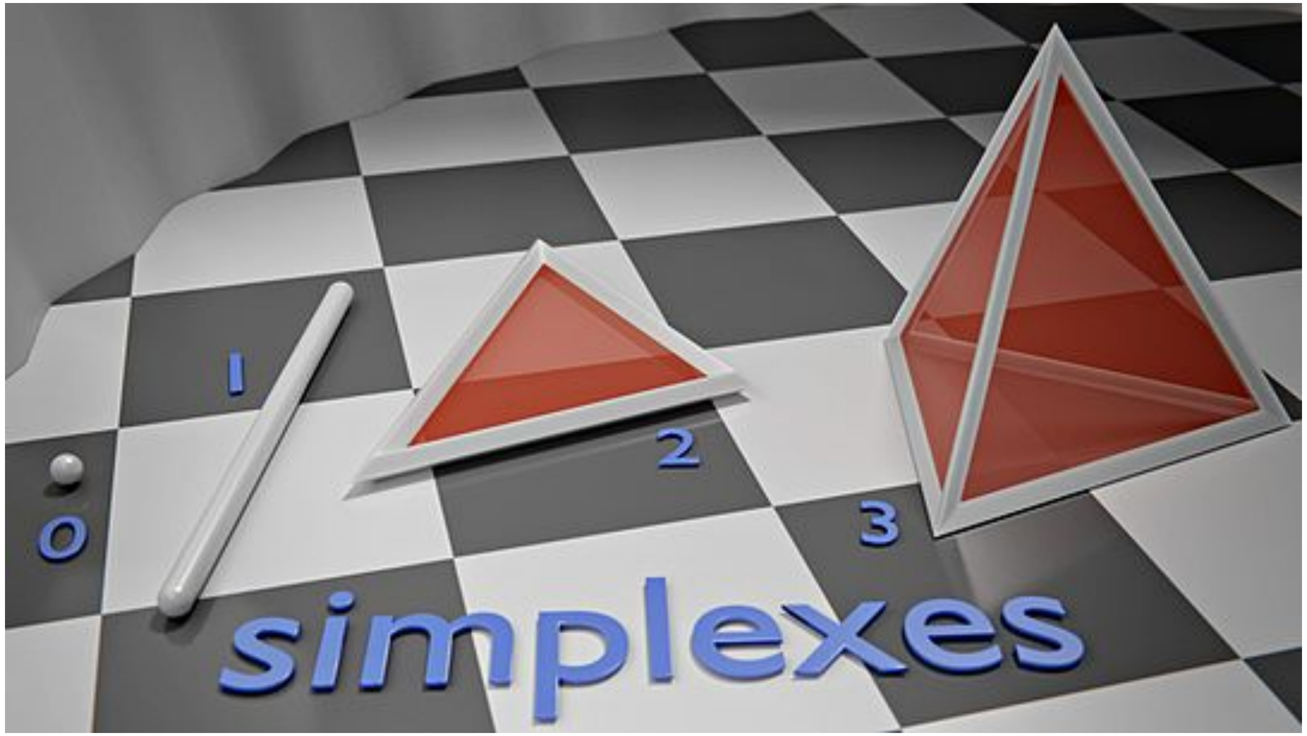
Affine Quantum Geometry: Lesson from the Ising Sphere



Richard Brower Boston University Lattice 2023 FNAL Aug 1 2023

with collaborators Venkitesh Ayyar, Cameron Cogburn, George Fleming, Anna-Maria Gluck, Jin-Yun Lin, Nobuyuki Matsumoto, Evan Owen. Tim Raben, Chung-I Tan,

