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Comparing phenomenological estimates of dilepton decays of pseudoscalar mesons with lattice QCD

based on Phys. Rev. Lett. **128** (2022) 172004, and in preparation

in collaboration with M. Hoferichter and J. Ruiz de Elvira

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August 3rd, 2023

Outline

Introduction & motivation

$\pi^0 \to e^+e^-$

$K_L \to \ell^+\ell^-$

Conclusion & outlook

Introduction & motivation

Neutral pion main decay modes & branching ratios (BRs): PDG, 2022

Decay modes	$\pi^0 \to \gamma\gamma$	$\pi^0 \to e^+e^-\gamma$	$\pi^0 \to e^+e^-e^+e^-$	$\pi^0 \to e^+e^-$
BR	98.823%	1.174%	3.34×10^{-5}	6.46×10^{-8}

- All listed decays described by pion transition form factor $F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$

Introduction & motivation

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K_L relevant decay modes: PDG, 2022

Decay modes		$K_L \to \gamma\gamma$	$K_L \to \ell^+\ell^-\gamma$	$K_L \to e^+e^-\ell^+\ell^-$	$K_L \to \ell^+\ell^-$
BR	$\ell = e$	5.47×10^{-4}	9.4×10^{-6}	3.56×10^{-8}	9×10^{-12}
	$\ell = \mu$		3.59×10^{-7}	2.69×10^{-9}	6.84×10^{-9}

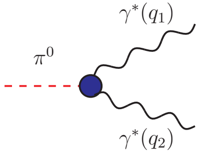
- All listed decays described by kaon transition form factor $F_{K_L\gamma^*\gamma^*}(q_1^2, q_2^2)$

Introduction & motivation

Pion transition form factor (TFF) $F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$:

- Defined by the matrix element of two electromagnetic currents $j_\mu(x)$

$$i\int \mathrm{d}^4x\, e^{iq_1\cdot x}\, \langle 0 | T\{j_\mu(x) j_\nu(0)\} | \pi^0(q_1+q_2)\rangle$$
$$= \epsilon_{\mu\nu\rho\sigma}\, q_1^\rho q_2^\sigma F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$$



- Normalization fixed by the Adler–Bell–Jackiw anomaly:

$$F_{\pi^0\gamma^*\gamma^*}(0,0) = \frac{1}{4\pi^2 F_\pi} \equiv F_{\pi\gamma\gamma}$$

$F_\pi = 92.28(10)\,\mathrm{MeV}$: pion decay constant — PDG, 2022

Introduction & motivation

Kaon TFF $F_{K_L\gamma^*\gamma^*}(q_1^2, q_2^2)$:

- Defined by the matrix element of two currents $j_\mu(x)$ & $\mathcal{L}_{\Delta S=1}(0)$

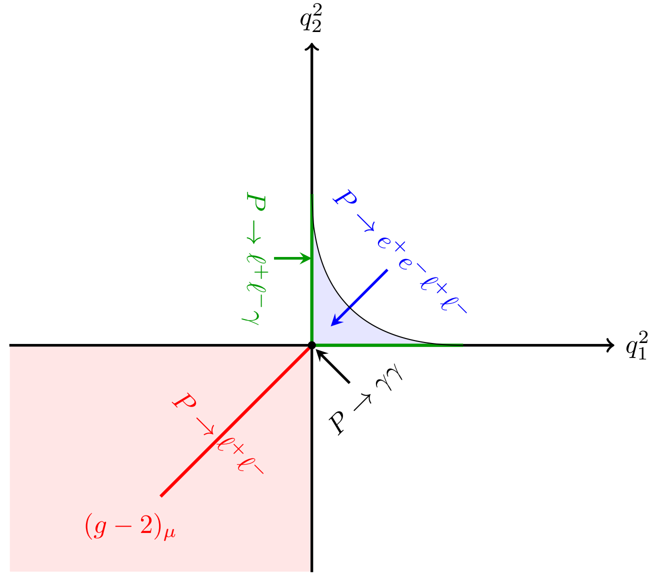
$$\int \mathrm{d}^4x\, e^{iq_1x} \int \mathrm{d}^4y\, e^{iq_2y} \langle 0 \,| T\{J_\mu(x)J_\nu(y)\mathcal{L}_{\Delta S=1}(0)\}|\, K_L(q_1+q_2)\rangle$$
$$= \epsilon_{\mu\nu\rho\sigma}\, q_1^\rho q_2^\sigma F_{K_L\gamma^*\gamma^*}(q_1^2, q_2^2)$$

