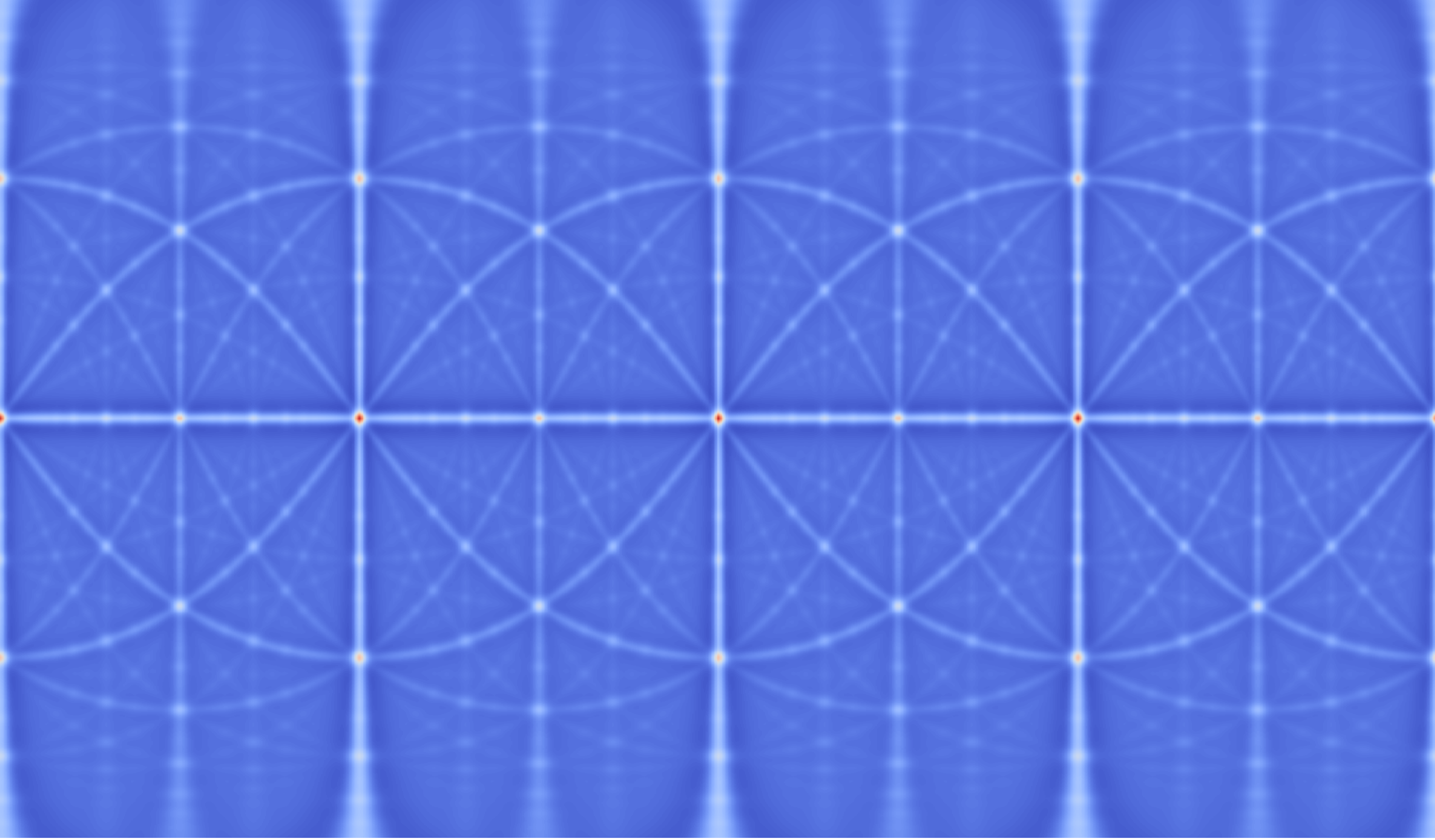


Finite-volume collinear divergences in radiative corrections to meson leptonic decays

Antonin Portelli — 04/08/2023

Lattice 2023, FermiLab, IL, USA

THE UNIVERSITY of EDINBURGH

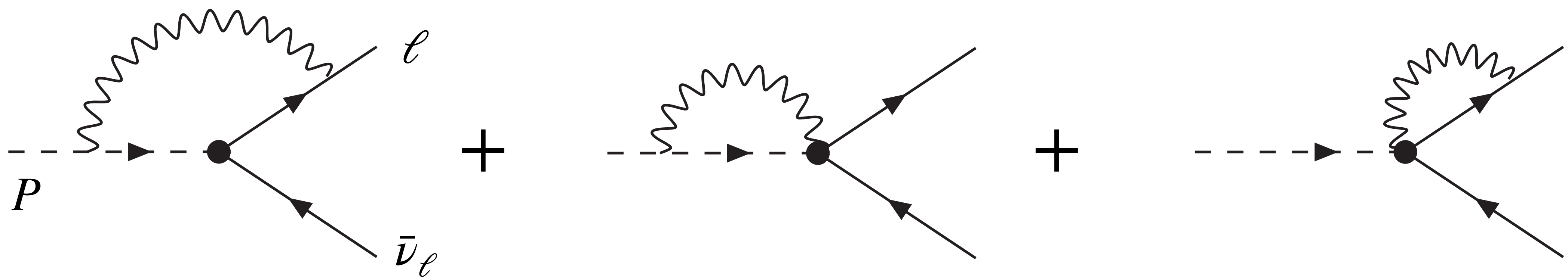
- Generalities
- Finite-volume collinear divergences
- Mitigation strategies
- Outlook

In collaboration with

M Di Carlo, M T Hansen, and N Hermansson-Truedsson

Generalities

Hard collinear divergences in leptonic decays



$$= \frac{\alpha}{4\pi}\left[2A_1(|\mathbf{v}_\ell|) + \text{finite} + \text{IR logs}\right]$$

$$A_1(|\mathbf{v}_\ell|) = \frac{\operatorname{arctanh}(|\mathbf{v}_\ell|)}{|\mathbf{v}_\ell|} \sim \log\left(\frac{m_\ell}{M_P}\right)$$

- **Divergent** for $|\mathbf{v}_\ell| \to 1$ or equivalently $m_\ell \to 0$
- Independent from soft-photon IR divergences

Boyle, AP, et al. JHEP23 242 (2023)

Hard collinear divergences in leptonic decays

- **Not strictly speaking a divergence** since $m_\ell > 0$
- However $r_\ell = m_\ell / M_P$ can be **small**
- *e.g.* for **muonic decays**

P	r_ℓ	$\lvert \mathbf{v}_\ell \rvert$
π^+	0.757	0.271
K^+	0.214	0.912
D^+	0.057	**0.994**
D_s^+	0.054	**0.994**

$$\lvert \mathbf{v}_\ell \rvert = \frac{1 - r_\ell^2}{1 + r_\ell^2}$$

Finite-volume collinear divergences

Finite-volume expansion

- In $\mathrm{QED_L}$: expansion in powers and log of L (same for $\mathrm{QED_C}$, $\mathrm{QED_r}$ and IR-improved variants)