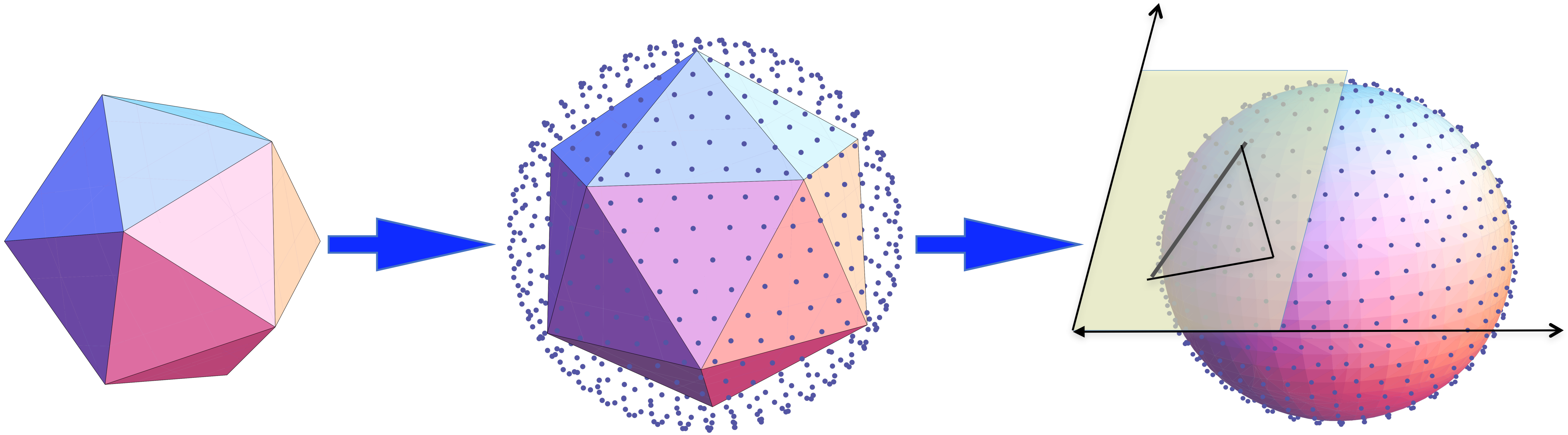
David Hilbert's Advice

The art of doing mathematics consists finding that special case which contains all the germs of generality.

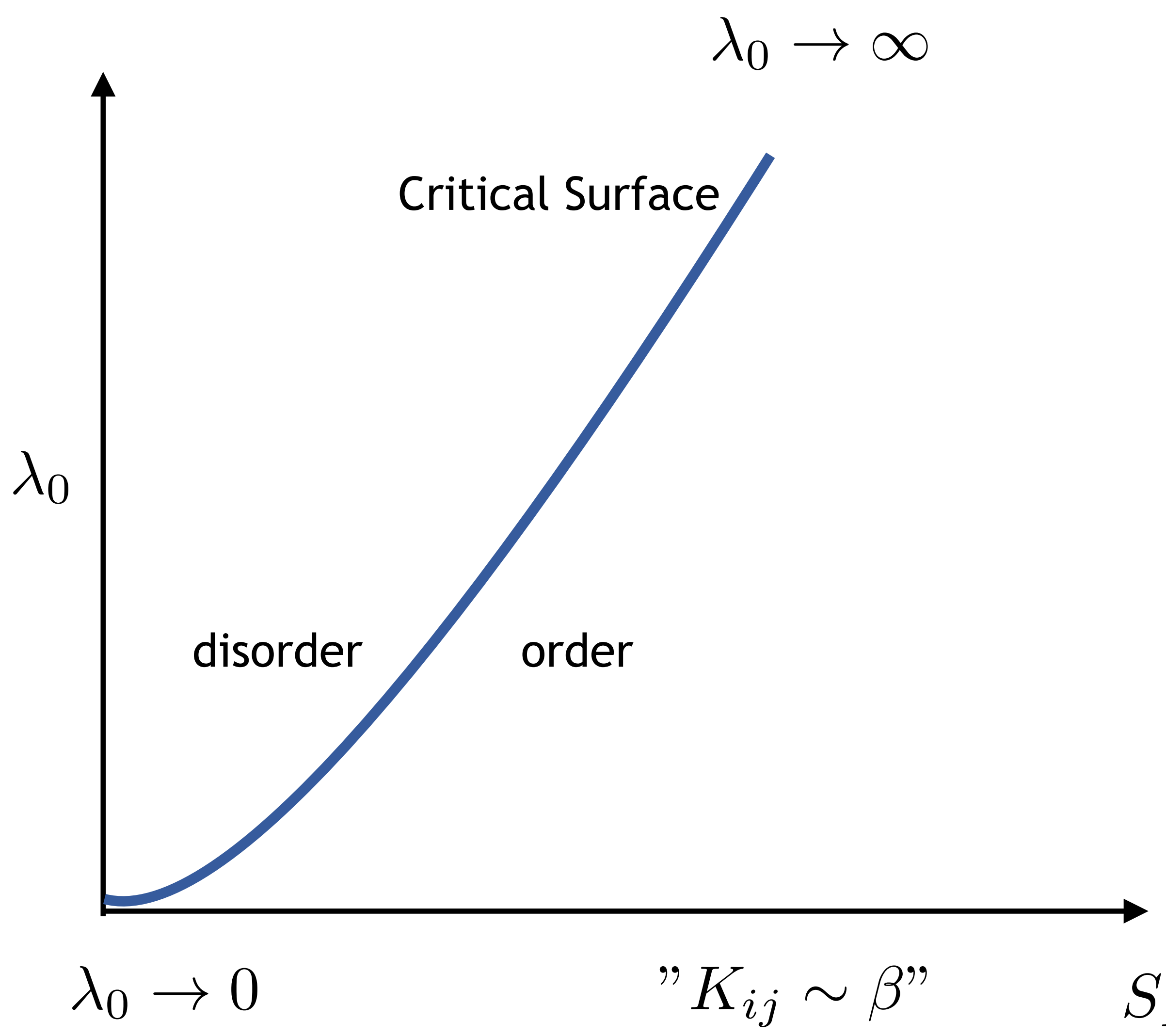
Mathematician, Physicist, Philosopher

Author of Geometry and the Imagination (1932)