- No low-energy theorem dictates normalization

- Normalization can be fixed from the real-photon decay $K_L \to \gamma\gamma$

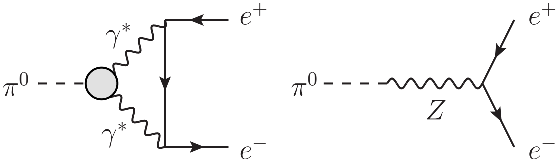
Introduction & motivation

$F_{P\gamma^*\gamma^*}(q_1^2, q_2^2)$ kinematic regions:



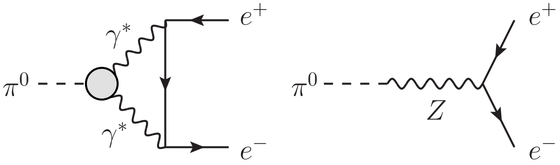
$\pi^0 \to e^+e^-$

Leading Standard-Model contributions to $\pi^0 \to e^+e^-$:



$\pi^0 \to e^+e^-$

Leading Standard-Model contributions to $\pi^0 \to e^+e^-$:



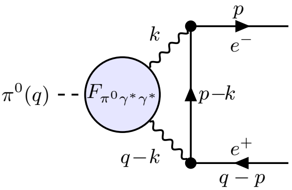
- QED loop contribution dominates
- Loop- and helicity-suppressed rare decay

$$\frac{\text{BR}[\pi^0 \to e^+e^-]}{\text{BR}[\pi^0 \to \gamma\gamma]} \sim \left(\frac{\alpha}{\pi}\right)^2 \frac{m_e^2}{M_{\pi^0}^2} \pi^2 \log^2 \frac{m_e}{M_{\pi^0}} \sim \mathcal{O}(10^{-8})$$

Drell, 1959

$\pi^0 \to e^+e^-$

Reduced amplitude $\mathcal{A}(q^2)$:



$$\frac{\text{BR}[\pi^0 \to e^+e^-]}{\text{BR}[\pi^0 \to \gamma\gamma]} = 2\sigma_e(q^2)\left(\frac{\alpha}{\pi}\right)^2 \frac{m_e^2}{M_{\pi^0}^2}\left|\mathcal{A}(q^2)\right|^2$$

$$q^2 = M_{\pi^0}^2,\ \sigma_e(q^2) = \sqrt{1 - \frac{4m_e^2}{q^2}}$$

$$\mathcal{A}(q^2) = \frac{2i}{\pi^2 q^2}\int \mathrm{d}^4k \frac{q^2k^2 - (q\cdot k)^2}{k^2(q-k)^2[(p-k)^2 - m_e^2]} \times \tilde{F}_{\pi^0\gamma^*\gamma^*}\big(k^2, (q-k)^2\big)$$

$\tilde{F}_{\pi^0\gamma^*\gamma^*}\big(k^2, (q-k)^2\big) = F_{\pi^0\gamma^*\gamma^*}\big(k^2, (q-k)^2\big)/F_{\pi\gamma\gamma}$, normalized pion TFF

$\pi^0 \to e^+e^-$

We build the form factor double-spectral representation:

Hoferichter et al., 2018

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = F^{\text{disp}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) + F^{\text{eff}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) + F^{\text{asym}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$$

- Reconstructed from the lowest-lying singularities 2π & 3π
- Fulfills the asymptotic constraints at $\mathcal{O}(1/Q^2)$
- Suitable for muon $g_\mu - 2$ & $\pi^0 \to e^+e^-$ loop-integral evaluation

$\pi^0 \to e^+e^-$

Imaginary part from the $\gamma\gamma$ cut:

$$\mathrm{Im}\,\mathcal{A}(q^2) = \frac{\pi}{2\sigma_e(q^2)} \log\left[\frac{1-\sigma_e(q^2)}{1+\sigma_e(q^2)}\right] = -17.52$$

$\mathrm{Re}\,\mathcal{A}(q^2)$? $\Rightarrow$ need to perform the integral with the form factor

- $\mathcal{A}^{\text{eff}}$: standard reduction — Passarino, Veltman, 1979, 't Hooft, Veltman, 1979
- $\mathcal{A}^{\text{asym}}$: integration by parts — Chetyrkin, Tkachov, 1981
- $\mathcal{A}^{\text{disp}}$: integration-kernel method — Masjuan, Sánchez-Puertas, 2016