- Expansion driven by the FV coefficients

$$c_j(\mathbf{v}) = \left(\sum_{\mathbf{n}\neq\mathbf{0}} - \int \mathrm{d}^3\mathbf{n} \right) \frac{1}{|\,\mathbf{n}\,|^j} \frac{1}{1-\mathbf{v}\cdot\hat{\mathbf{n}}} + \underbrace{\frac{1}{6} \sum_{|\mathbf{n}|=1} \frac{1}{1-\mathbf{v}\cdot\mathbf{n}}}_{\mathrm{QED_r}\ \text{term}}$$

- For more details
  - *Matteo Di Carlo* 03/08 10:00
  - *Nils Hermansson-Truedsson* 04/08 09:20

Finite-volume coefficients

$$c_j(\mathbf{v}) = \left(\sum_{\mathbf{n}\neq\mathbf{0}} - \int \mathrm{d}^3\mathbf{n} \right) \frac{1}{|\mathbf{n}|^j} \frac{1}{1 - \mathbf{v}\cdot\hat{\mathbf{n}}} + \frac{1}{6} \sum_{|\mathbf{n}|=1} \frac{1}{1 - \mathbf{v}\cdot\mathbf{n}}$$

- Contains **finite-volume** colinear divergences
- At $|\mathbf{v}| = 1$, diverges if a **lattice vector** is collinear with $|\mathbf{v}|$
- Encodes **breaking of rotational symmetry**
- Related to **number-theoretical properties** of $\mathbf{v}$

QedFv package

- **C++ library** for computing FV coefficients using exp-exp sum acceleration

  Davoudi, AP, et al. PRD99(3), 034510 (2019)

  https://github.com/aportelli/QedFvCoef

- Features
  - Fast multi-threaded sums
  - Auto-tuning
  - Python binding

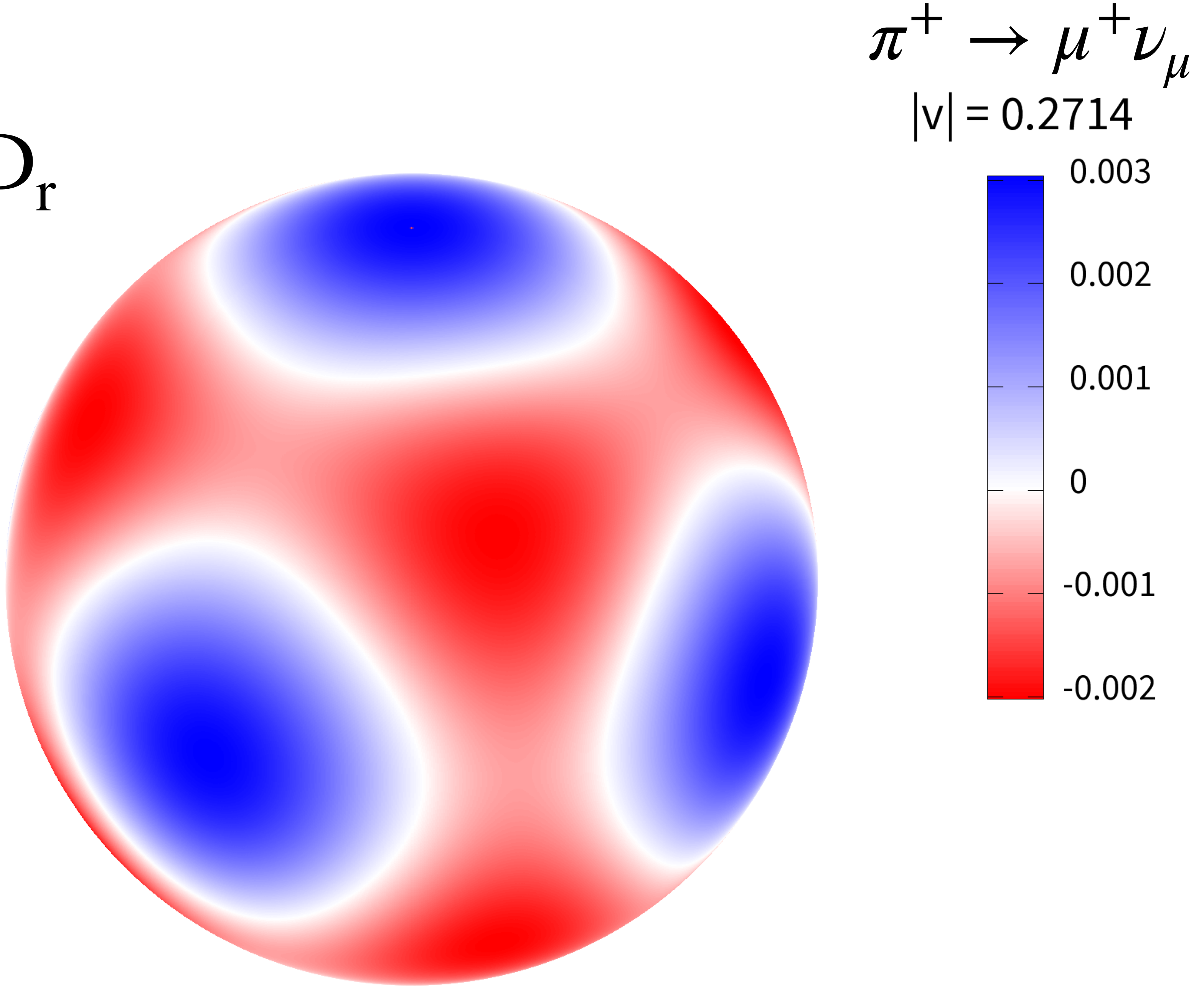
```
[2]  cqedl = qedfv.Coef()
     cqedr = qedfv.Coef(qed=qedfv.Qed.r)

[3]  v = [0.2,0.1,0.1]
     pl = cqedl.tune(2., v)
     pr = cqedr.tune(2., v)
     [pl,pr]
...  [{ eta: 0.386924, nmax: 15 }, { eta: 0.386924, nmax: 15 }]

[4]  [cqedl(2, v, pl), cqedr(2, v, pr)]
...  [-9.099156120172816, -8.07853322454992]
```

