The 40th International Symposium on Lattice Field Theory

Universal scaling and the asymptotic behaviour of Fourier coefficients of the baryon number density

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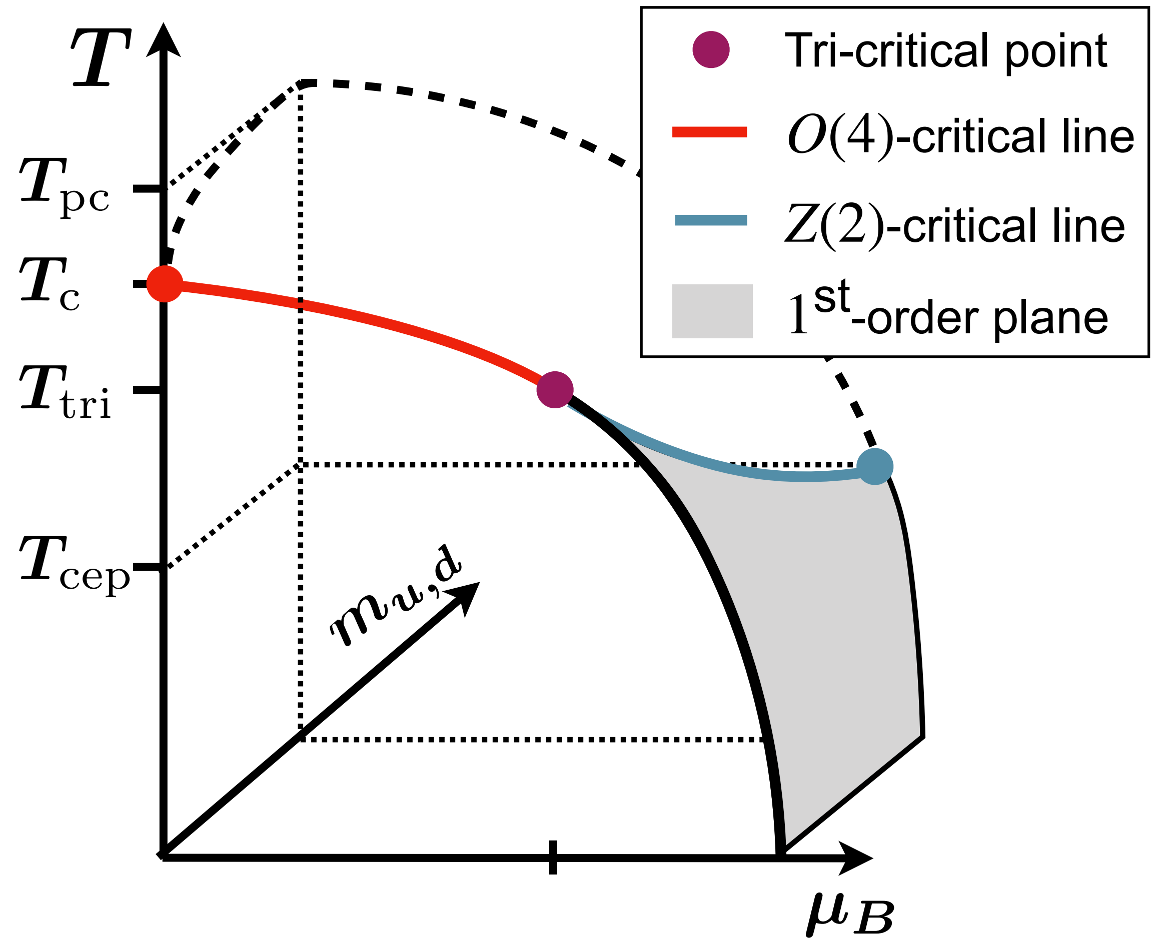
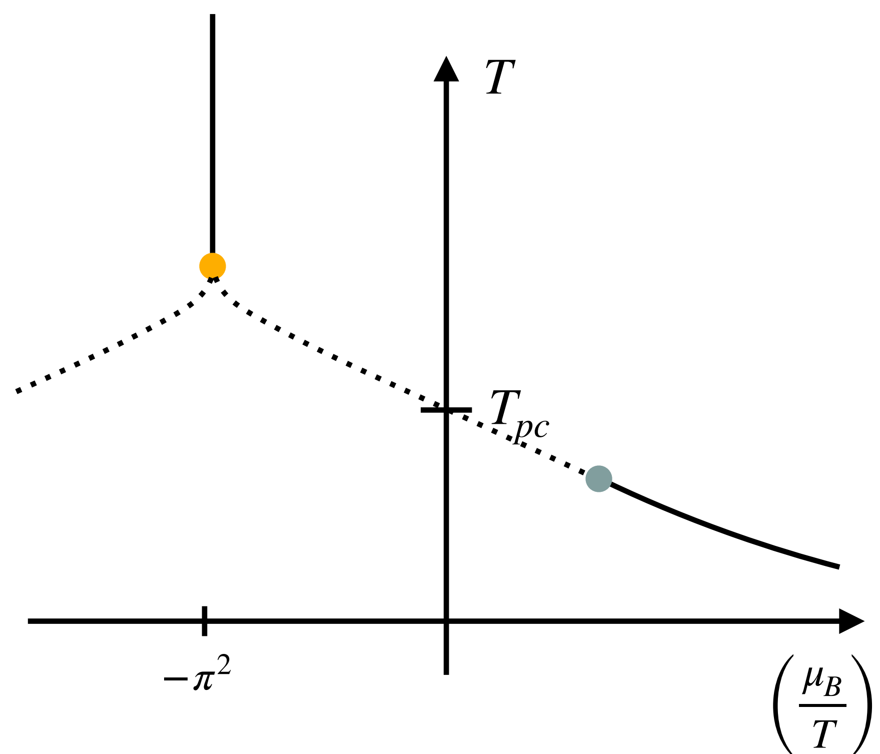
We want to investigate the analytic structure of the partition function ($\ln Z$)

- Locating singularities in the complex μ_B/T -plane might help to find the elusive QCD critical point
- We have theoretical guidance by the Lee-Yang theorem, which predicts branch-cuts in the scaling function of the order parameter, located at universal positions.

We use lattice QCD data of the baryon number density ($\partial_\mu \ln Z$) at imaginary chemical potential





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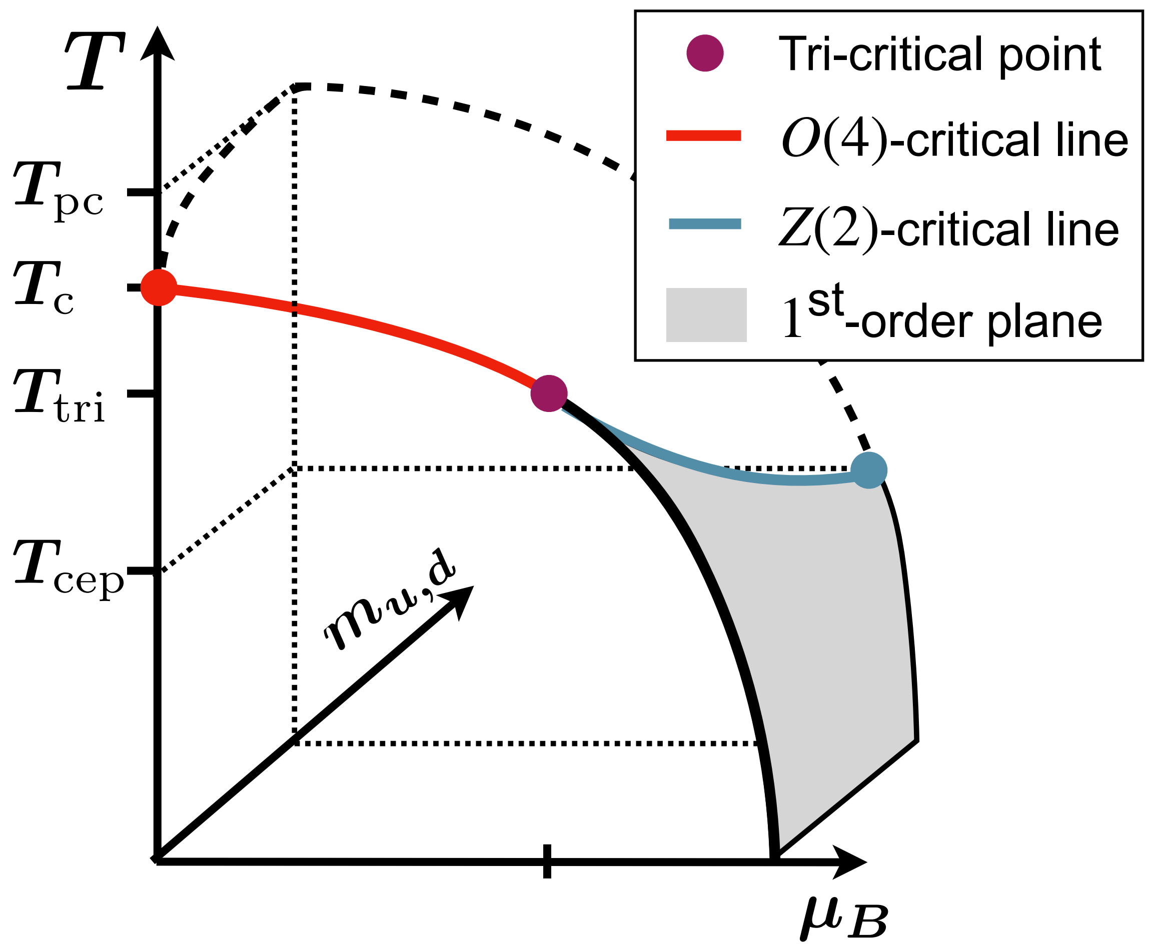
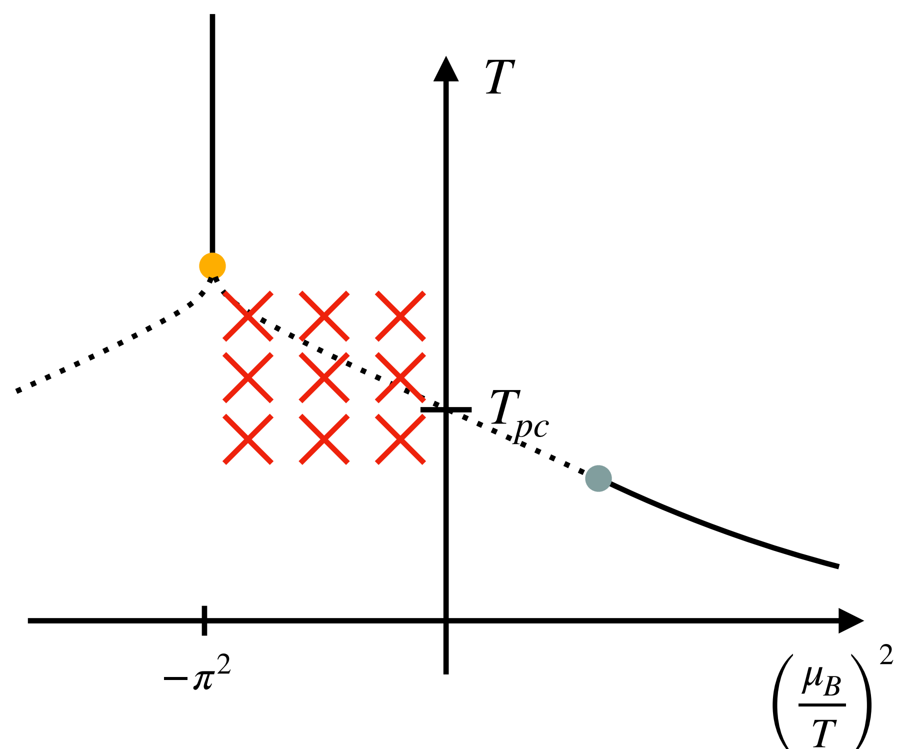