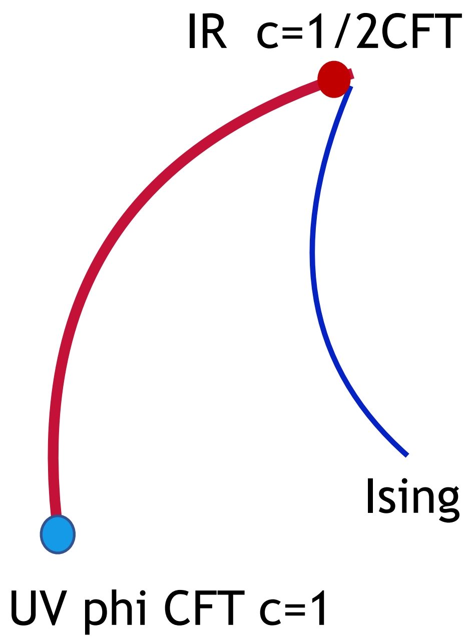
The Special Case Ising (explain by Evan Owen) and Phi4 (next Anna Gluck)



2d Phi 4& Ising lattice give universal equivalent CFT



$$S_{ising} = \frac{1}{2}\sum_{\langle i,j \rangle} K_{ij}(s_i - s_j)^2$$



$$S_{FEM} = \frac{1}{2}[K_{ij}(\phi_i - \phi_j)^2 + \lambda_0(\phi_i^2 - 1)^2]$$

First step: Construct the Classical (Regge) Manifold & Simplicial (FEM) Action

$$S = \frac{1}{2}\int_{\mathcal{M}} d^d x \sqrt{g(x)}\,[g^{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x) + m^2\phi^2(x) + \lambda\phi^4(x)]$$

$g_{\mu\nu}(x)$

Quantum field $\phi(x)$

Regge Calc Geometry

Finite Element Method

Classical Action on Simplicial Lattice (Complex)

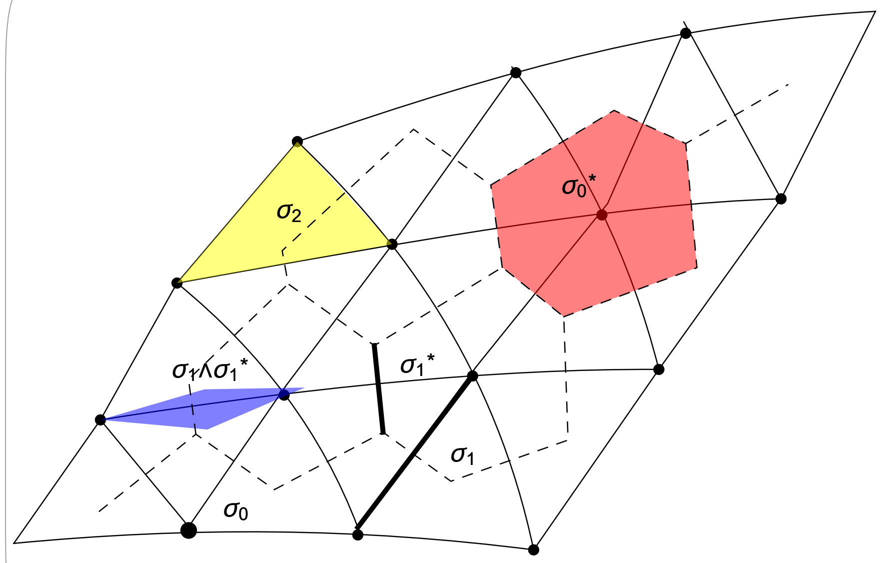
$$S_{FEM} = \frac{1}{2}[K_{ij}(\phi_i - \phi_j)^2 + \lambda_0\sqrt{g_i}\lambda_0(\phi_i^2 - 1)^2]$$

Start with Classical Simplicial Lattice

Gravitation Metric Manifold

REGGE: Piecewise linear metric

$$(\mathcal{M}, g_{\mu\nu}(x)) \leftrightarrow (\mathcal{M}_\sigma, g_\sigma = \{l_{ij}\})$$