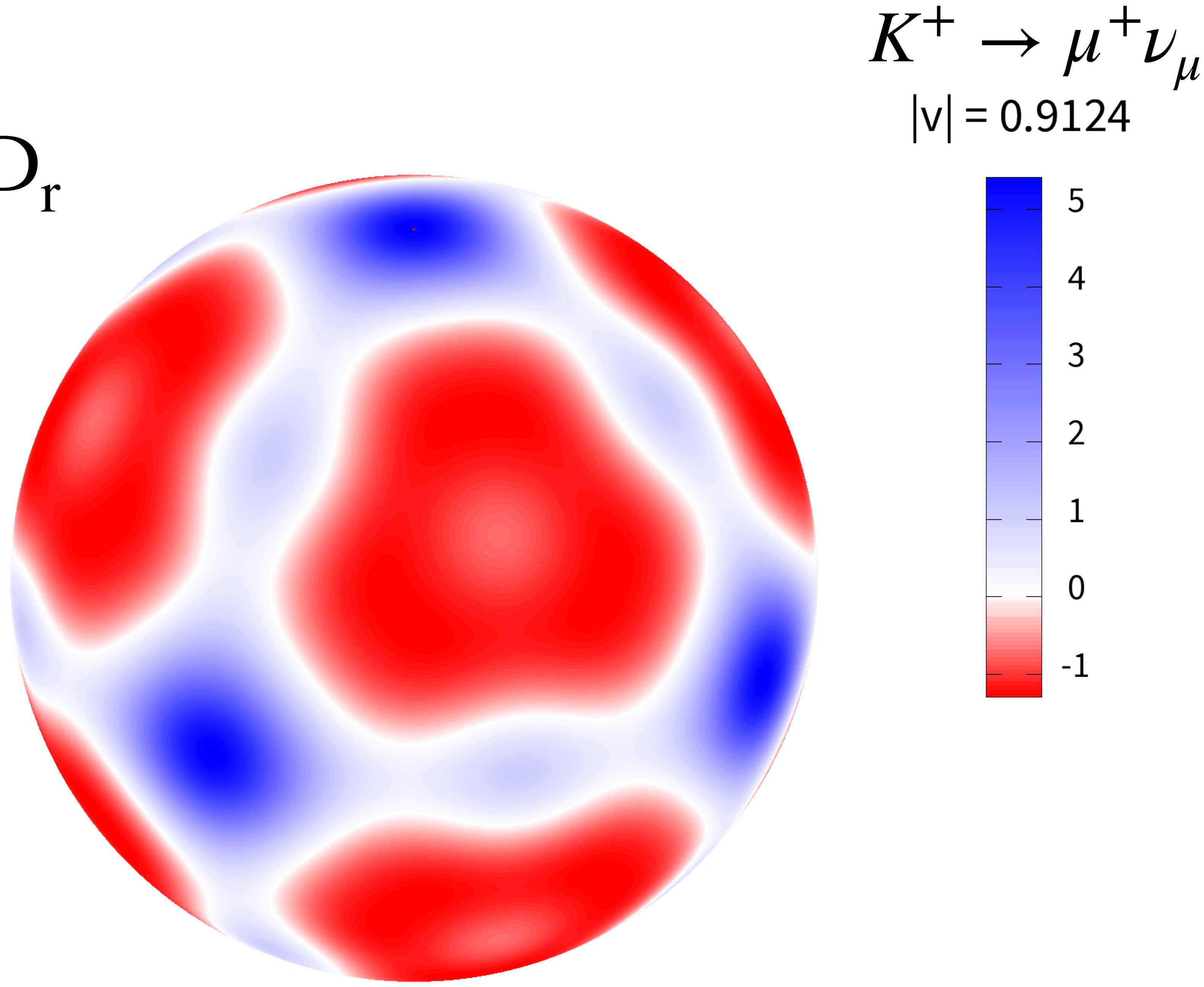
Angular dependence

$c_0(\mathbf{v})$ in $\mathrm{QED_r}$



Angular dependence

$c_0(\mathbf{v})$ in $\mathrm{QED_r}$



Angular dependence

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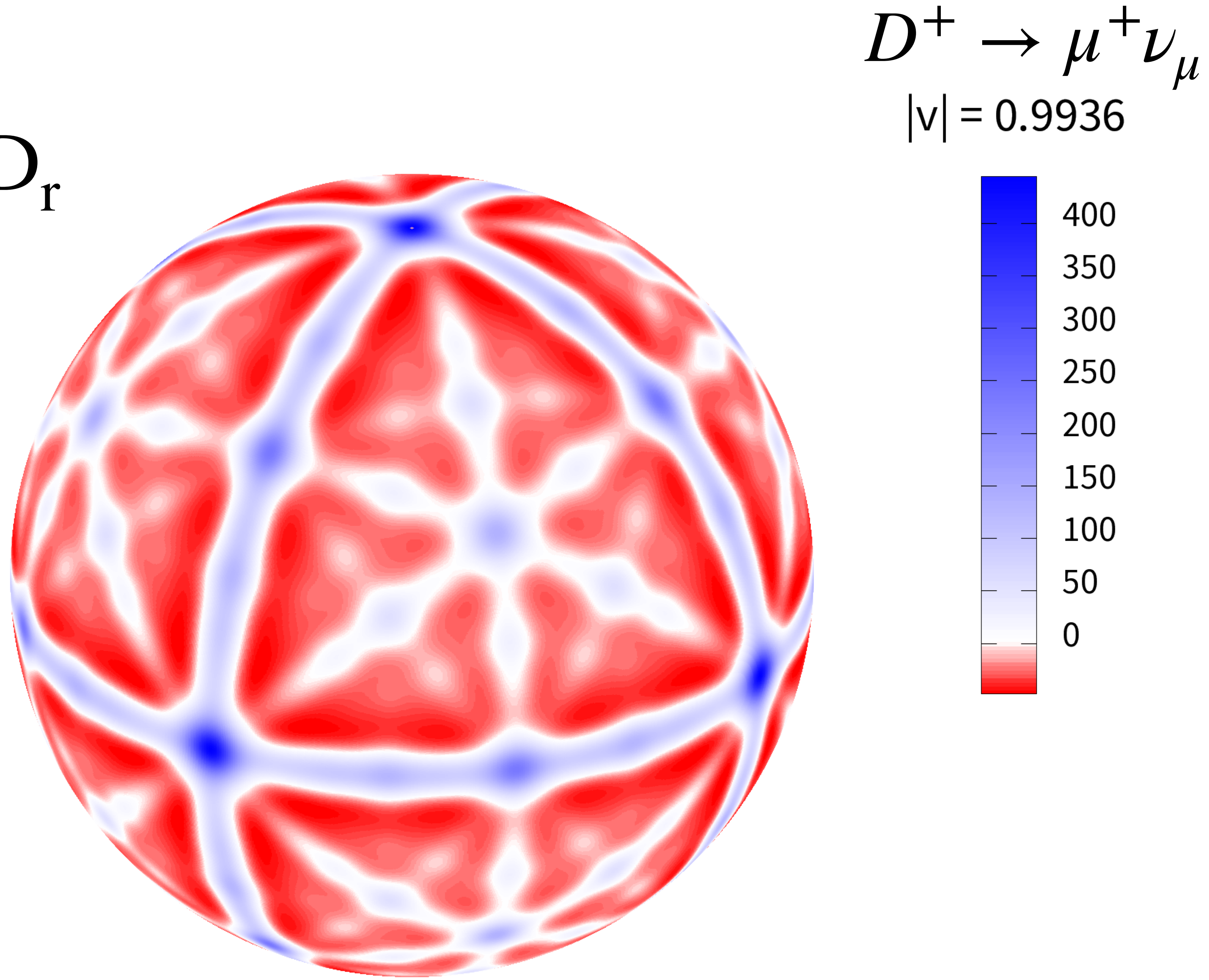