We use lattice QCD data of the baryon number density ($\partial_\mu \ln Z$) at imaginary chemical potential

- Perform analytic continuation by mean of a multi-point Padé analysis **(see Talk by F. Di Renzo)**
- Inspect Fourier coefficients of the baryon number density **(this talk)**

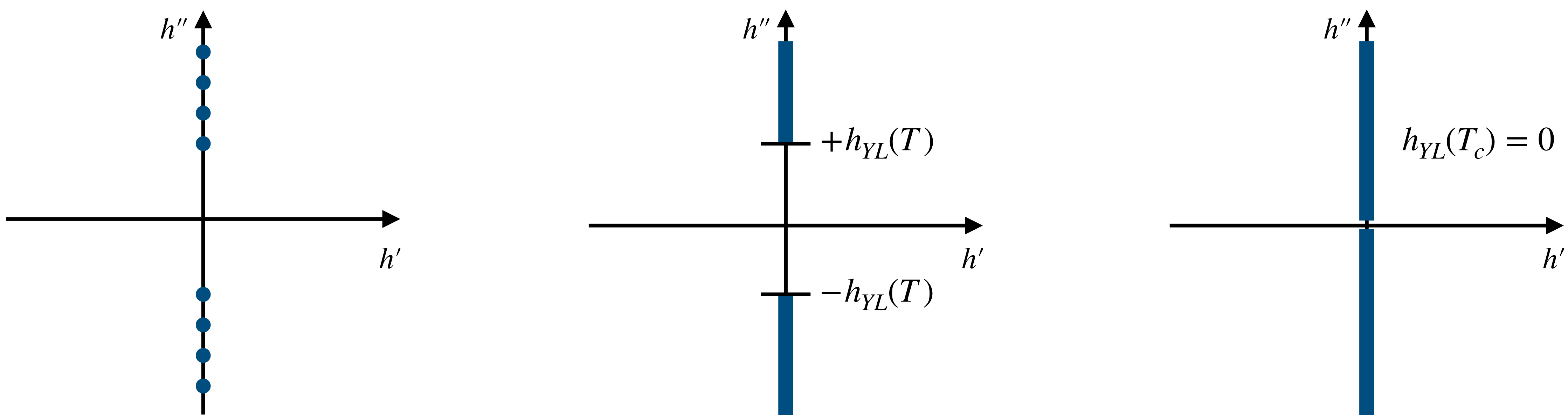


- **Recap on the Lee-Yang edge singularity**
  - Lee-Yang theorem (circle theorem)
  - A branch cut in the universal scaling function
  - Scaling variables for Roberge-Weiss, chiral transition and QCD critical point
- **An update on the Roberge-Weiss critical temperature**
  - A preliminary continuum extrapolation from $N_\tau = 4{,}6{,}8$
- **Fourier Coefficients, an alternative to the multi-point Padé?**
  - Expected asymptotic behaviour
  - Fourier coefficients in the quark-meson model
  - Fourier coefficients from lattice QCD data

Consider a generic Ising or O(N) model:

- One finds zeros of the partition function only at imaginary values of the symmetry breaking field [Lee, Yang 1952]
- In the thermodynamic limit the zeros become dense on the line $h' = 0$

$Z(V, T, h) \equiv 0, \quad h = h' + ih''$



V finite, $T > T_c$ | $V \to \infty$, $T > T_c$ | $V \to \infty$, $T \to T_c$

→ a brach cut is observed in the free energy density and order parameter

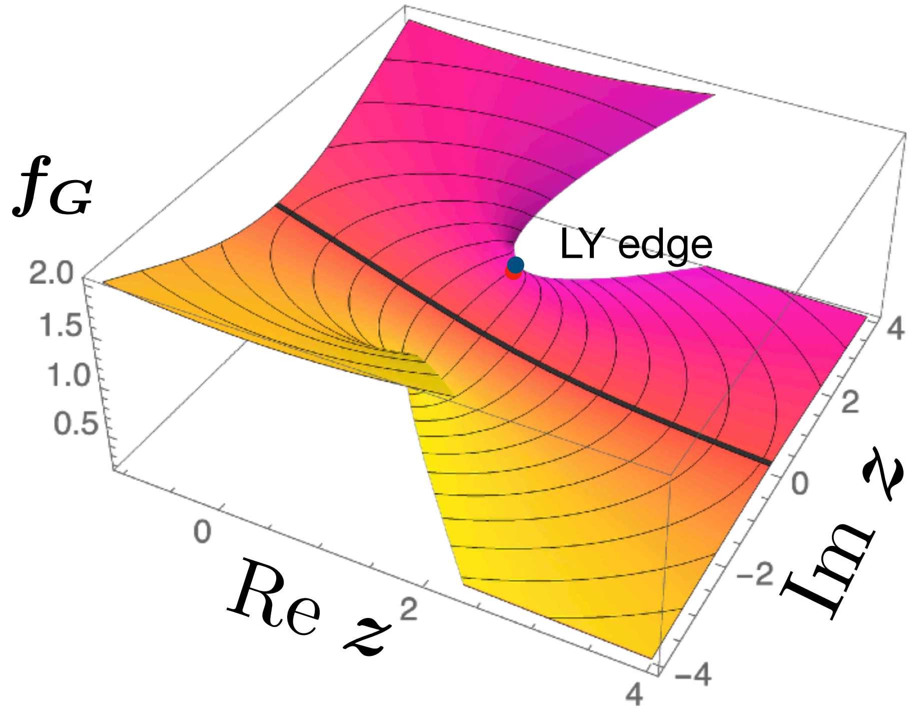
- The position of the branch cut singularity is **universal**. In terms of the scaling variable $z = t/h^{1/\beta\delta}$, we have $z_{LY} = |z_c| e^{i\frac{\pi}{2\beta\delta}}$, where $t = T/T_c - 1$.

[Connelly et al. PRL 125 (2020) 19]

[Johnson et al. PRD 107 (2023) 116013]

$$M(t,h) = h^{1/\delta} f_G(z)$$

- Mean field: $f_G(z + f_G^2) = 1$



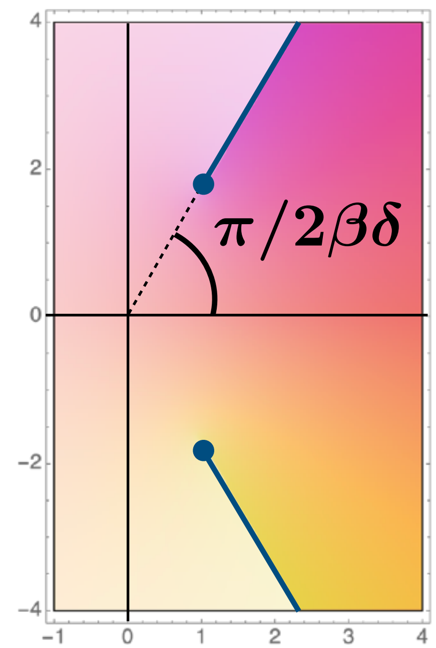
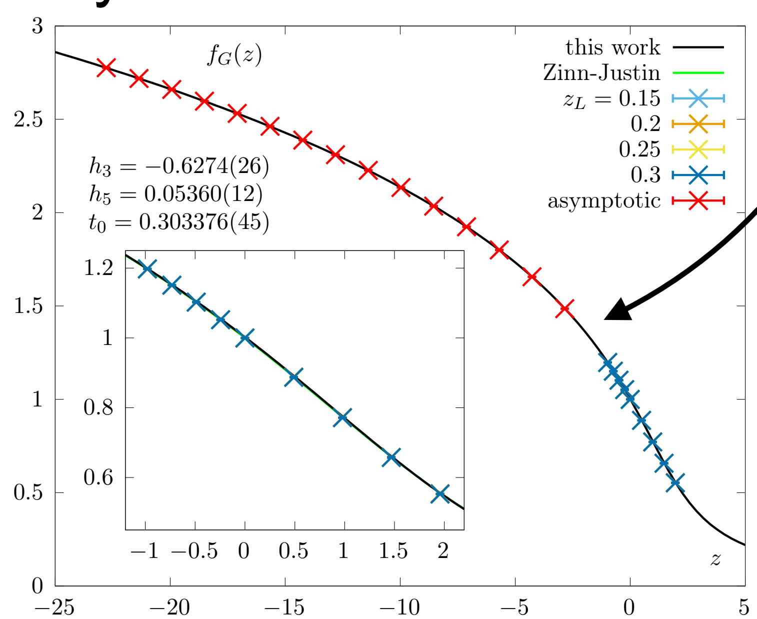


Fig: complex scaling function $f_G(z)$ over the complex z pane, courtesy to V. Skokov

- Beyond mean field: FRG or lattice methods



[Karsch et al. PRD 108 (2023) 014505]

Fit with Schofield type of parametrisation

$$M = m_0 R^\beta \theta$$
$$t = R(1-\theta^2)$$
$$h = h_0 R^{\beta\delta} h(\theta)$$

⇒ Find LY edge also in the parametrisation with quite similar results for $|z_c|$

[Karsch, CS, Singh, work in progress]

Solve for $z_{LY} = |z_c| e^{i\frac{\pi}{2\beta\delta}}$

with different scaling fields and non-universal parameters

Roberge-Weiss transition:

$$t = t_0 \left(\frac{T_{RW} - T}{T_{RW}} \right) \quad \text{and} \quad h = h_0 \left(\frac{\hat{\mu}_B - i\pi}{i\pi} \right)$$