$\pi^0 \to e^+e^-$

Long-range contribution from the final representation:

$$\text{Re}\,\mathcal{A}(q^2)\big|_{\gamma^*\gamma^*} = 10.16(5)_{\text{disp}}(8)_{\text{BL}}(2)_{\text{asym}}$$

Z-boson contribution:

$$\text{Re}\,\mathcal{A}(q^2)\big|_{Z} = -\frac{F_\pi G_F}{\sqrt{2}\,\alpha^2 F_{\pi\gamma\gamma}} = -0.05(0)$$

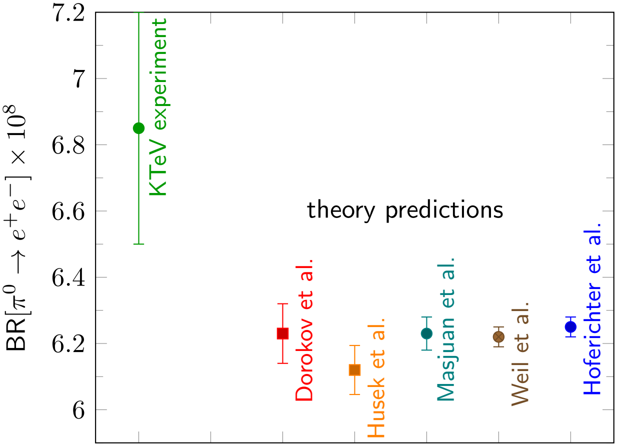
Final Standard-Model prediction:

$$\text{Re}\,\mathcal{A}(q^2)\big|_{\text{SM}} = 10.11(10)$$
$$\text{BR}[\pi^0 \to e^+e^-]\big|_{\text{SM}} = 6.25(3) \times 10^{-8}$$

- Fully controlled uncertainty estimates
- Mild 1.8σ tension with experiment

$\pi^0 \to e^+e^-$

Experiment vs theory: KTeV, 2006



- $\sim 2\sigma$ discrepancy between experiment and theory

$\pi^0 \to e^+e^-$

Comparison to lattice result: RBC/UKQCD, 2022

$$\mathrm{Re}\,\mathcal{A} = 18.60(1.19)(1.04)\mathrm{eV}$$

$$\mathrm{Im}\,\mathcal{A} = 32.59(1.50)(1.65)\mathrm{eV}$$

$$\frac{\mathrm{Re}\,\mathcal{A}}{\mathrm{Im}\,\mathcal{A}} = 0.571(10)(4)$$

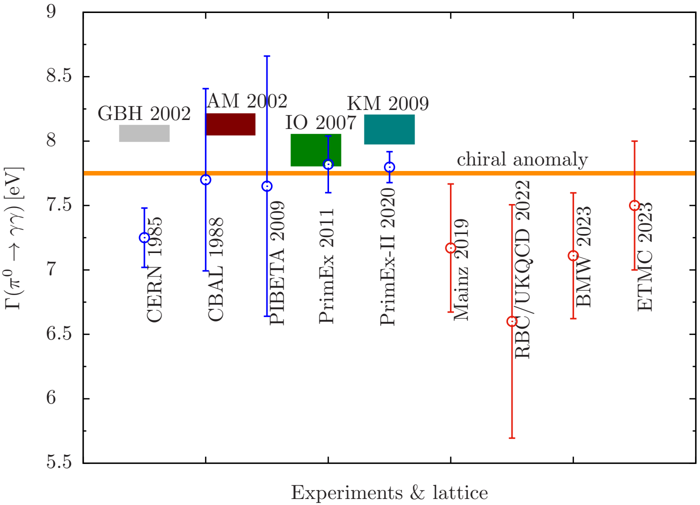
$$\Gamma\left(\pi^0 \to \gamma\gamma\right) = 6.60(0.61)(0.67)\mathrm{eV}$$

Instead using experimental decay width $\Gamma\left(\pi^0 \to \gamma\gamma\right) = 7.802(0.052)(0.105)\mathrm{eV}$:

$$\mathrm{Re}\,\mathcal{A} = 20.2(0.4)_{\mathsf{stat}}\,(0.1)_{\mathsf{syst}}(0.2)_{\mathsf{expt}}\mathrm{eV}$$
$$\mathsf{BR}[\pi^0 \to e^+e^-] = 6.22(5)_{\mathsf{stat}}\,(2)_{\mathsf{syst}} \times 10^{-8}$$

$\pi^0 \to e^+e^-$

Comparison of $\Gamma\left(\pi^0 \to \gamma\gamma\right)$ as byproducts:



- Lattice results systematically below experiment, finite-volume corrections?