Angular dependence

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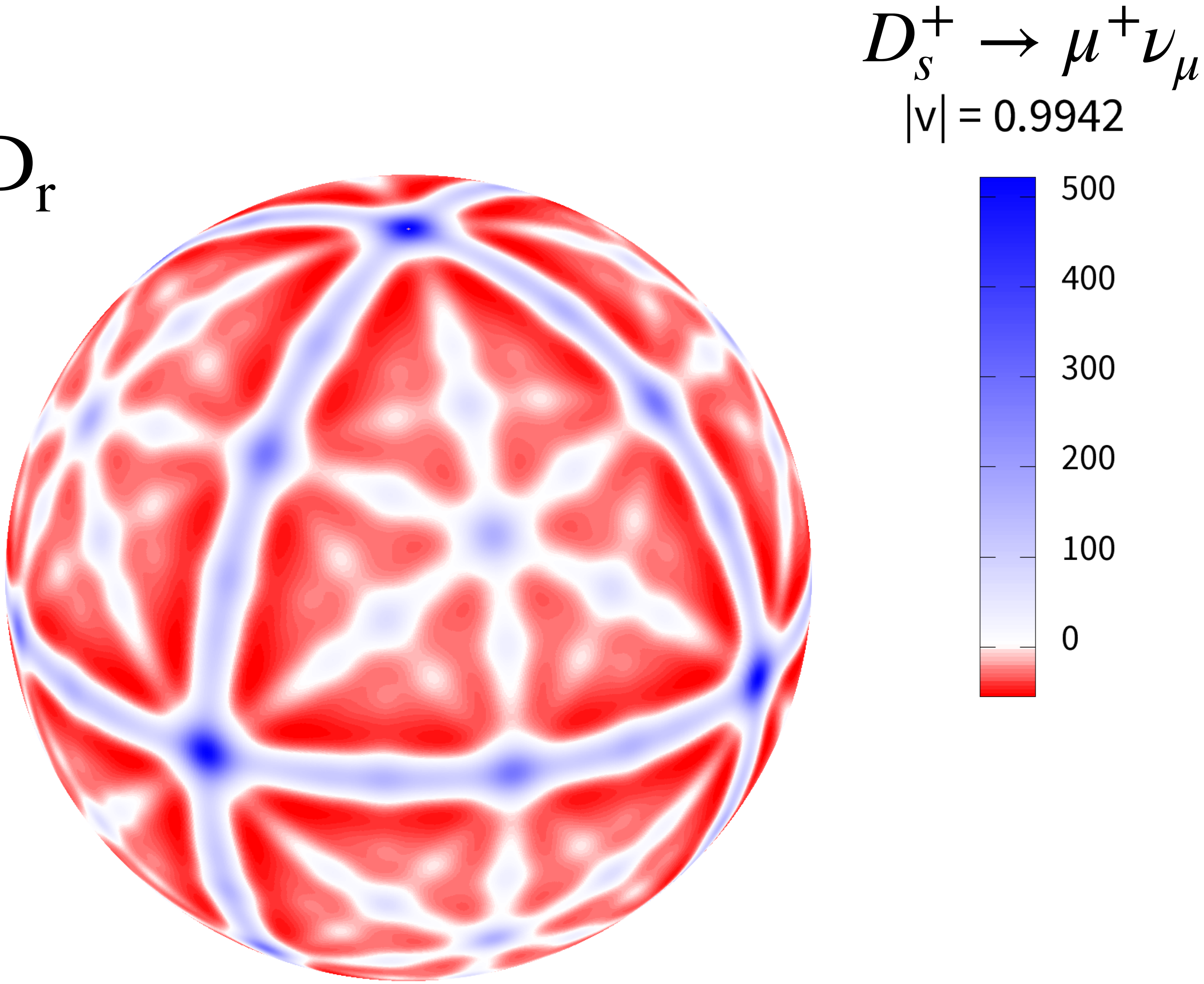
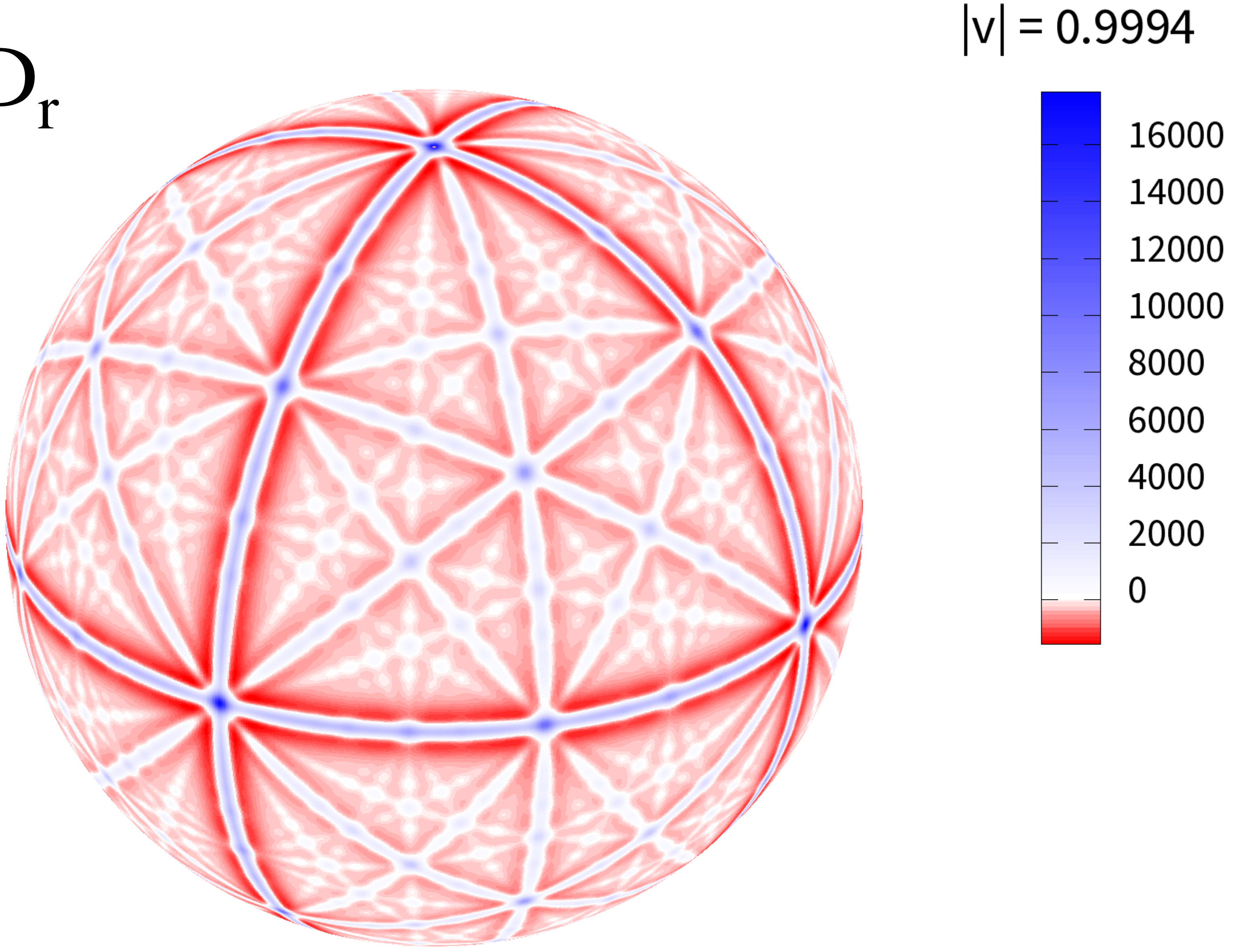


