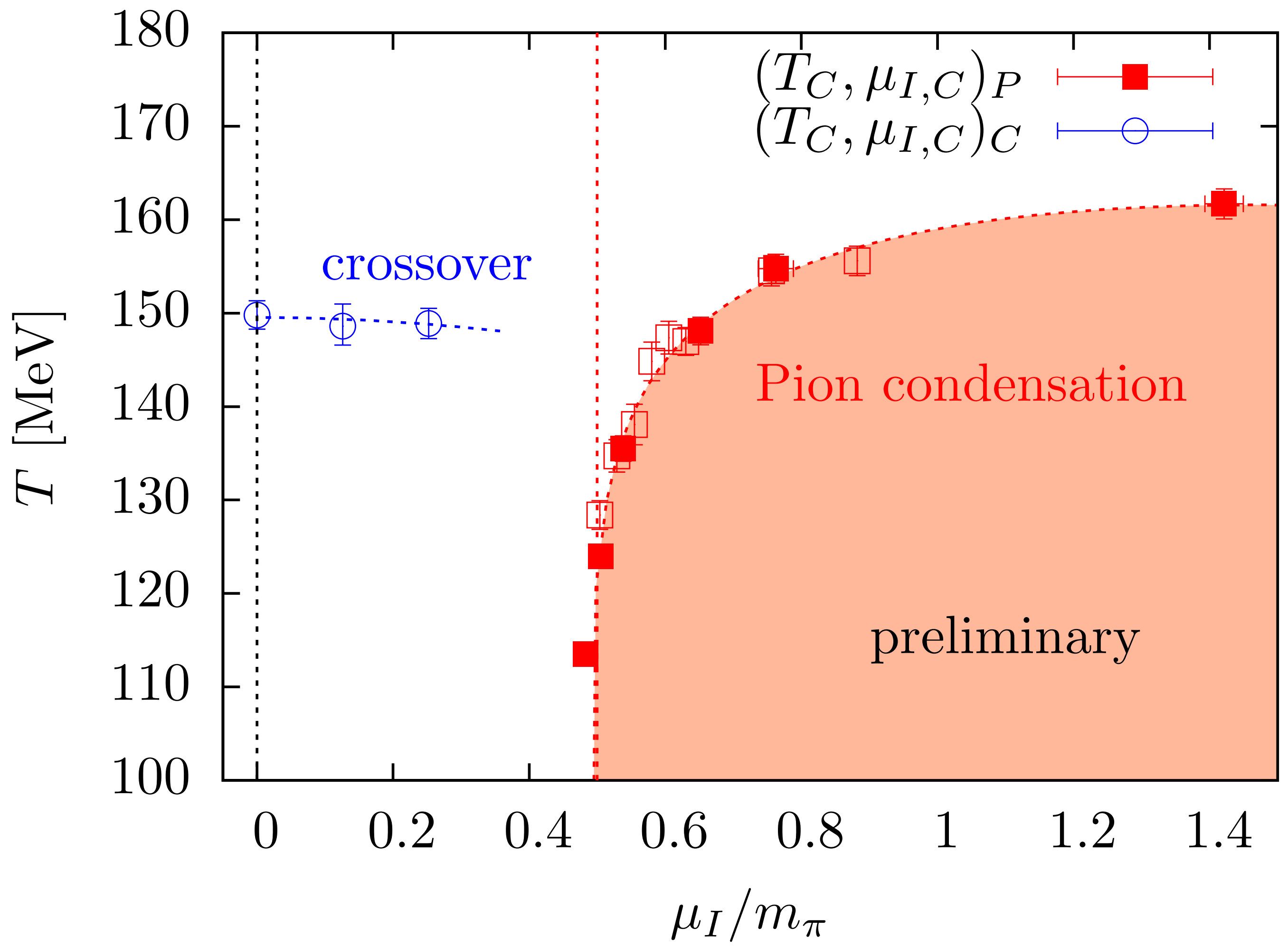
A fascinating playground

- Symmetric polynomial algorithm allows extension of canonical approach to large μ_I
- Enormous scale variation breaks the central limit theorem
    - Analysis based around empirically observed log-normality
- Clear signal for transition to pion BEC and eventually to BCS superconducting state predicted by pQCD
    - Large $\mu_I \sim 15 m_\pi \sim 2$ GeV needed to reach pQCD
    - What about at baryon chemical potential?
- Conformal bounds from holographic models clearly exceeded as in $N_c = 2$ QCD - analogue would have interesting consequences for neutron star equation of state