Chiral transition:

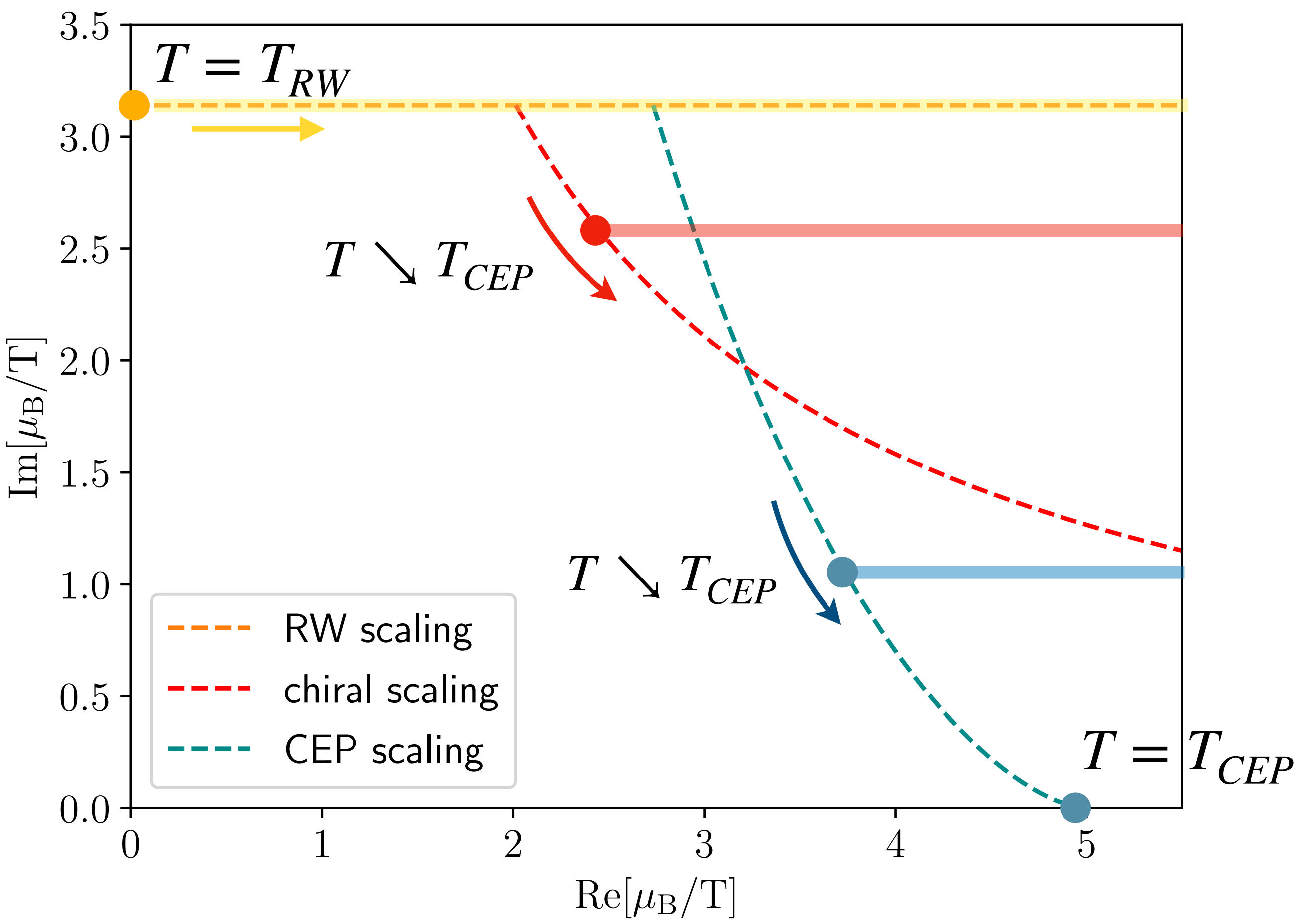
$$t = t_0 \left[\frac{T - T_c}{T_c} + \kappa_2^B \left(\frac{\mu_B}{T} \right)^2 \right] \quad \text{and} \quad h = h_0 \frac{m_l}{m_s^{\text{phys}}}$$

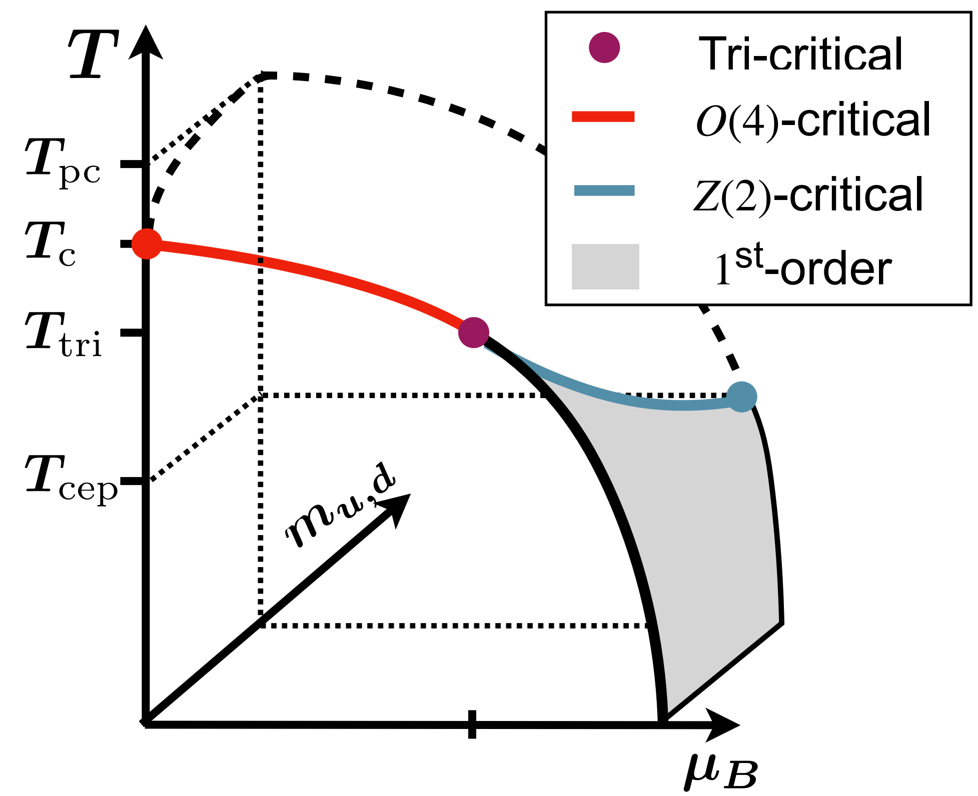
QCD critical point:

$$t = \alpha_t (T - T_{cep}) + \beta_t (\mu_B - \mu_{cep}) \quad \text{and}$$

$$h = \alpha_h (T - T_{cep}) + \beta_h (\mu_B - \mu_{cep})$$

⇒ See next talk by D. Clarke



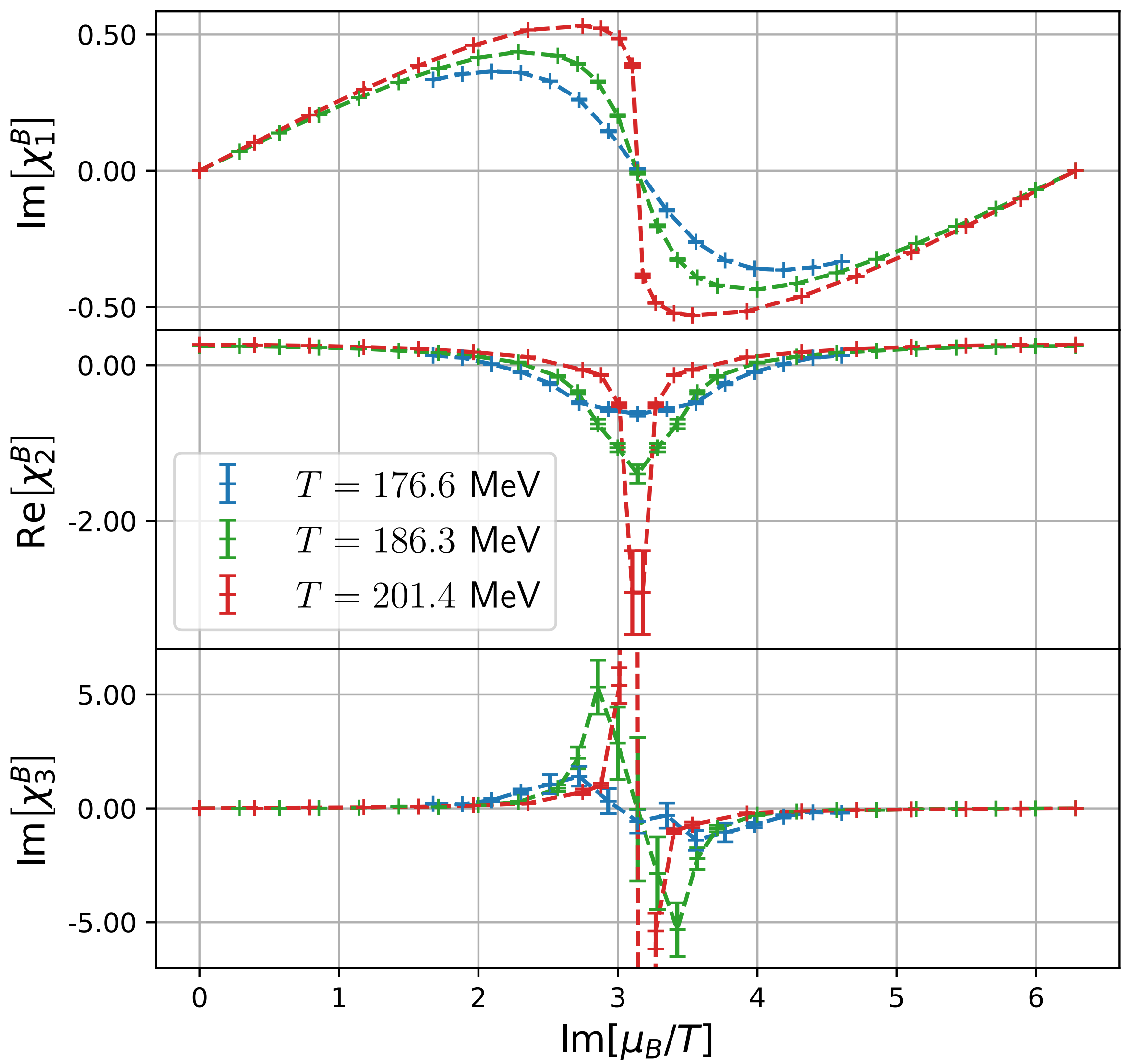


→ different temperature intervals are sensitive to different scaling of the Lee-Yang edge singularity