$K_L \to \ell^+\ell^-$

Aim to build the form factor double-spectral representation:

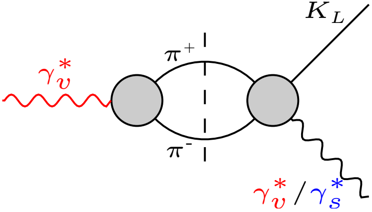
$$F_{K_L\gamma^*\gamma^*}(q_1^2, q_2^2) = F^{\text{disp}}_{K_L\gamma^*\gamma^*}(q_1^2, q_2^2) + F^{\text{asym}}_{K_L\gamma^*\gamma^*}(q_1^2, q_2^2)$$

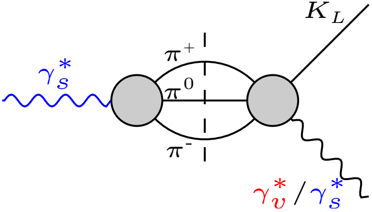
- Reconstructed from the lowest-lying singularities 2π, 3π, & $K\pi$
- Fulfills the asymptotic constraints at $\mathcal{O}\big(1/Q^2\big)$
- Suitable for $K_L \to \ell^+\ell^-$ loop-integral evaluation

$K_L \to \ell^+\ell^-$

Dispersive reconstruction from the lowest-lying hadronic intermediate states:

$$F^{\text{disp}}_{K_L\gamma^*\gamma^*}(q_1^2, q_2^2) = F_{vv}(q_1^2, q_2^2) + F_{vs}(q_1^2, q_2^2) + F_{vs}(q_2^2, q_1^2) + F_{ss}(q_1^2, q_2^2)$$





$K_L \to \ell^+\ell^-$

Dispersive reconstruction from the lowest-lying hadronic intermediate states:

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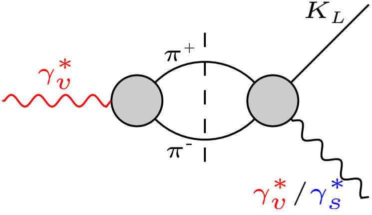
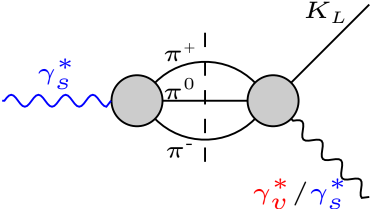


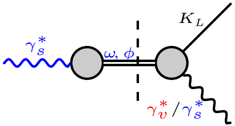


Isovector photon: 2 pions

- $\gamma_v^* \to \pi^+\pi^- \to \gamma_v^*/\gamma_s^* K_L$
- disc $\propto$ pion vector form factor $\times$ $K_L \to \pi^+\pi^-\gamma_v^*/\gamma_s^*$ amplitude

Isoscalar photon: 3 pions

- $\gamma_s^* \to \pi^+\pi^-\pi^0 \to \gamma_v^*/\gamma_s^* K_L$
- Dominated by ω, ϕ



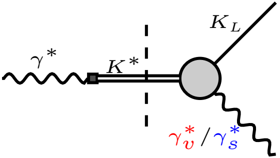
$K_L \to \ell^+\ell^-$

Dispersive reconstruction from the lowest-lying hadronic intermediate states:

- Relative weights of different contributions not known:
- Reliance on VMD estimates gives $vv : vs : ss = 1 : -1/3 : 5/9$
- Input from lattice QCD?

Slope parameter of TFF suggests additional $K\pi$ cut:

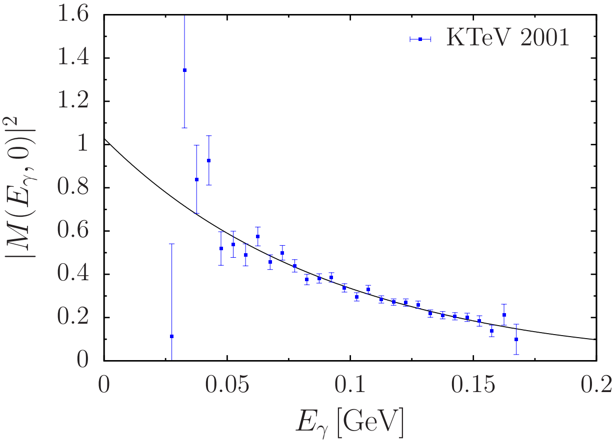
$$F^{\text{disp}}_{K_L\gamma^*\gamma^*}(q_1^2, q_2^2) \supset F_{vK^*}(q_1^2, q_2^2) + F_{K^*v}(q_1^2, q_2^2) + F_{sK^*}(q_1^2, q_2^2) + F_{K^*s}(q_1^2, q_2^2)$$