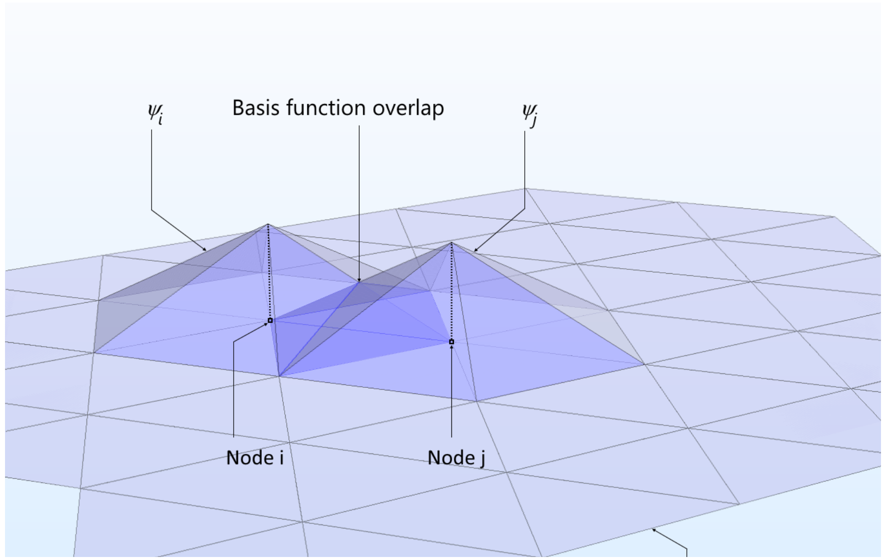


Simplicial Complex/Delaunay Dual Complex + Regge flat metric on each Simplex

Classical Fields: PDEs

FEM: Piecewise linear fields

$$\phi(x) \leftrightarrow \phi = \sum_i \phi_i W_i(\xi)$$



Actually fancier methods: **Discrete Exterior Calculus** (scalar), Spin connection (Fermion), Wilson links (gauge) , etc.

Basic Observation

1. Regge's Calculus for Einstein Hilbert and Finite Elements Method share EXACTLY the same simplicial lattice
2. Both give exact results in the classical continuum limit
3. BUT QFTs choose an emergent geometry at IR fixed point.
4. Must match the geometry (Regge edge length) to lattice field theories (coupling parameter)
5. There is a finite Affine parameterization on each (flat)tangent plane

Classical Gravity and Fields Exactly the Same Lattice Geometry!

Einstein Classical Gravity
(i.e. PDEs for metric)

Lattice: **REGGE**: Triangulated (Simplicial) Geometry

CLASSICAL

Classical Fields Theory
(i.e PDE's for equation of motion)

Lattice: FEM: (Finite Element on triangulated shapes)

QFE:
Quantum Geometry

Quantum Gravity (???)

REGGE: Dynamical triangulation: Maybe?

QUANTUM

Quantum Field Theory (QFT)

continuum limit of Simplicial lattice YES

REGGE:
"General Relativity without Coordinates" 1960

- The Simplicial Approximation Theory (L.E.J. Brouwer 1927?)

In mathematics, the **simplicial approximation theorem** is a foundational result for algebraic topology, guaranteeing that continuous mappings can be (by a slight deformation) approximated by ones that are piecewise of the simplest kind. It applies to mappings between spaces that are built up from simplices—that is, finite simplicial complexes. The general continuous mapping between such spaces can be represented approximately by the type of mapping that is (*affine*-) linear on each simplex into another simplex, at the cost (i) of sufficient barycentric subdivision of the simplices of the domain, and (ii) replacement of the actual mapping by a homotopic one.

Einstein: $\{\mathcal{M}, g_{\mu\nu}\}$

$$S_{EH} = \int d^d x \sqrt{g(x)} R(x)$$

Regge: $\{G, \ell_{ij}\}$

$$S_{Regge}[\ell_{ij}] = \sum_{i \in G} \epsilon_i \; A_i^*$$

GEOMETRIC TUG OF WAR

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}$$

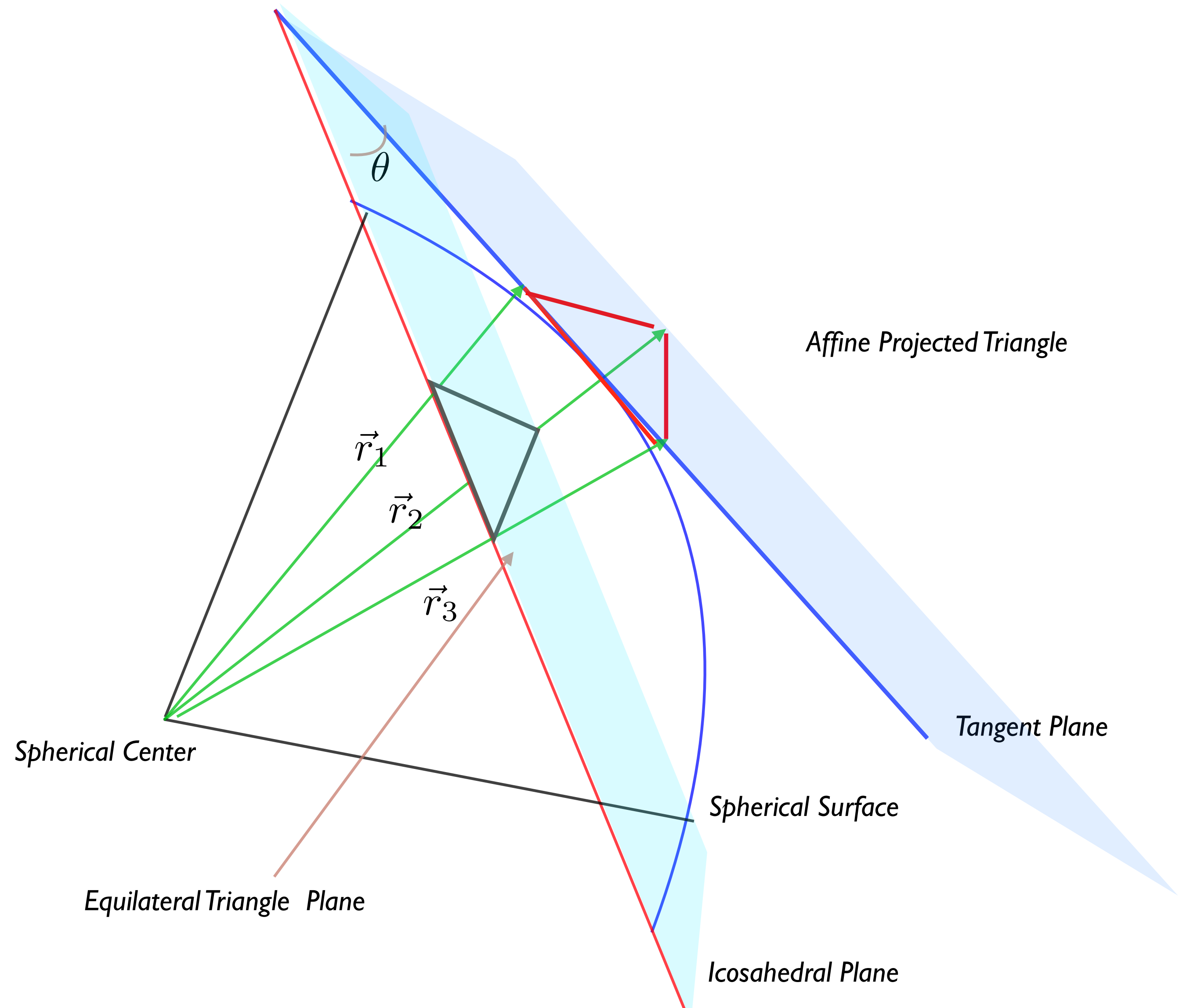
$$\frac{\delta[\cdots]}{\delta g^{\mu\nu}}$$