Angular dependence

$c_0(\mathbf{v})$ in $\mathrm{QED_r}$



Spherical harmonic expansion

$$c_j(\mathbf{v}) = A_1(|\mathbf{v}|)\, c_j(\mathbf{0}) + \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} a_{klm}(\mathbf{v})\, y_{jlm}$$

- s-wave similar to infinite-volume
- $a_{klm}(\mathbf{v})$ is $\mathcal{O}(|\mathbf{v}|^l)$ **velocity-suppressed**
- *cf.* proof in
  Davoudi, AP, et al. PRD99(3), 034510 (2019)
- Expansion **not very useful** for $|\mathbf{v}| \to 1$

Finite-volume collinear divergences

- *Lattice-aligned vectors*:

$$\Gamma = \{\mathbf{x} \in \mathbb{R}^3 \,|\, \exists \alpha \in \mathbb{R} \text{ s.t. } \alpha\mathbf{x} \in \mathbb{Z}^3\}$$

  Γ is **dense** in $\mathbb{R}^3$ (proof: $\mathbb{Q}^3 \subset \Gamma$)

- *Primitive direction*: for $\mathbf{x} \in \Gamma$, $\mathbf{x}^* \in \mathbb{Z}^3$ such that

$$|\mathbf{x}^*| = \min\{\,|\mathbf{n}|\;|\;\mathbf{n} \in \mathbb{Z}^3\backslash\{\mathbf{0}\},\ \alpha > 0 \text{ s.t. } \mathbf{x} = \alpha\mathbf{n}\}$$

  unique and has **co-prime components**

$$\mathrm{GCD}(\mathbf{x}_1^*, \mathbf{x}_2^*, \mathbf{x}_3^*) = 1$$

Finite-volume collinear divergences

- Preliminary **new result**, for $j > 3$ (UV-finite integrands)

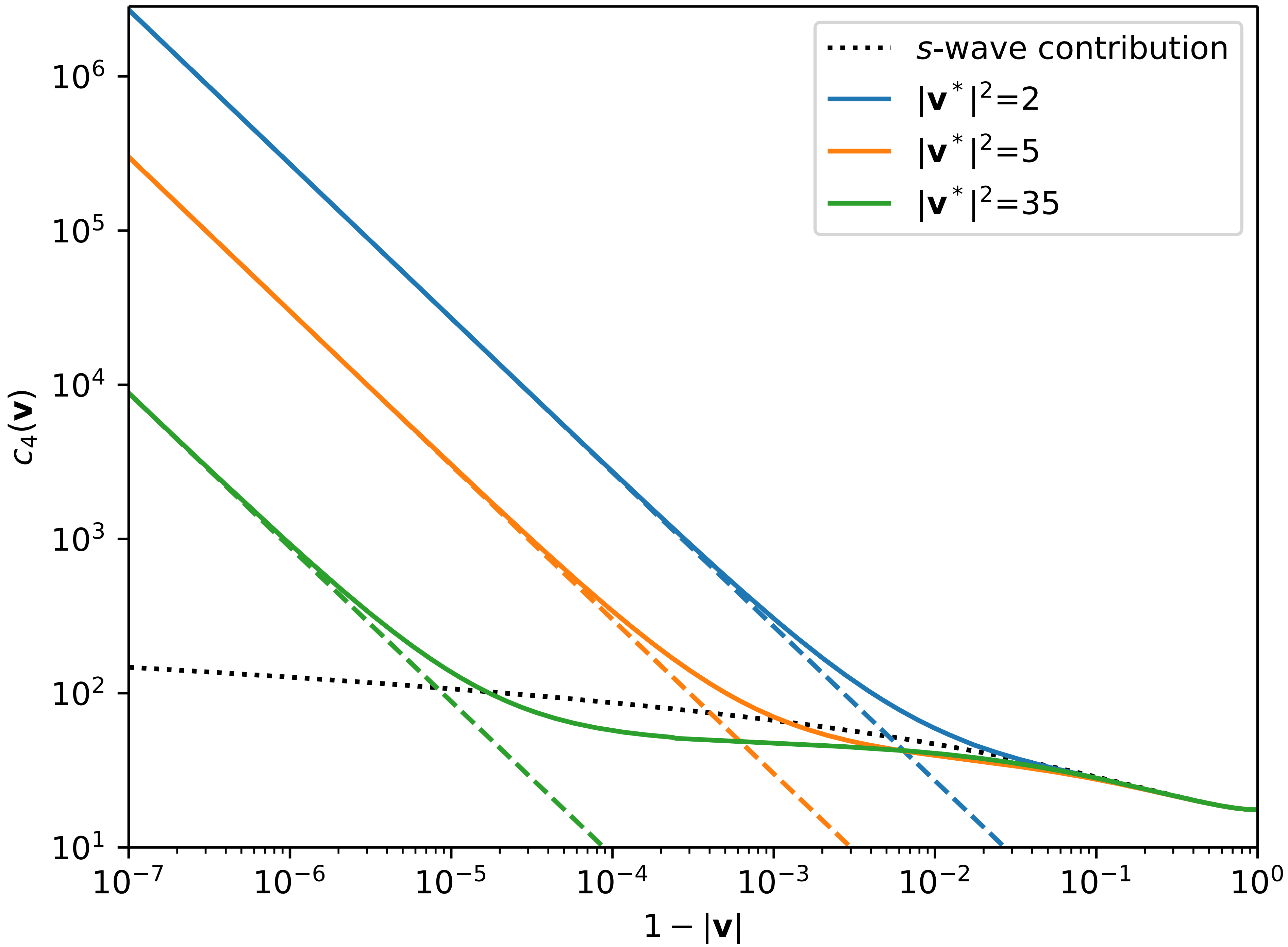
$$c_j(\mathbf{v}) \underset{|\mathbf{v}|\to 1}{\sim} \frac{\zeta(j)}{|\mathbf{v}^*|^j}\frac{1}{1-|\mathbf{v}|}$$