- We use (2+1)-flavor of highly improved staggered quarks (HISQ)
- Simulations at $\mu_B > 0$ are not possible due to the infamous sign problem
- Instead we perform calculations at imaginary chemical potential $\mu_B = i\mu_B^I$
  [De Frorcrand, Philipsen (2002); D'Elia, Lombardo (2003)]
- The temperature scale and line of constant physics is taken from previous HotQCD calculations
  [see e.g., Bollweg et al. PRD 104 (2021)]
- We measure cumulants of net baryon number in the interval $i\mu_B^I/T \in [0,\pi]$
  [Allton et al. PRD 66 (2002)]
- The cumulants χ_n^B are odd and imaginary for n odd and even and and real for n even

→ see talk by F. Di Renzo

Lattice size: $24^3 \times 4$



[Dimopoulos et al., *PRD* 105 (2022)]

$$\chi_n^B(T,V,\mu_B) = \left(\frac{\partial}{\partial\hat{\mu}_B}\right)^n \frac{\ln Z(T,V,\mu_l,\mu_s)}{VT^3}$$

$$= \left(\frac{1}{3}\frac{\partial}{\partial\hat{\mu}_l} + \frac{1}{3}\frac{\partial}{\partial\hat{\mu}_s}\right)^n \frac{\ln Z(T,V,\mu_l,\mu_s)}{VT^3}$$

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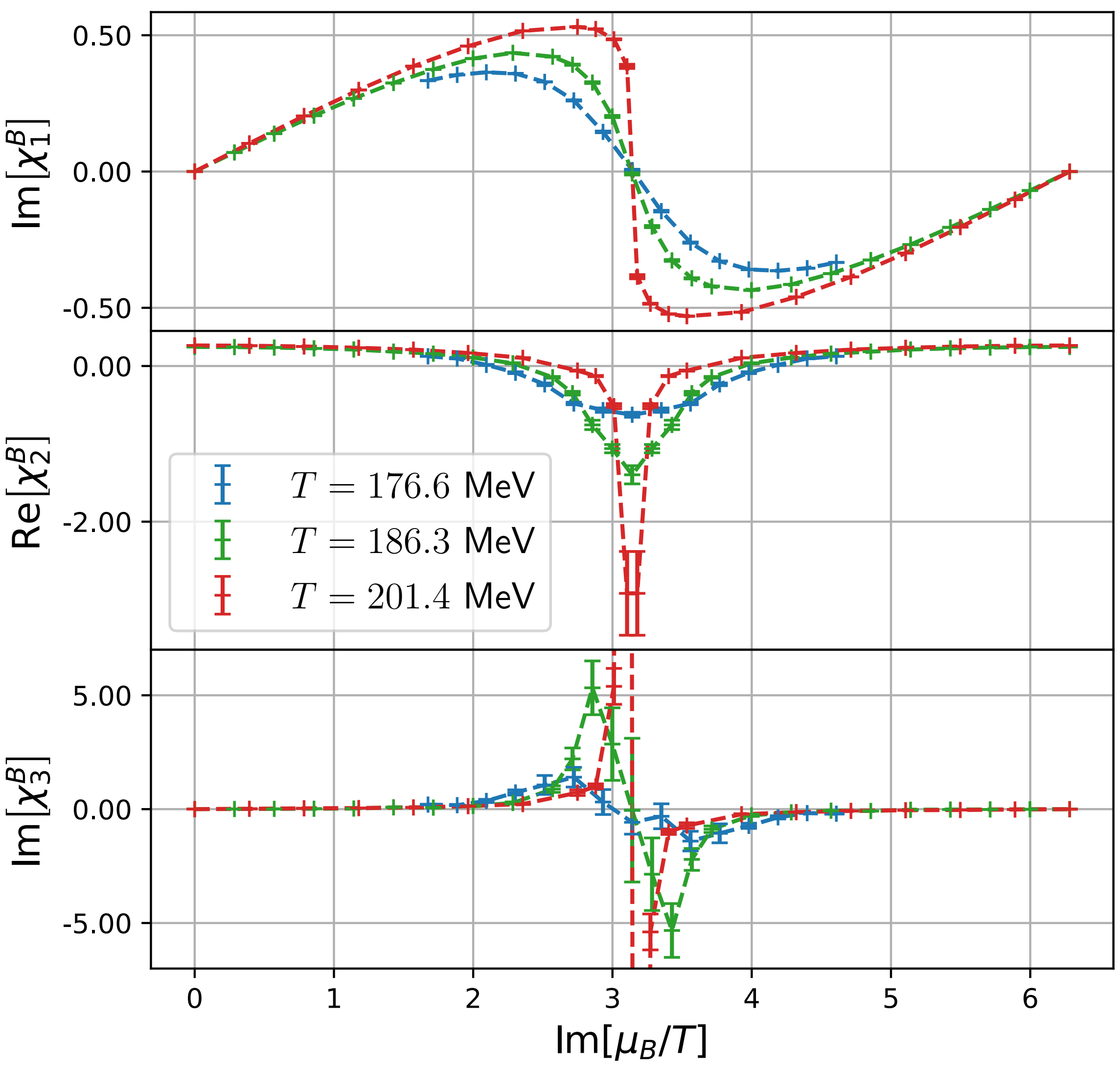
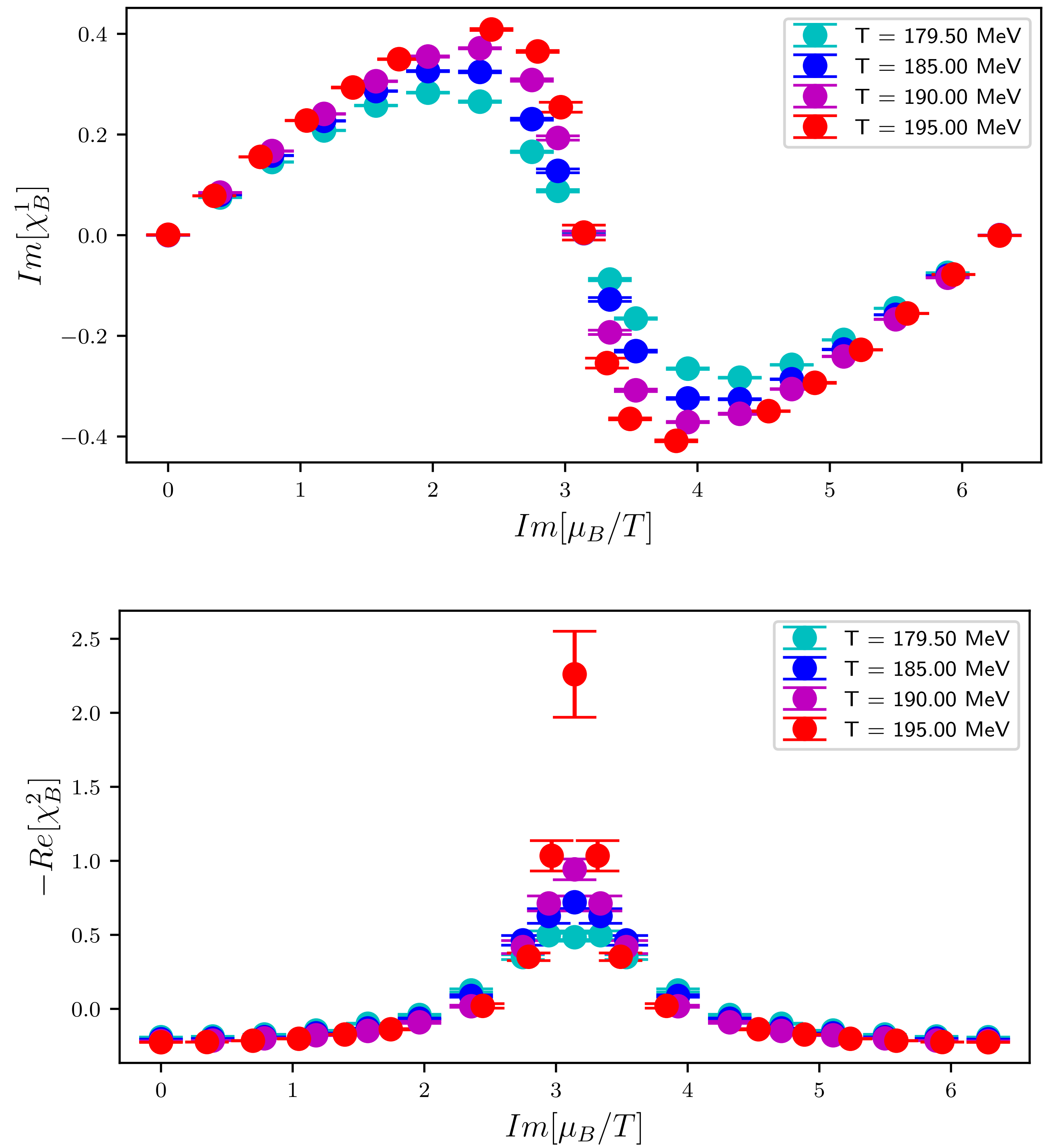


[Dimopoulos et al., *PRD* 105 (2022)]

Lattice size: $36^3 \times 6$



[Zambello et al., *PoS* LATTICE2022 (2023) 164]

$$\chi_n^B(T, V, \mu_B) = \left(\frac{\partial}{\partial \hat{\mu}_B}\right)^n \frac{\ln Z(T, V, \mu_l, \mu_s)}{VT^3}$$