$$S_{EH} = \int d^dx\sqrt{g(x)}R(x)$$

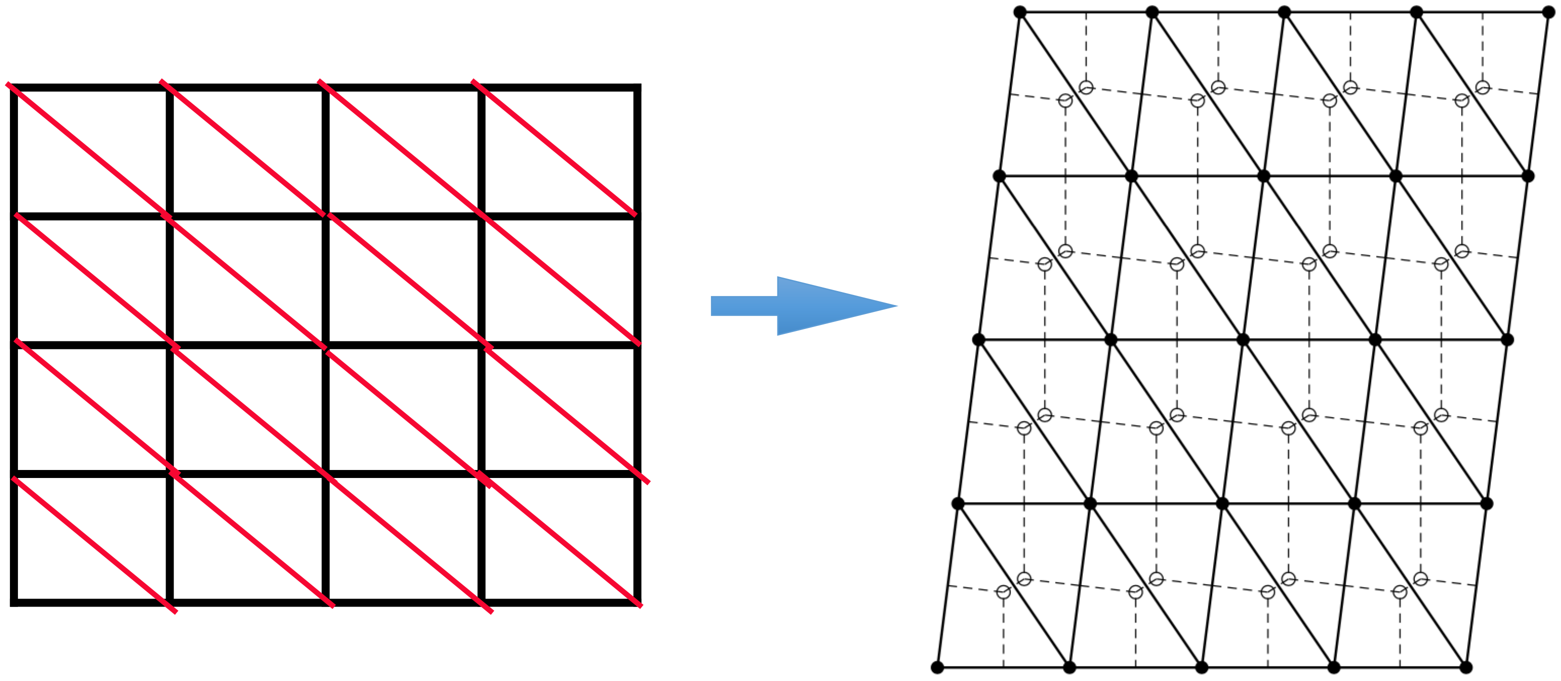
$$S_{QFT} = \int d^dx\sqrt{g(x)}\mathcal{L}(\phi, \psi, A_\mu ..)$$