- **Power divergence** vs. log in infinite volume

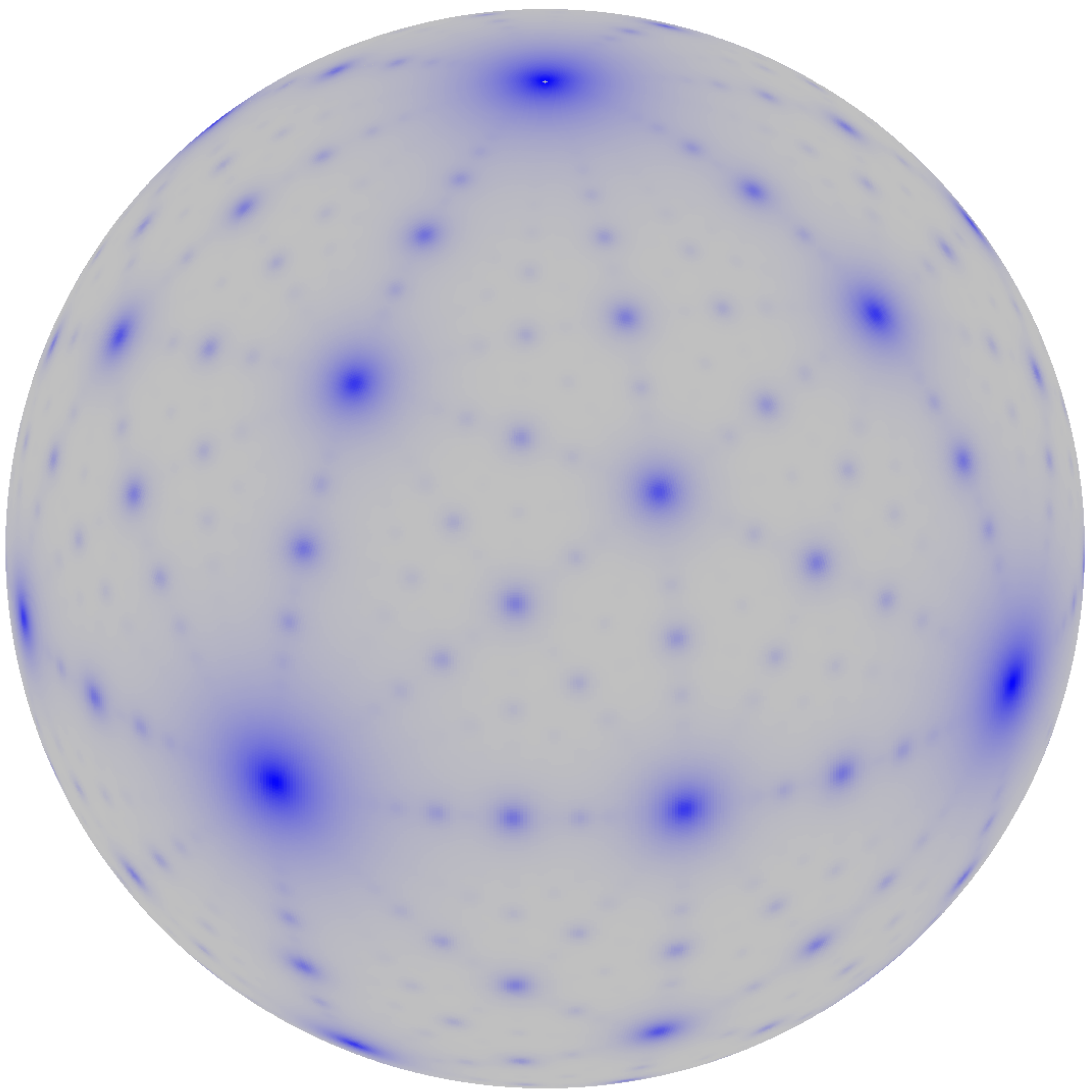
- Divergence **suppressed by primitive norm**

- Working on **generalising** to $j < 3$

Numerical comparison

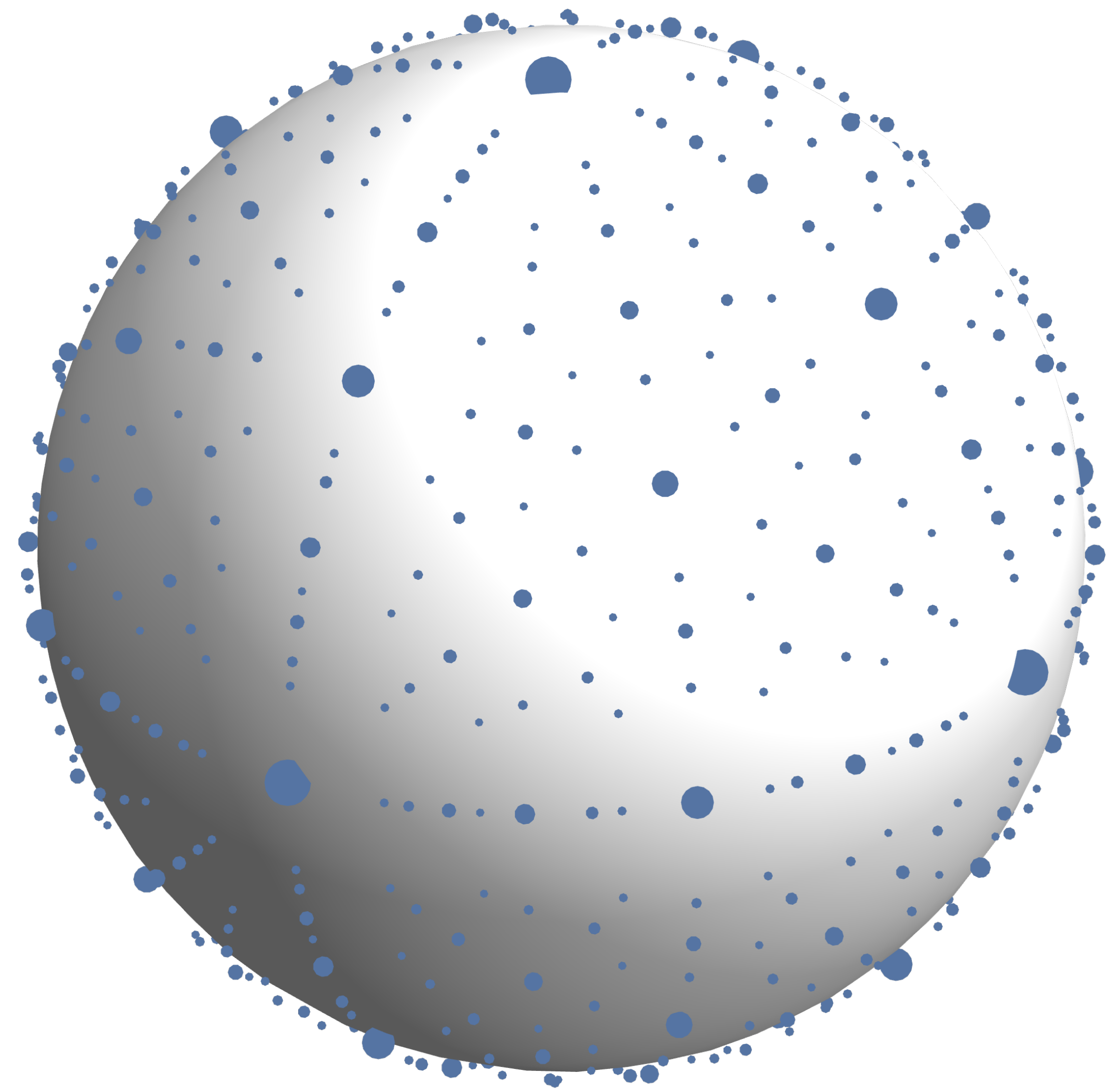


Geometrical structure



$c_4(\mathbf{v})$ for $|\mathbf{v}| = 0.9999$

colour in log scale



directions $\hat{\mathbf{n}}$ for $|\mathbf{n}^*|^2 \leq 30$

point size proportional to inverse primitive norm

Implications for lattice calculations

- “Naive” momentum directions e.g. $(1,0,0)$ are associated with **huge FV collinear enhancement** at high velocity
- Potentially degrading effects on **statistical signal**
  - Very large **analytical subtractions** of FV effects
  - **Variance** might suffer from same phenomena