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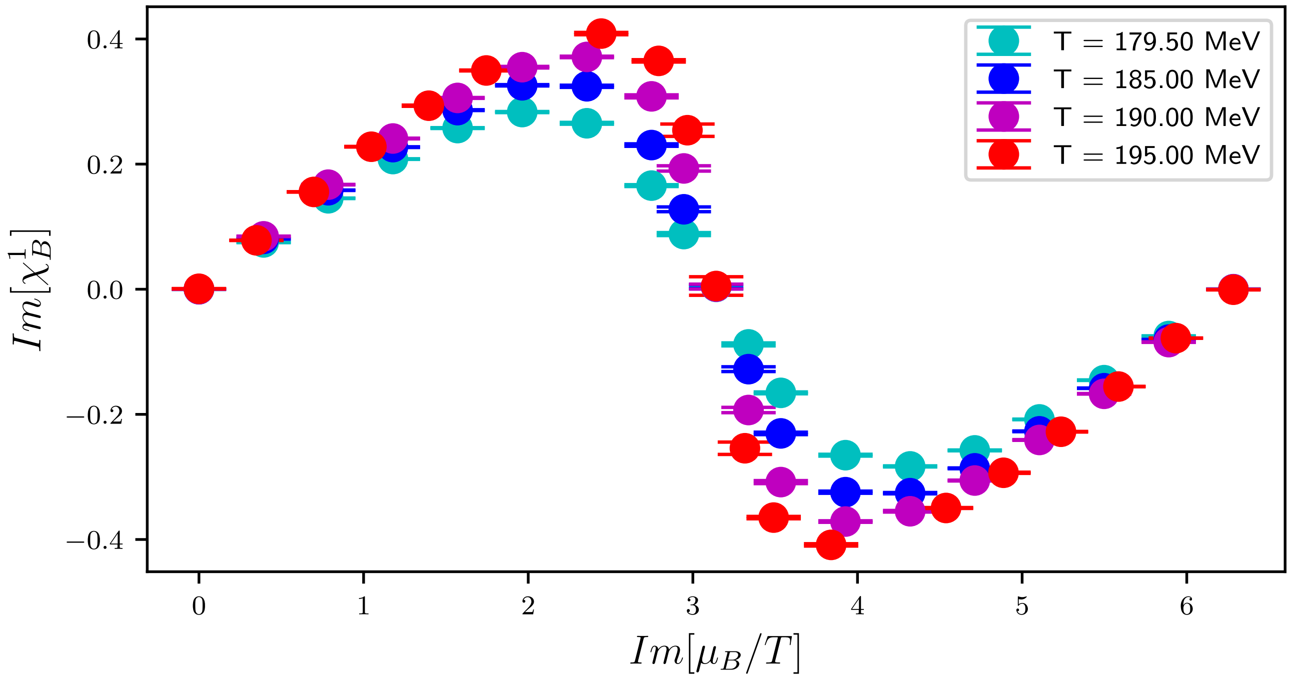
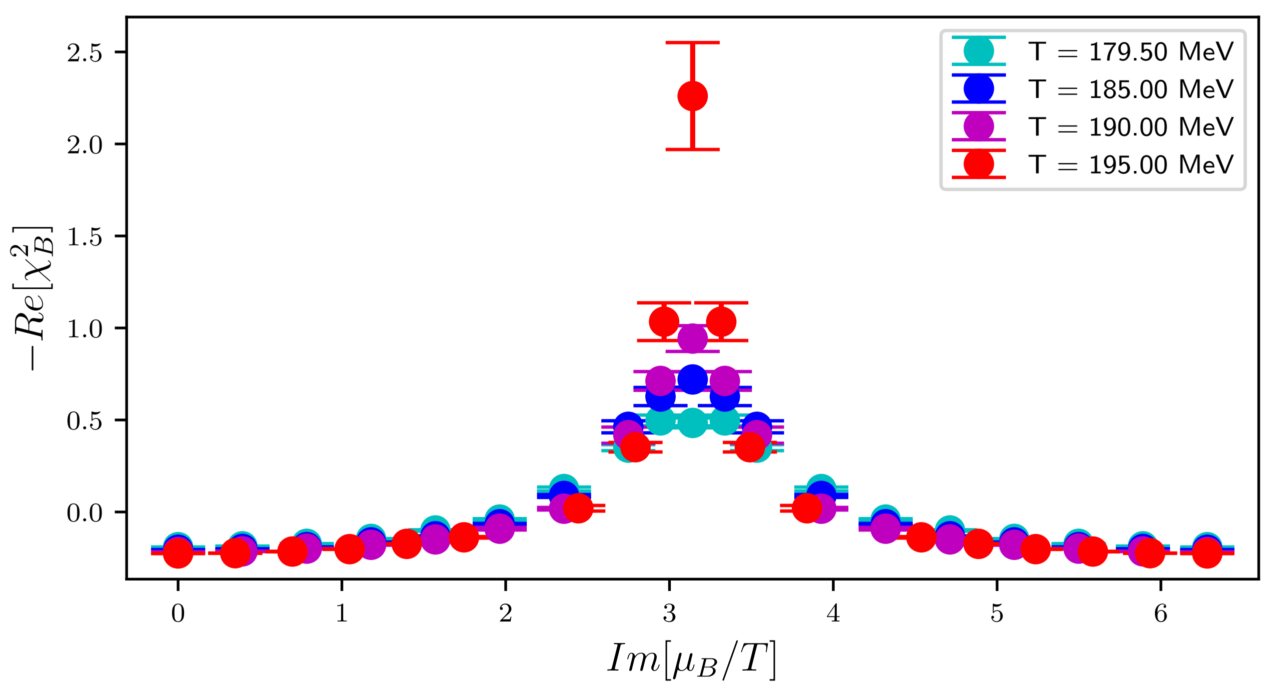
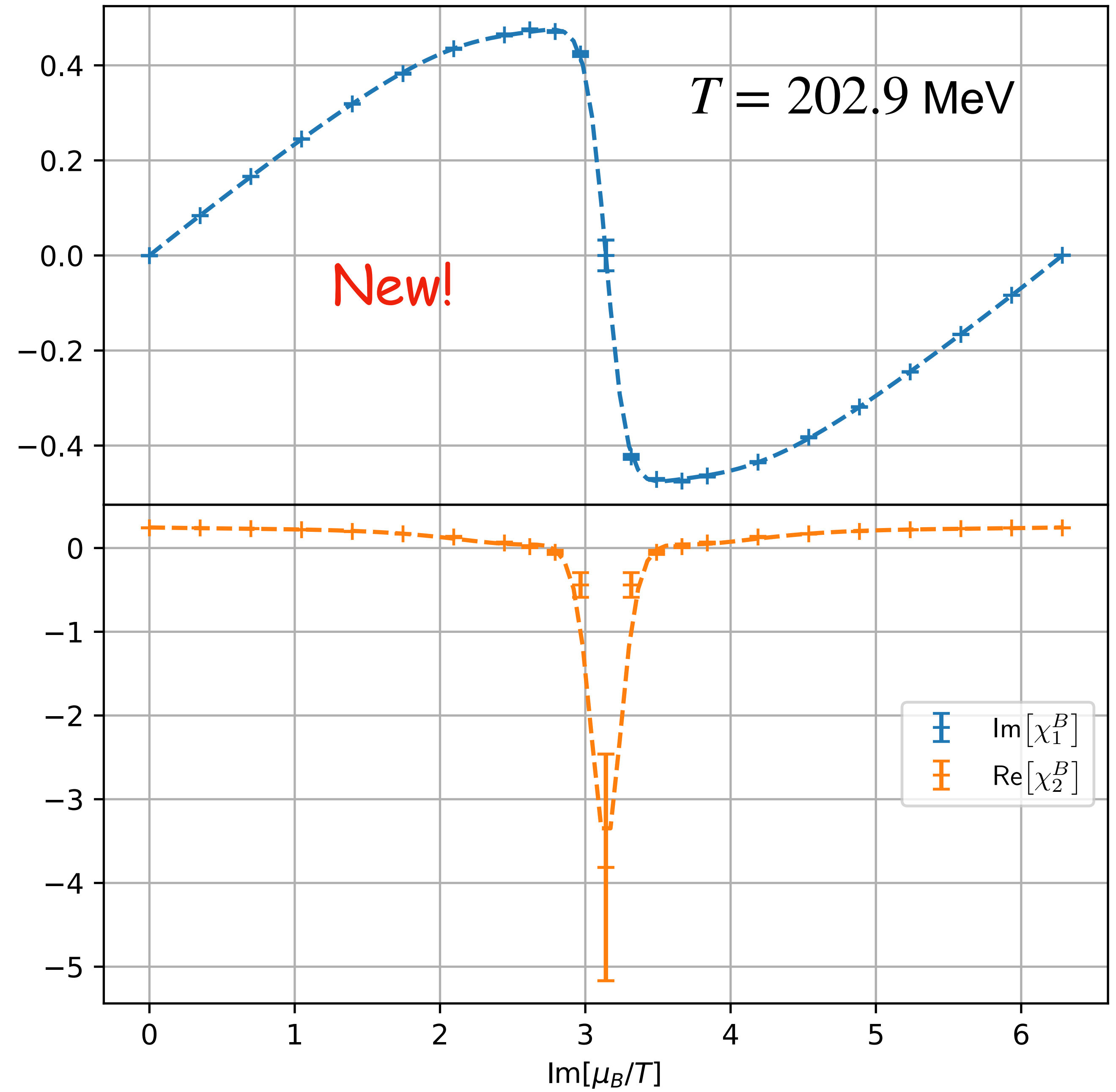




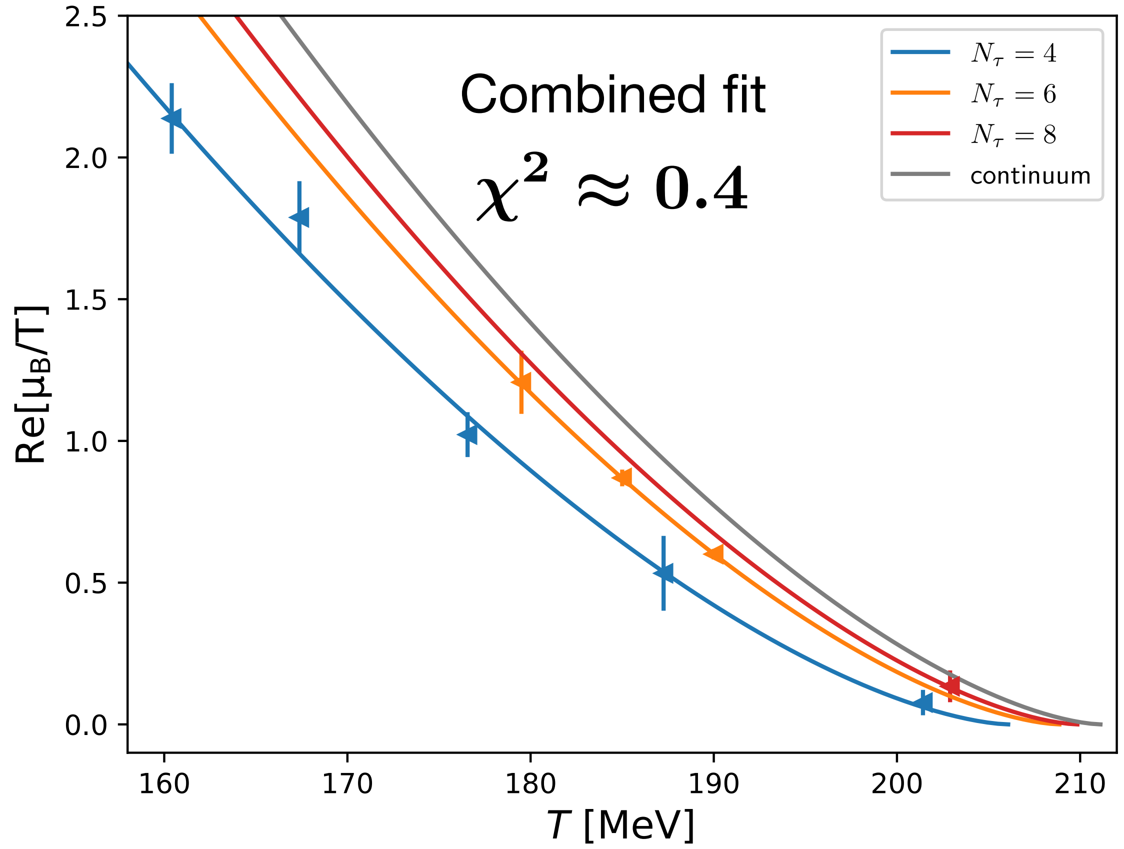
[Zambello et al., *PoS* LATTICE2022 (2023) 164]