$K_L \to \ell^+\ell^-$

Fit to the $K_L \to \pi\pi\gamma$ magnetic amplitude M with representation:

$$M(t, q^2) = \sum_{i=v,s} a_i(q^2) \left[N_i(1 + \alpha_i t) + \frac{a_{K^*}(t)}{a_{K^*}(0)} \right] \Omega_1(t),$$



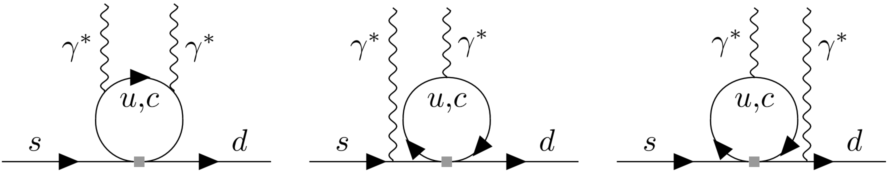
$K_L \to \ell^+\ell^-$

Short-distance constraints:

Simma, Wyler, 1990, Herrlich, Kalinowski 1992
Isidori, Unterdorfer, 2004

$$F^{\text{asym}}_{K_L\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{16\alpha G_F V_{us}^* V_{ud} F_K}{9\pi\sqrt{2}} \left[C_2(\mu) + 3C_1(\mu)\right] \left[I(r_1, r_2) + T(r_1) + T(r_2)\right]$$

- F_K: kaon decay constant
- Difference between c- and u-quark contributions



$K_L \to \ell^+\ell^-$

Preliminary results of $K_L \to \mu^+\mu^-$:

- Imaginary part from the complete analysis:

$$\text{Im}\,\mathcal{A}(q^2)[\mu^+\mu^-] = -5.20$$

- Imaginary part from the $\gamma\gamma$ cut:

$$\text{Im}\,\mathcal{A}(q^2)[\mu^+\mu^-] = \frac{\pi}{2\sigma_\mu(q^2)} \log\left[\frac{1-\sigma_\mu(q^2)}{1+\sigma_\mu(q^2)}\right] = -5.21$$

- Higher intermediate-state contributions found to be small!

Conclusion & outlook

- $\pi^0 \to e^+e^-$ decay
  - Reduced amplitude with form factor representation
  - Standard-Model prediction with 0.5% precision
  - Experiment and lattice progress NA62, RBC/UKQCD, 2022
- $K_L \to \ell^+\ell^-$ decay
  - Reduced amplitude with form factor representation
  - Standard-Model prediction with well-controlled uncertainties
  - Experiment and lattice progress talk by Chao

Much obliged for your attention!

"Rare is the union of beauty and purity."

Juvenal

Pion transition form factor

Dispersive form factor:

$$F^{\text{disp}}_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2) = \frac{1}{\pi^2}\int_{4M_\pi^2}^{s_{\text{iv}}} \text{d}x \int_{s_{\text{thr}}}^{s_{\text{is}}} \frac{\rho^{\text{disp}}(x,y)\,\text{d}y}{\left(x+Q_1^2\right)\left(y+Q_2^2\right)},$$

$$\rho^{\text{disp}}(x,y) = \frac{q_\pi^3(x)}{12\pi\sqrt{x}}\text{Im}\left[\left(F_\pi^V(x)\right)^* f_1(x,y)\right] + [x \leftrightarrow y]$$

$F_\pi^V(s)$ pion vector form factor, $f_1(s,q^2) = \gamma_s^*(q) \to 3\pi$ P-wave amplitude

Pion transition form factor

Effective pole term:

$F^{\text{disp}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$ fulfills the chiral anomaly $F_{\pi\gamma\gamma}$ by around 90%

⇒ Introduce an effective pole term

$$F^{\text{eff}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{g_{\text{eff}}}{4\pi^2 F_\pi} \frac{M_{\text{eff}}^4}{(M_{\text{eff}}^2 - q_1^2)(M_{\text{eff}}^2 - q_2^2)}$$

g_{eff} fixed by fulfilling the chiral anomaly

$g_{\text{eff}} \sim 10\%$, ⇒ small

M_{eff} fit to singly-virtual data excluding BaBar above $5\,\text{GeV}^2$ Gronberg et al., 1998, Aubert et al., 2009, Uehara et al., 2012