- Regge Calculus and Finite Elements share the same piece-wise flat simplicial (triangular) complex
- Actually for general Elegant **"Discrete Exterior Calculus"**

Even Simpler Example:
Affine Geometry of TANGENT PLAN



Affine Square to general triangles



All Triangle (simplicies) are Affine Equivalent
Poincare d(d+1)/2 plus general flat metric d(d+1)/2

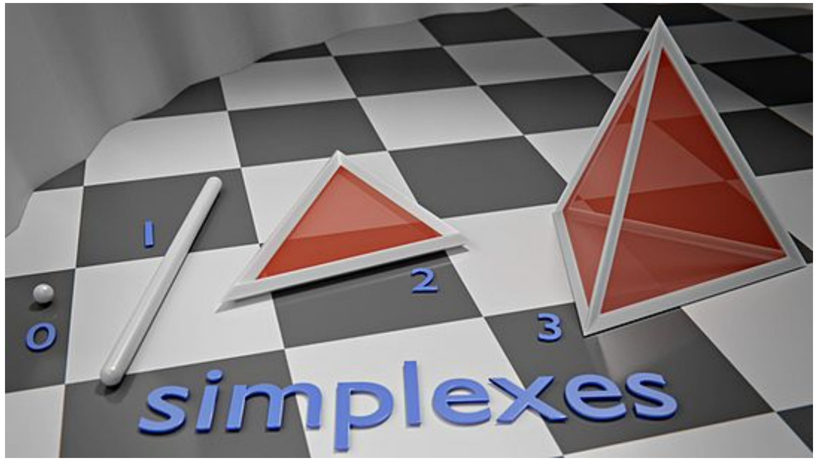
Affine Transform == Metric == Simplex Manifold

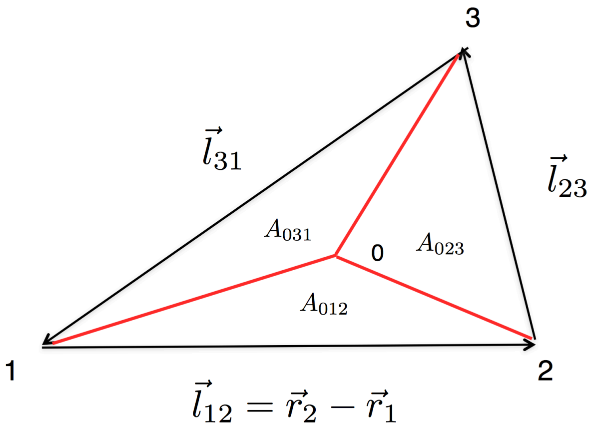
$$x = A\xi + b \implies x^\mu = A^\mu_i \xi^i + b^\mu$$

$$ds^2 = dx^T dx = (A^T A)_{\mu\nu} d\xi^\mu d\xi^\nu$$
$$\implies g_{\mu\nu} = (A^T A)_{\mu\nu} = \vec{e}_\mu \cdot \vec{e}_\nu$$

d(d+1)/2 Poincare plus d(d+1)/2 Metric

- All simplexes are affine equivalent.





In a simplex $\vec{X} = \vec{x}_i \xi_i + \xi_0 \vec{x}_0$ with $i = 1, .., d$ and $\xi_0 = 1 - \sum_i \xi_i$

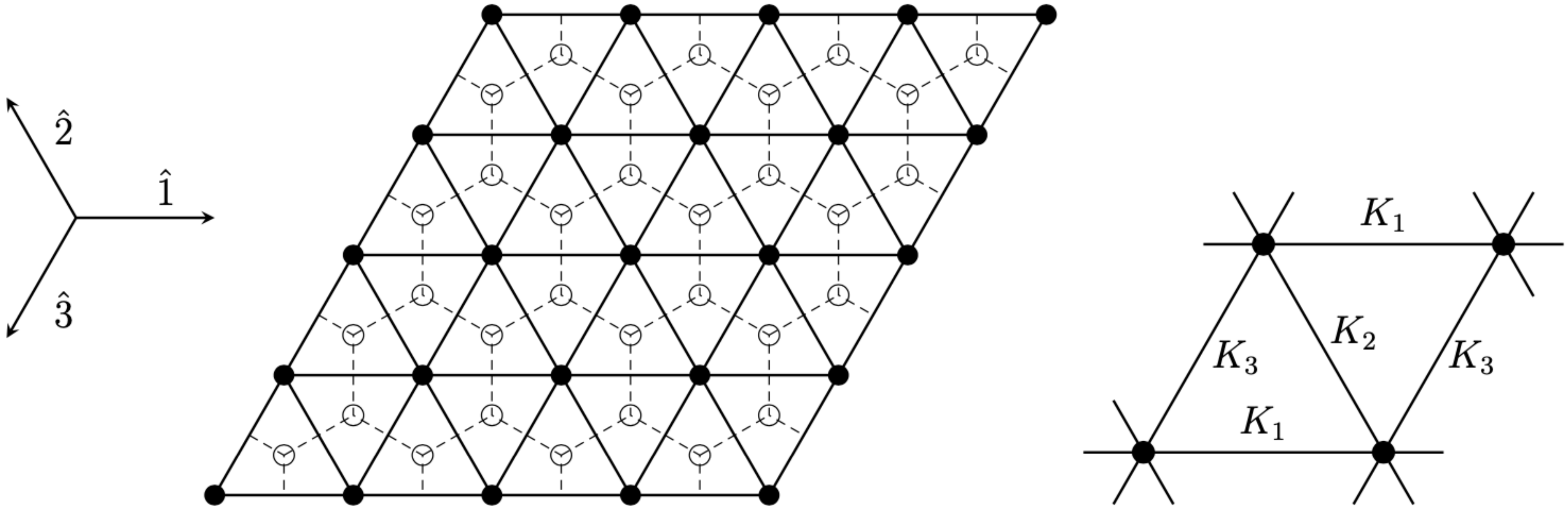
Affine Parameters:

2d Affine transformation takes circle to ellipse:

$$\langle\phi(x,y)\phi(0)\rangle = \frac{1}{(x^2+y^2)^{\Delta_\phi}} \leftrightarrow \frac{1}{(ax^2+bxy+cy^2)^{\Delta_\phi}}$$

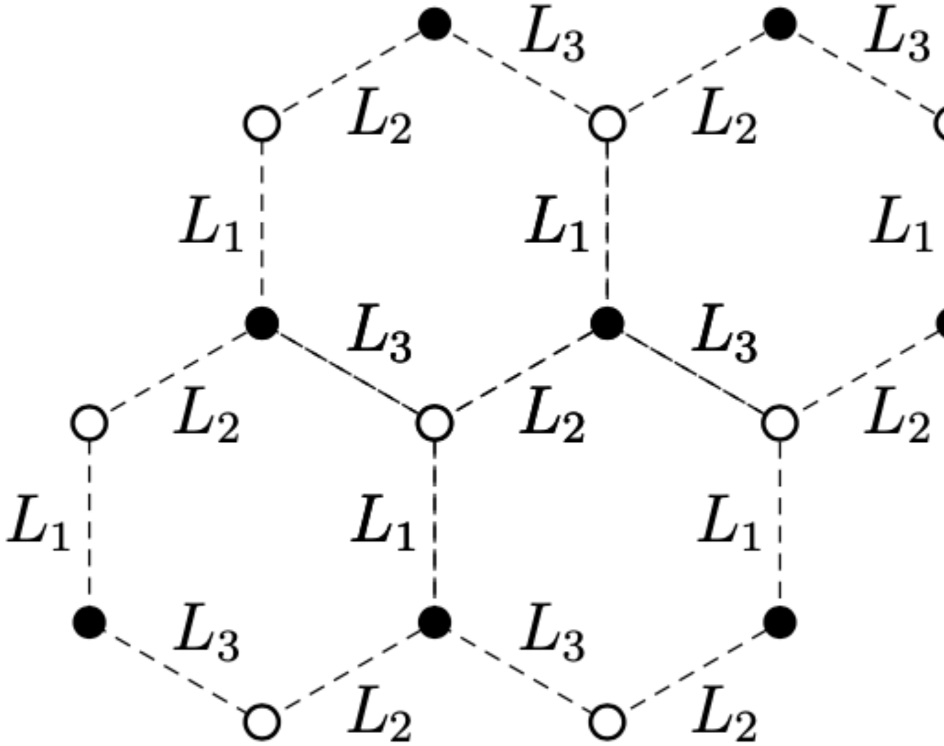
- d = 2 Poincare 1 rotation 2 translation
- New Affine plus 1 major/minor + 1 orientation + 1 scaling
- General Poincare d(d+1)/2 plus d(d+1)/2 the number of edge in d-simplex - local metric

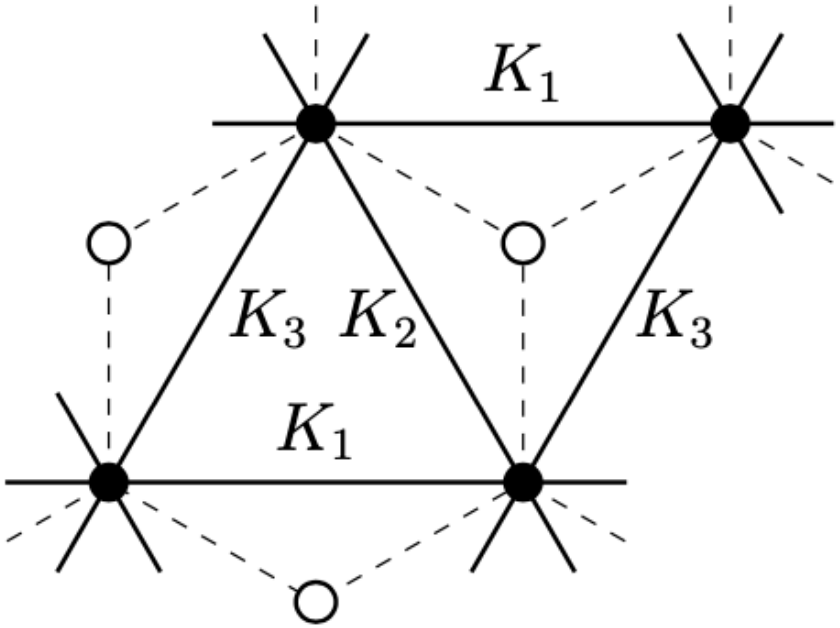
BUT lattice Ising (or Phi 4th) on "affine graphs" has only coupling constants with NO edge length given!