Lattice size: $48^3 \times 8$



$\Rightarrow$ more $N_\tau = 8$ ensembles are work in progress.

$\Rightarrow$ analytic structure (poles and roots) are similar to $N_\tau = 6$ data



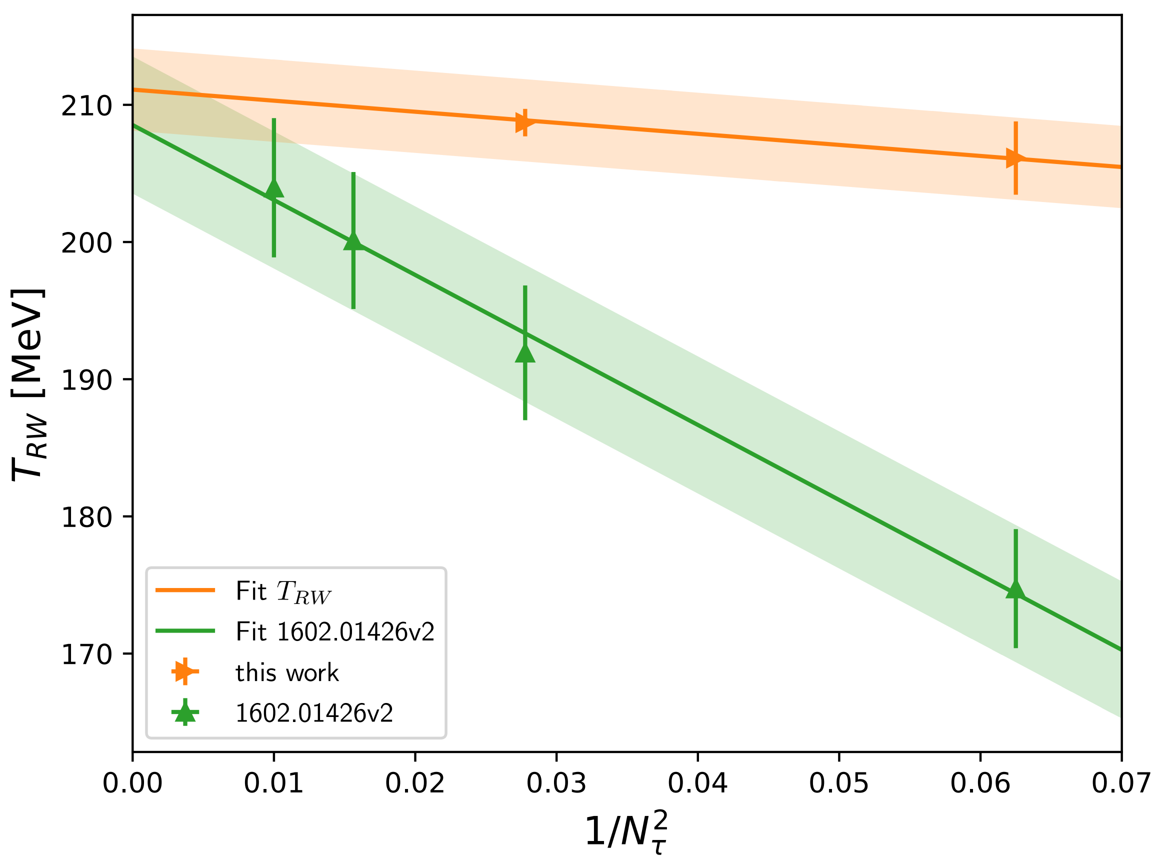
* The approach of the LY edge to the RW critical point: By solving $z = t/h^{1/\beta\delta} \equiv z_c$ we find

$$\hat{\mu}^R_{LY} = a(N_\tau)\left(\frac{T_{RW}(N_\tau) - T}{T_{RW}(N_\tau)}\right)^{\beta\delta}$$

with $\hat{\mu}^R_{LY} = \mathrm{Re}[\mu_B/T]$

We assume $T_{RW} = T^{(0)}_{RW} + T^{(2)}_{RW}/N^2_\tau$

$$a = a^{(0)} + a^{(2)}/N^2_\tau$$



* Obtain continuum result

$$T^{(0)}_{RW} = 211.1 \pm 3.1 \text{ MeV}$$

⇒ in good agreement with previous results from the Pisa group

[Bonati et al., PRD 93 (2016) 074504]

Definition:

$$b_k(T) = \frac{1}{\pi}\int_0^{2\pi} d\theta_B \, \mathrm{Im}\,\chi_1^B(T, i\theta_B)\,\sin(k\theta_B)$$

$\mu_B/T = i\theta_B$, with $\theta_B \in \mathbb{R}$

highly oscillatory for large k

Data only defined on a discrete set of points

A Fourier interpolation of the data is periodic by construction!

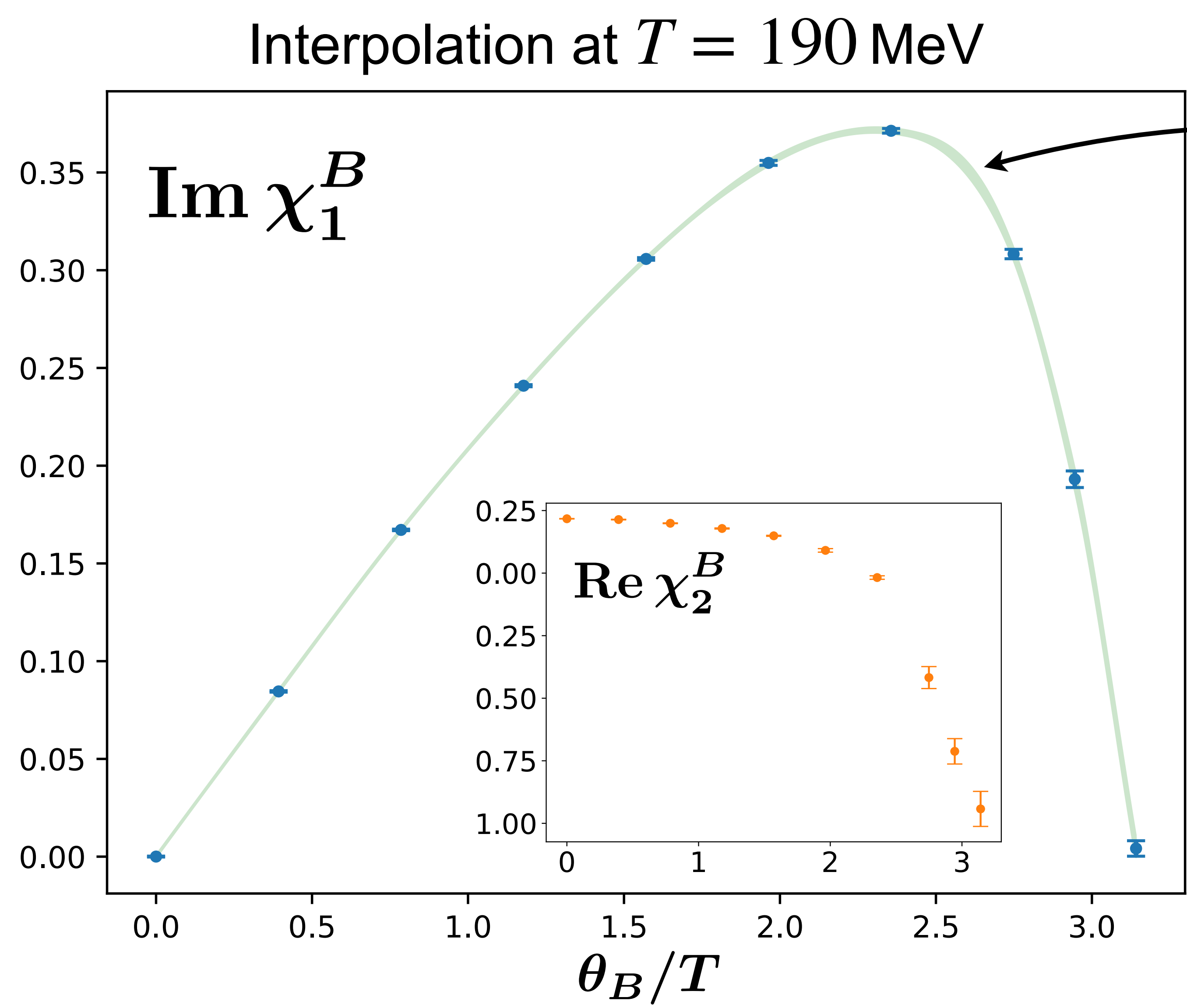
Method:

- Interpolate Im χ_1^B, take also it's derivative Re χ_2^B and eventually higher derivatives up to order s into account → Hermite-interpolation (spline)
- Piecewise integration can be done analytically

$$b_k = \frac{2}{\pi}\sum_{i=0}^{N-1}\int_{\theta_B^{(i)}}^{\theta_B^{(i+1)}} d\theta_B \, p(\theta_B)\sin(k\theta_B) \quad \text{with} \quad 0 = \theta_B^{(0)} < \theta_B^{(1)} < \cdots < \theta_B^{(N)} = \pi$$