$M_{\text{eff}} \sim 1.5\text{–}2\,\text{GeV}$, ⇒ reasonable

Pion transition form factor

Asymptotically, $F_{\pi^0\gamma^*\gamma^*}$ should fulfill Brodsky, Lepage, 1979-1981

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = -\frac{2F_\pi}{3}\int_0^1 \mathrm{d}x \frac{\phi_\pi(x)}{xq_1^2 + (1-x)q_2^2} + \mathcal{O}\left(\frac{1}{q_i^4}\right),$$

Pion distribution amplitude $\phi_\pi(x) = 6x(1-x) + \cdots$

Brodsky–Lepage (BL) limit:

$$\lim_{Q^2\to\infty} F_{\pi^0\gamma^*\gamma}(-Q^2, 0) = \frac{2\,F_\pi}{Q^2}$$

Operator product expansion (OPE): Nesterenko, Radyushkin, 1983, Novikov et al., 1984, Manohar, 1990

$$\lim_{Q^2\to\infty} F_{\pi^0\gamma^*\gamma^*}(-Q^2, -Q^2) = \frac{2\,F_\pi}{3\,Q^2}$$

Pion transition form factor

Rewrite the asymptotic form into a double-spectral representation:

$$F^{\mathsf{asym}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{1}{\pi^2}\int_{s_\mathrm{m}}^{\infty}\int_{s_\mathrm{m}}^{\infty} \mathrm{d}x\mathrm{d}y \frac{\rho^{\mathsf{asym}}(x,y)}{(x-q_1^2)(y-q_2^2)},$$

$$\rho^{\mathsf{asym}}(x,y) = -2\pi^2 F_\pi xy\delta''(x-y)$$

This defines the asymptotic contribution:

$$F^{\mathsf{asym}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = 2F_\pi \int_{s_\mathrm{m}}^{\infty} \mathrm{d}x \frac{q_1^2 q_2^2}{(x-q_1^2)^2(x-q_2^2)^2}$$

- Does not contribute for the singly-virtual kinematics
- Restores the asympotics for singly-/doubly-virtual kinematics

$\pi^0 \to e^+e^-$

For the dispersive part we can write   Masjuan, Sánchez-Puertas, 2016

$$\mathcal{A}^{\mathrm{disp}}(q^2) = \frac{2}{\pi^2} \int_{4M_\pi^2}^{s_{\mathrm{iv}}} \mathrm{d}x \int_{s_{\mathrm{thr}}}^{s_{\mathrm{is}}} \mathrm{d}y \frac{\tilde{\rho}^{\mathrm{disp}}(x,y)}{xy} K(x,y)$$

Integration kernel

$$K(x,y) = \frac{2i}{\pi^2 q^2} \int \mathrm{d}^4k \frac{q^2k^2 - (q\cdot k)^2}{k^2(q-k)^2[(p-k)^2 - m_e^2]} \times \frac{xy}{(k^2-x)[(q-k)^2-y]}$$

- Checked with several standard techniques

$\pi^0 \to e^+e^-$

Experiment: KTeV, 2006

$$\text{BR}[\pi^0 \to e^+e^-(\gamma), x_D > 0.95]\big|_{\text{KTeV}} = 6.44(25)(22) \times 10^{-8}$$

$$x_D = \frac{m_{e^+e^-}^2}{M_{\pi^0}^2} = 1 - 2\frac{E_\gamma}{M_{\pi^0}}$$

- With old radiative corrections Bergström, 1982

$$\text{BR}[\pi^0 \to e^+e^-]\big|_{\text{KTeV}} = 7.48(29)(25) \times 10^{-8}$$

- With reexamined radiative corrections Vaško, Novotný, 2011, Husek et al., 2014

$$\boxed{\text{BR}[\pi^0 \to e^+e^-]\big|_{\text{KTeV}} = 6.85(27)(23) \times 10^{-8}}$$

$\pi^0 \to e^+e^-$

Theory predictions	$\text{BR}[\pi^0 \to e^+e^-] \times 10^8$
Dorokhov et al.	$6.23(9)$
Husek et al.	$6.12(7)$
Masjuan et al.	$6.23(5)$
Weil et al.	$6.22(3)$
This work	$6.25(3)$