Mitigation strategies

General problem

- For weak decays, effects up to $\mathcal{O}(1/L^2)$ are well known and **can be subtracted analytically**
- Issues with **large subtraction** can still arise
- The $\mathcal{O}(1/L^3)$ coefficient is **not well known**, and a linear combination of $c_0(\mathbf{0})$ and $c_0(\mathbf{v})$

Matteo Di Carlo 03/08 10:00
Nils Hermansson-Truedsson 04/08 09:20
Di Carlo, AP, et al. PRD 105(7), 074509 (2022)
Boyle, AP, et al. JHEP23 242 (2023)

Magic angles

$$c_0(\mathbf{v}) = A_1(\,|\,\mathbf{v}\,|\,)\,c_0(\mathbf{0}) \;+\; \sum_{l=1}^{+\infty}\sum_{m=-l}^{l} a_{klm}(\mathbf{v})\,y_{0lm}$$

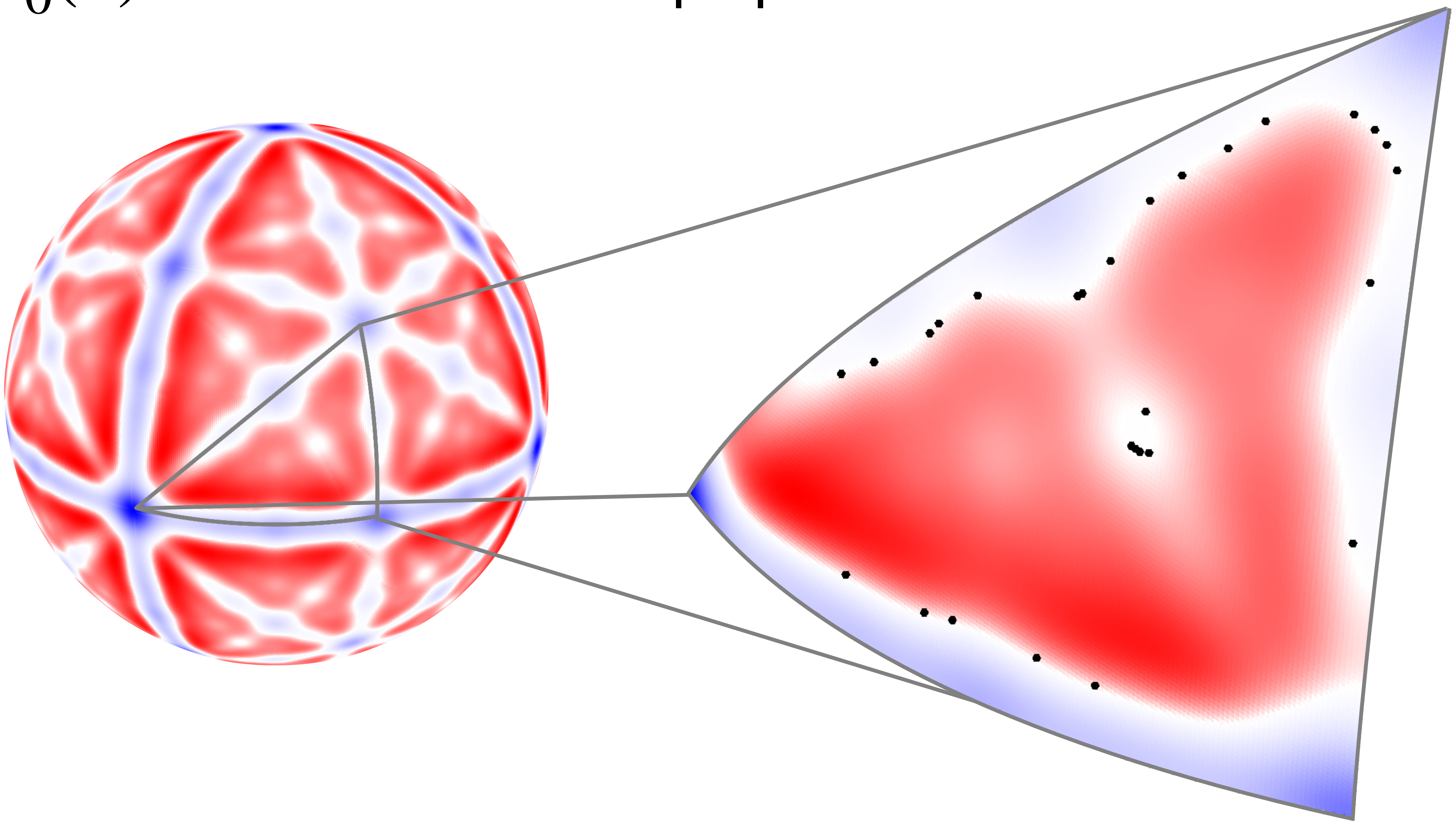
- $c_0(\mathbf{0}) = 0$ in $\mathrm{QED_r}$ or local theories, so

$$\int_{S^2} \mathrm{d}^2\hat{\mathbf{v}}\,c_0(\mathbf{v}) = 0$$

- It implies there exists $\bar{\mathbf{v}}$ such that $c_0(\bar{\mathbf{v}}) = 0$
- **“Magic angles”**, can be solved numerically

Magic angles

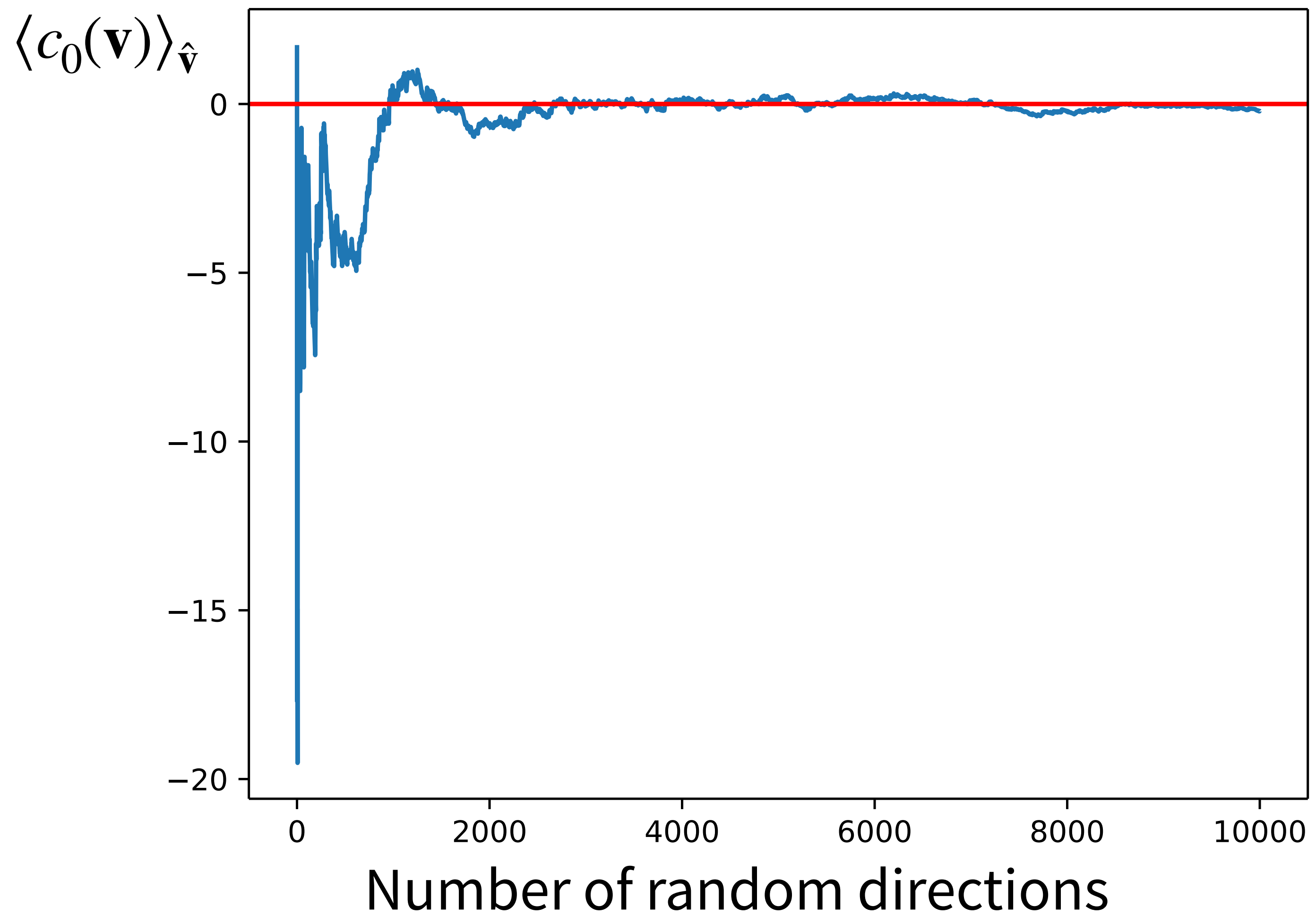
$c_0(\bar{\mathbf{v}}) = 0$ solutions at $|\mathbf{v}| = 0.995$