→ variant of a Filon-type quadrature: error decreases as $\mathcal{O}(k^{-s-2})$ (for exact data)

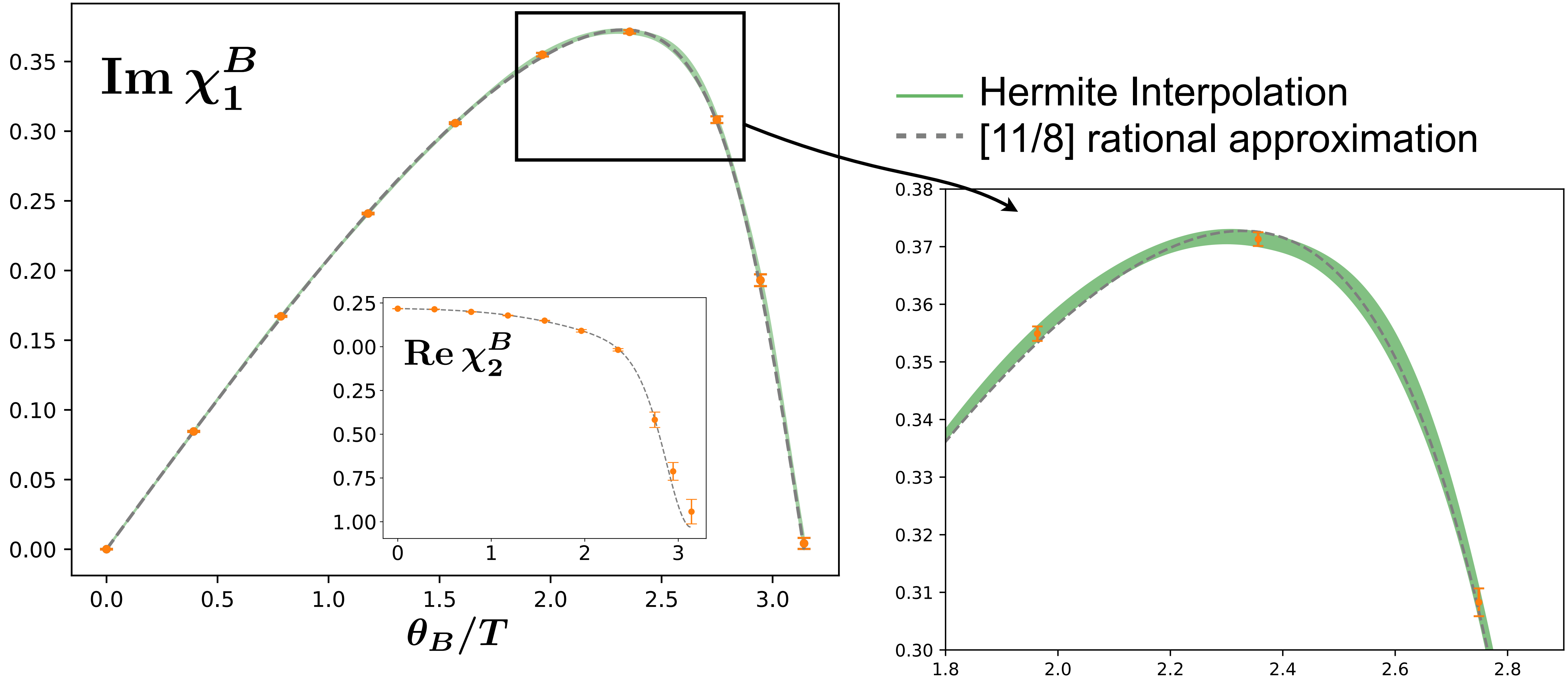
- Statistical error is estimated by bootstrapping over the error of Im χ_1^B and Re χ_2^B.



Error band of the Hermite-Interpolation (piecewise cubic, taking Im χ_1^B and Re χ_2^B into account)

- Note asymmetry of the data, w.r.t a sin function: data can be described by $\mathcal{O}(10)$ Fourier-coefficients

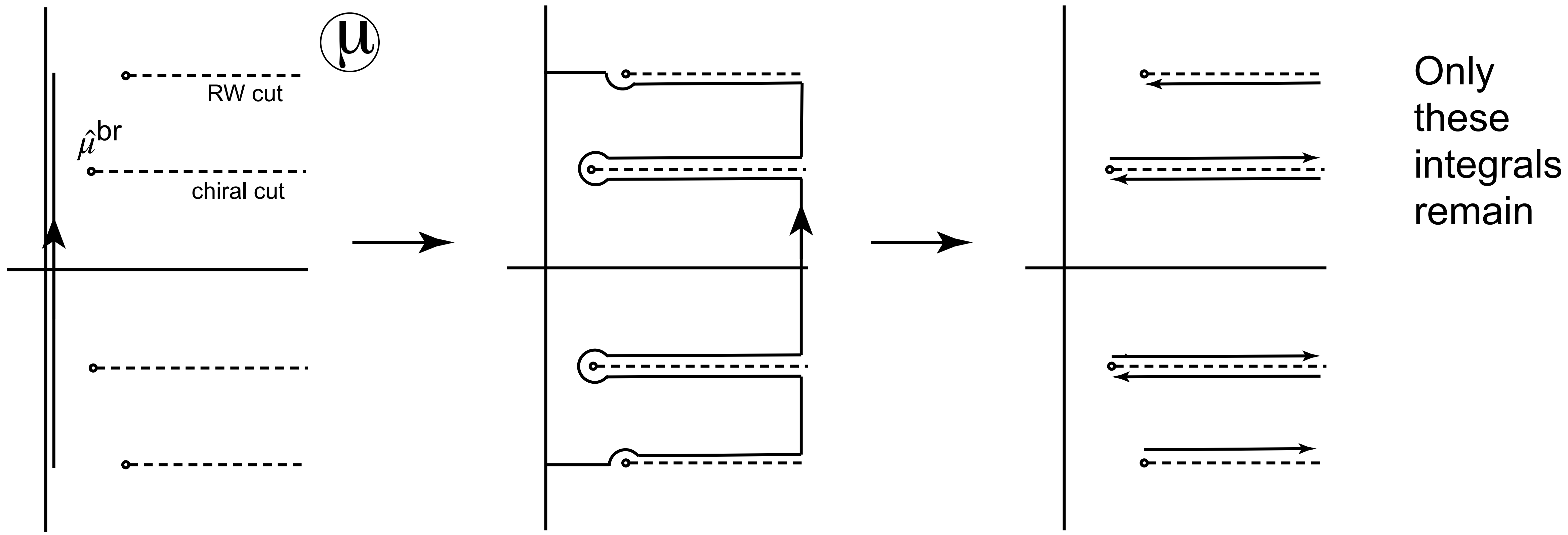
Interpolation at $T = 190$ MeV, $36^3 \times 6$



- Note asymmetry of the data, w.r.t a sin function: need $\mathcal{O}(10)$ Fourier-coefficients to describe the data

- Both interpolations agree within error.

- We can deform the integration contour to integrate along the cuts



- Assume that we can express the density along the cuts as

$$n_B(\hat{\mu}) = \underbrace{A(\hat{\mu}-\hat{\mu}^{\rm br})^{\sigma}}_{\text{Leading order non-analytic part}}(1 + B(\hat{\mu}-\hat{\mu}^{\rm br})^{\theta_c} + \ldots) + \underbrace{\sum_{n=0}^{\infty} a_n(\hat{\mu}-\hat{\mu}^{\rm br})^n}_{\text{analytic part}}$$

edge coefficient, $\sigma > -1$

Leading order non-analytic part

analytic part

- The final result for one cut is

$$b_k = \frac{e^{-\mu^{\mathrm{br}} k}}{i\pi} A \frac{\Gamma(1+\sigma)}{k^{1+\sigma}} \left(1 - e^{i2\pi\sigma} + \frac{B}{k^{\theta_c}} \left[1 - e^{i2\pi(\sigma+\theta_c)}\right] \frac{\Gamma(1+\sigma+\theta_c)}{\Gamma(1+\sigma)} + \ldots\right)$$

Absorbing k-independent factors into A and B we get

$$b_k = \tilde{A} \frac{e^{-\hat{\mu}^{\mathrm{br}} k}}{k^{1+\sigma}} \left(1 + \frac{\tilde{B}}{k^{\theta_c}} + \ldots\right)$$

Note that the regular part cancels completely

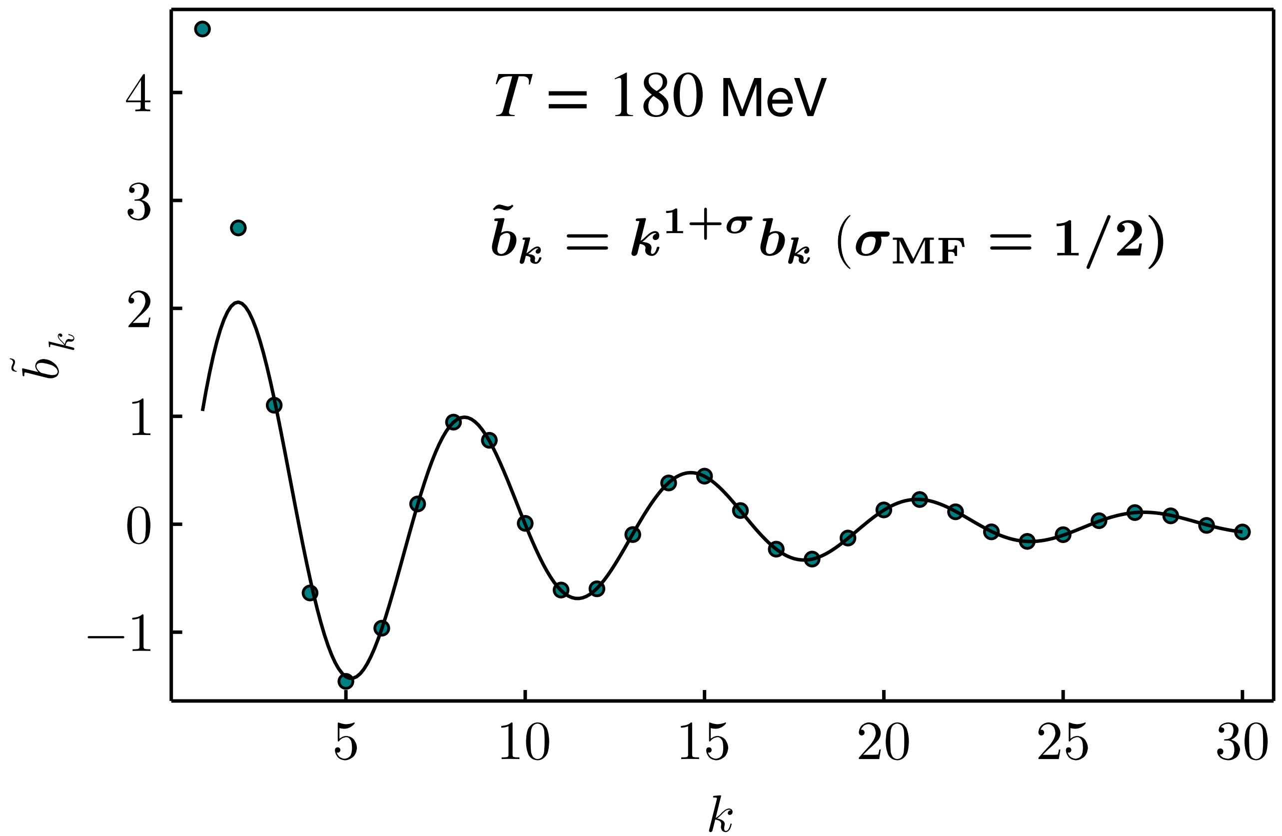
- The final result for both cuts is (dropping NLO)