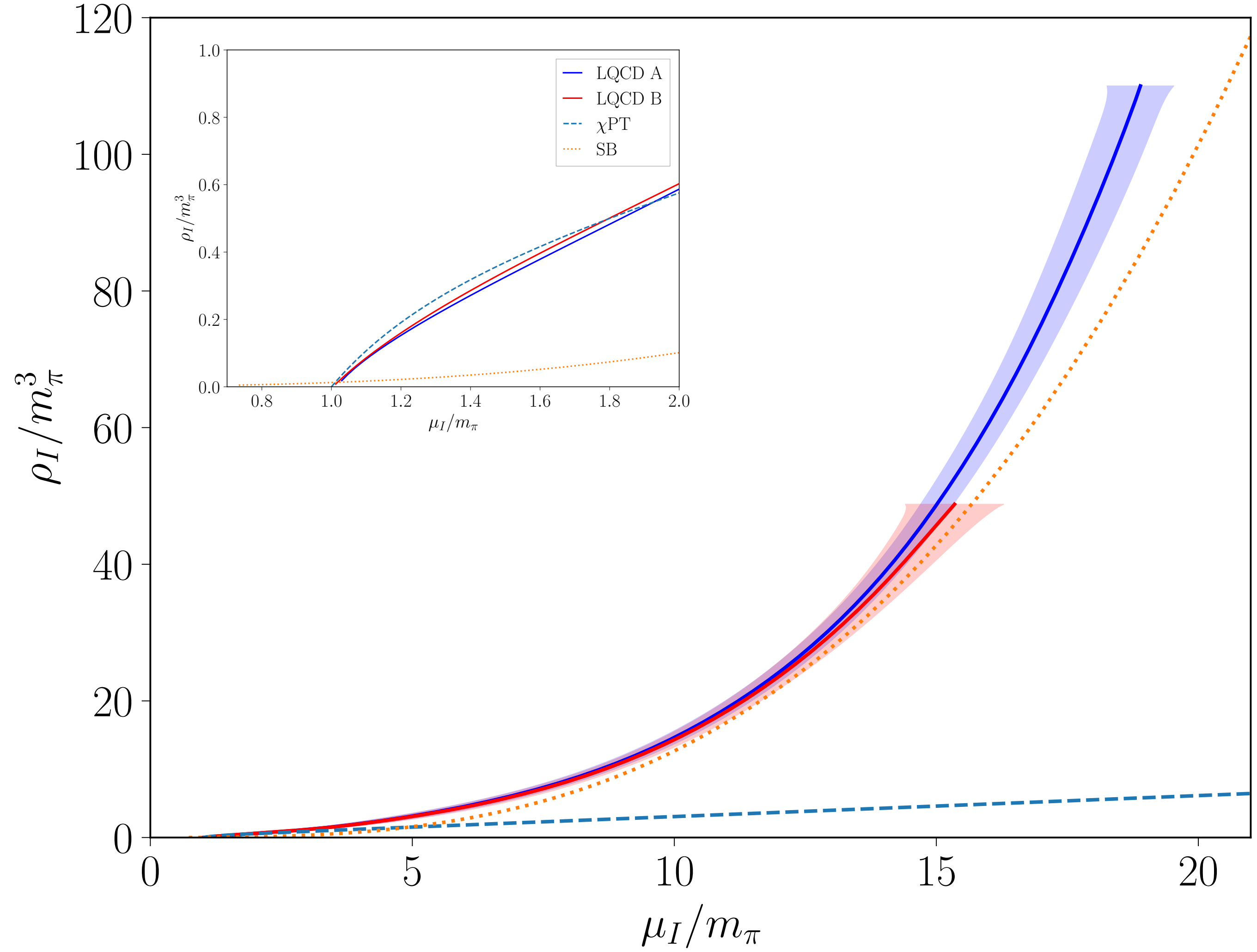
QCD at $\mu_I \neq 0$

Previous studies

- Brandt & Endrodi 1611.06758



QCD at $\mu_I \neq 0$
density vs chemical potential



QCD at $\mu_I \neq 0$

Log-normality test