Analysis and plot: M Di Carlo

Stochastic direction averaging (SDA)

- Angular dependence **can be removed stochastically** by drawing the **momentum direction randomly** for each measurement, i.e. $\langle c_j(\mathbf{v}) \rangle_{\hat{\mathbf{v}}} = A_1(\,|\,\mathbf{v}\,|\,)\, c_j(\mathbf{0})$



Implications for lattice calculations

- Using $\mathrm{QED_r}$ or a local theory + magic angles or SDA, **the $\mathcal{O}(1/L^3)$ effect can be removed entirely**
- FV effects on weak decay radiative corrections **controlled up to likely negligible $\mathcal{O}(1/L^4)$ effects**
- Both magic angles and SDA currently running with physical point $\mathrm{QCD+QED_r}$ for π, K, D, D_s leptonic decays
- **We will follow up!**

Outlook

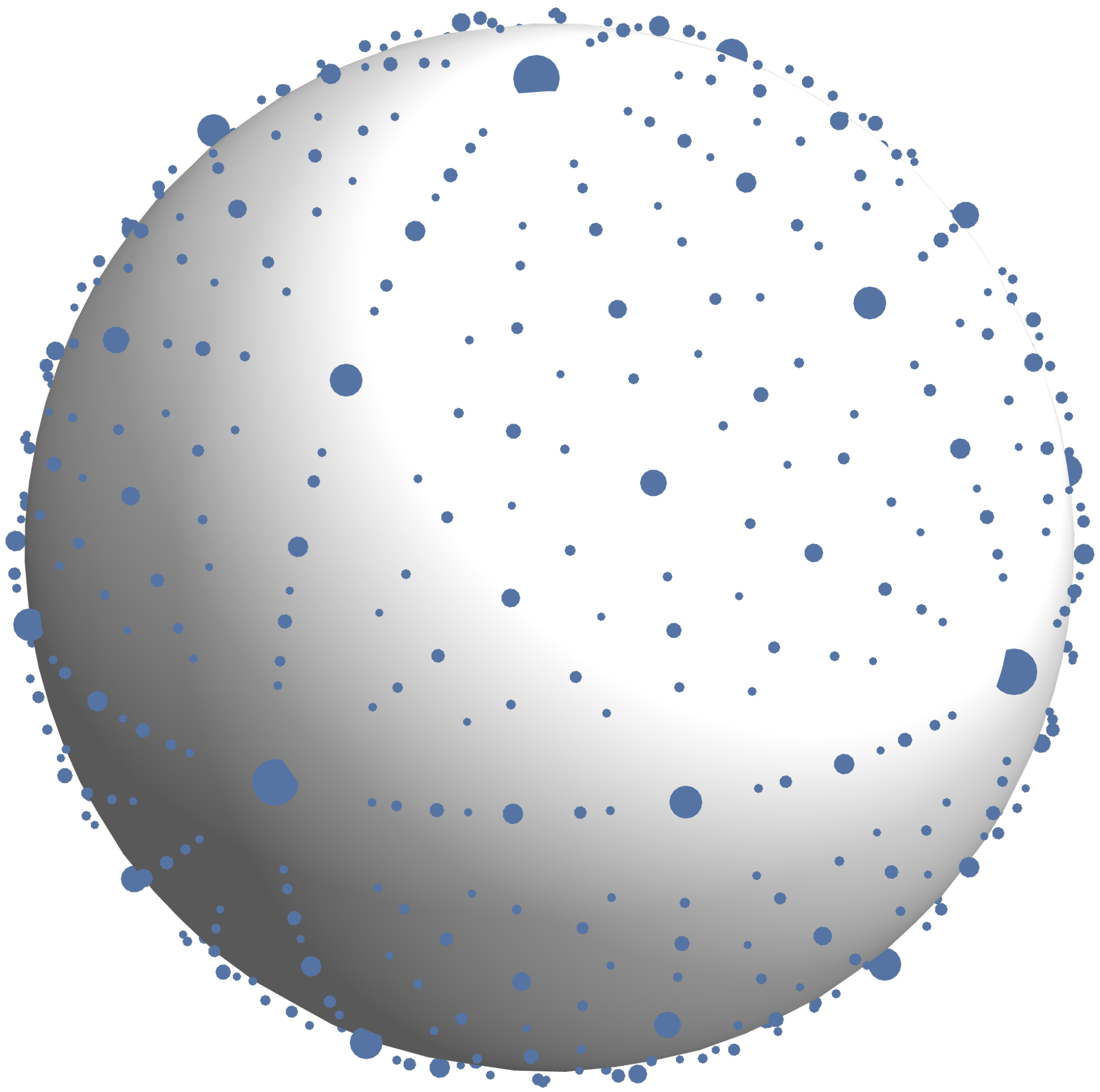
Summary

- Collinear divergences in finite-volume are **power-like and not logarithmic** as in infinite-volume

- The leading divergence coefficient is related to **number-theoretical properties** of the momentum direction

- **Behaviour understood analytically** for $j > 3$, likely generalises to $j \leq 3$

- **SDA can help restoring the gentler infinite-volume behaviour** in lattice measurements of leptonic decays



$$c_j(\mathbf{v}) \underset{|\mathbf{v}|\to 1}{\sim} \frac{\zeta(j)}{|\mathbf{v}^*|^j} \frac{1}{1-|\mathbf{v}|}$$

Thank you!

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