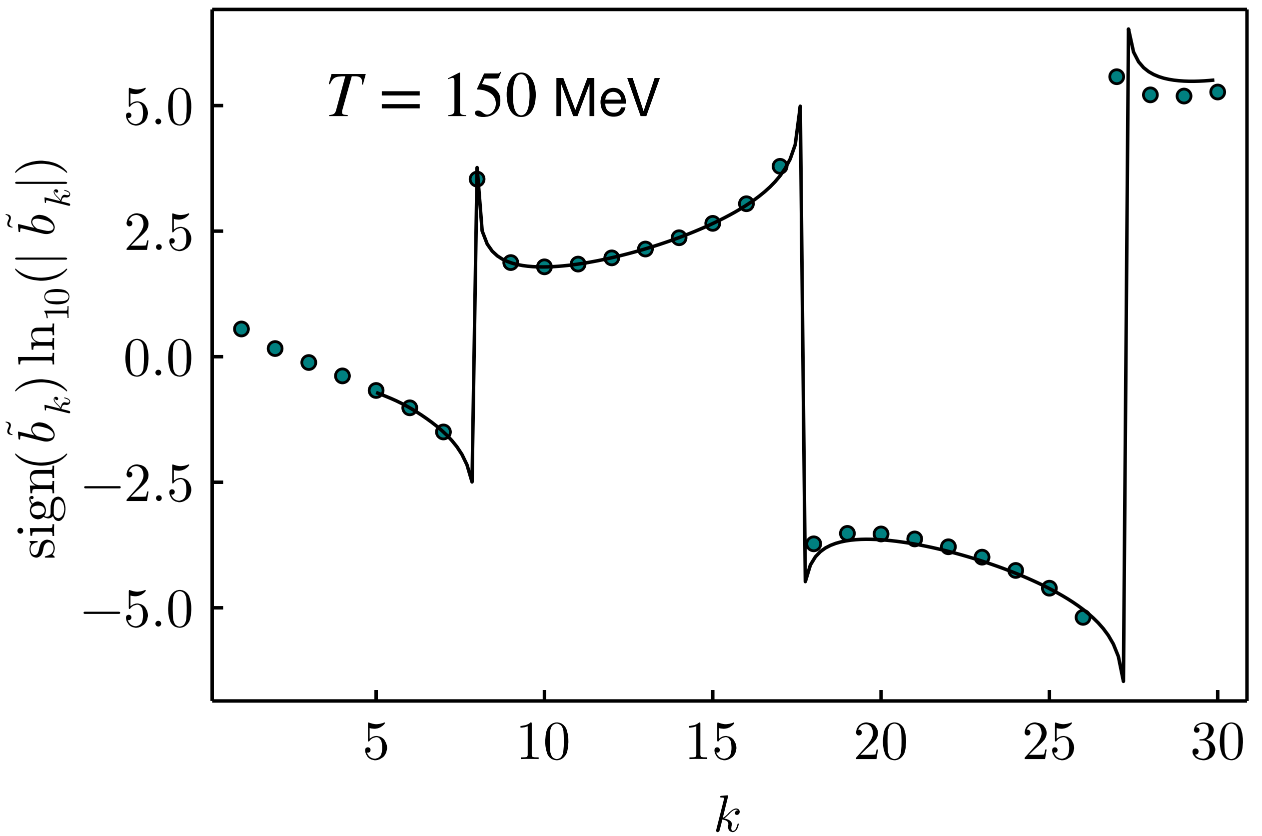
$$b_k = |\tilde{A}_{\mathrm{YLE}}| \frac{e^{-\hat{\mu}_r^{\mathrm{YLE}} k}}{k^{1+\sigma}} \cos(\hat{\mu}_i^{\mathrm{YLE}} k + \phi_a^{\mathrm{YLE}}) + |\hat{A}_{\mathrm{RW}}| (-1)^k \frac{e^{-\hat{\mu}_r^{\mathrm{RW}} k}}{k^{1+\sigma}}$$

$\hat{\mu}_i^{RW} = \pi$

- The analytic form fits the Fourier coefficients from the quark-meson model well. Details of the Model can be found here [Skokov et al., PRD 82 (2010) 034029]

- In Mean-Field and LAP approximation fits to the Fourier coefficients reproduce the correct location of the LY edge up to (5 – 7)%.

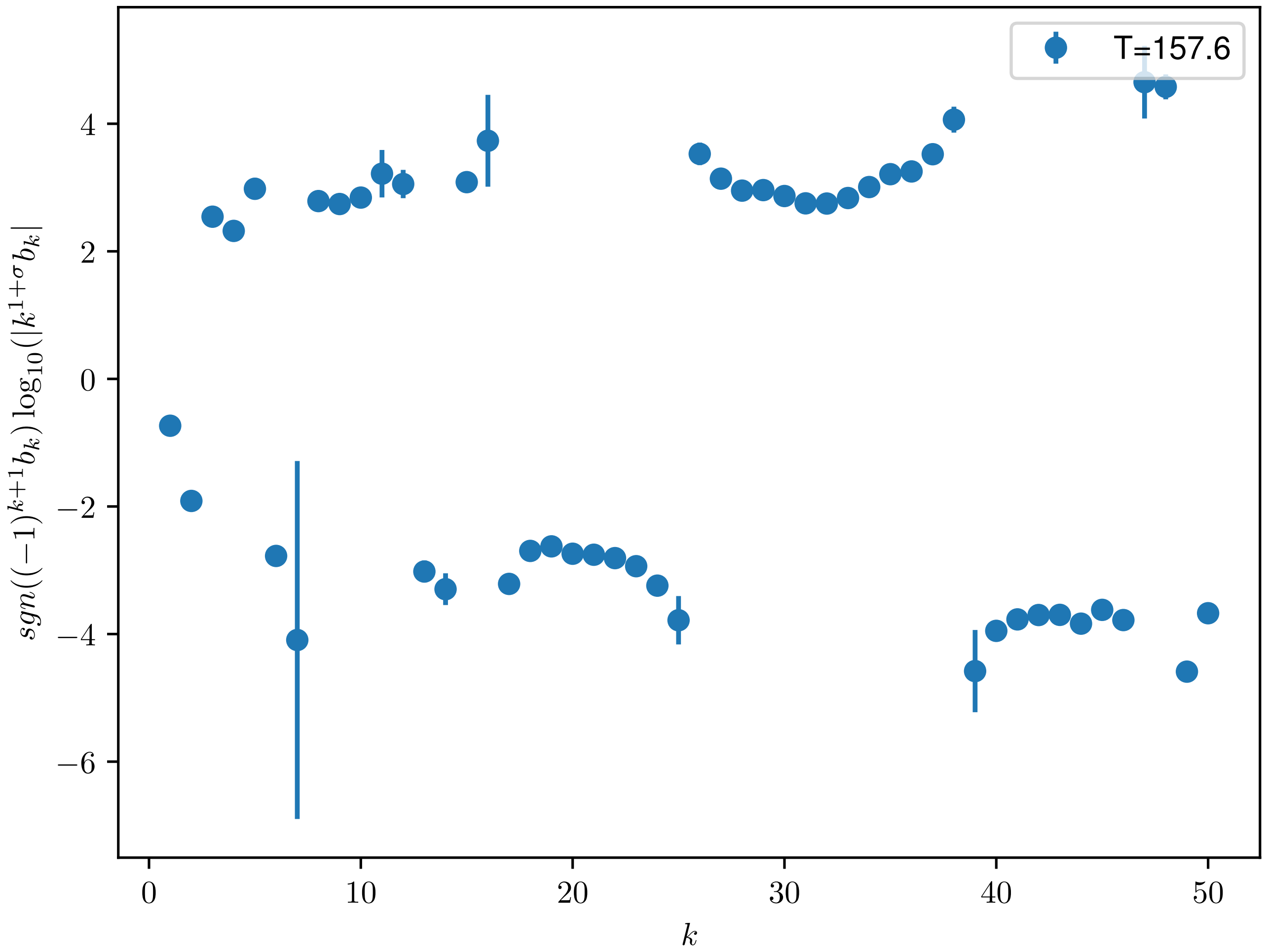




$$\hat{\mu}^{\mathrm{fit}}_{\mathrm{YLE}} = 0.1156(6) + i\,0.9952(5)$$
$$\hat{\mu}_{\mathrm{YLE}} = 0.118657 + i\,1.00256$$

$$\hat{\mu}^{\mathrm{fit}}_{\mathrm{YLE}} = 0.441(2) + i\,0.325(3)$$
$$\hat{\mu}_{\mathrm{YLE}} = 0.412884 + i\,0.342187$$

- Find some oscillating pattern. Need further investigation of the systematic error.



- Investigate also the analytic from for Fourier coefficients in the finite volume.

- At different temperature intervals we expect to be sensitive to different scaling behaviour of the Lee-Yang edge singularly (Roberge-Weiss, chiral, QCD critical point)
- At high temperature scaling of the Lee-Yang edge is in agreement with Roberge-Weiss Z(2) scaling. Find continuum $T_{RW} = 211.1 \pm 3.1$ MeV.
- Derived analytic form for the asymptotic behaviour of the Fourier coefficients of the baryon number density and tested the formula at the quark-meson model with success.
- Fourier analysis of QCD data in progress, need further check of systematics.

Outlook:

- perform more systematic fits to Fourier coefficients
- investigate asymptotic behaviour for finite volume
- performing smaller than physical mass calculations

→the real part of the branch cut singularity will be reduced!