$$Z^{\triangle} = \sum_{s_n=\pm 1} e^{\,K_1 s_n s_{n+\hat{1}} + K_2 s_n s_{n+\hat{2}} + K_3 s_n s_{n+\hat{3}}}\ ,$$

Step I : Star Triangle ID: Hex to Triangle Map





$$h\sinh(2K_1)\sinh(2L_1) = h\sinh(2K_2)\sinh(2L_2) = h\sinh(2K_3)\sinh(2L_3) = 1$$

$$h(K_1, K_2, K_3) = \frac{(1-v_1^2)(1-v_2^2)(1-v_3^2)}{4\sqrt{(1+v_1v_2v_3)(v_1+v_2v_3)(v_2+v_3v_1)(v_3+v_1v_2)}} \qquad \text{with } v_i = \tanh(K_i)$$

Flat Space Affine Ising vs free phi

- Free (FEM) scalar CFT.

$$S_{\text{free}} = \frac{1}{2}\sum_n [K_1(\phi_n - \phi_{n+\hat{1}})^2 + K_2(\phi_n - \phi_{n+\hat{2}})^2 + K_3(\phi_n - \phi_{n+\hat{3}})^2]$$

$$2K_1 = \ell_1^*/\ell_1 \quad , \quad 2K_2 = \ell_2^*/\ell_2 \quad , \quad 2K_3 = \ell_3^*/\ell_3 \ .$$

- Critical Ising

$$\sinh(2K_1) = \ell_1^*/\ell_1 \quad , \quad \sinh(2K_2) = \ell_2^*/\ell_2 \quad , \quad \sinh(2K_3) = \ell_3^*/\ell_3$$

$$p_1 p_2 + p_2 p_3 + p_3 p_1 = 1 \quad \text{with} \quad p_i = \exp(-2K_i)$$

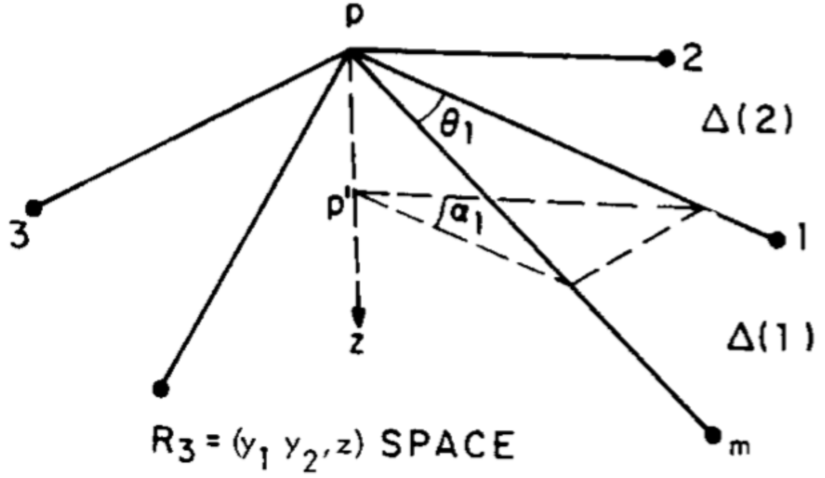
- 1960 Regge Calculus "General Relativity without Coordinates"

FINDING CO-ORDINATES ON THE QUATUM SPHERE

- Need smooth Regge affine between neighboring tangent planes
- The delta function deficit angle at points obey

$$\epsilon_p = \frac{A_p^*}{R^2}[1 + O(\ell^2/R^2)]$$

- Smooth Ricci curvature



- 1984 G. Feinberg,, R. Friedber, T.D. Lee and H C. Ren, "Lattice Gravity near the Continuum Limit"

- This gives local critical scaling and forces the EM trace to zero

$$\delta g^{\mu\nu} T_{\mu\nu} \implies Tr[T] = 0$$

- Second need the affine shape parameter give rotational co-ordinate for correlation function
- Operator description of continnum CFT

$$\psi^* \sigma_\mu \partial_\mu \psi + m^2 \epsilon(x) + h(x)\sigma(x) + g^{\mu\nu} T_{\mu\nu}$$

Free Majorana CFT

mass deformed primary

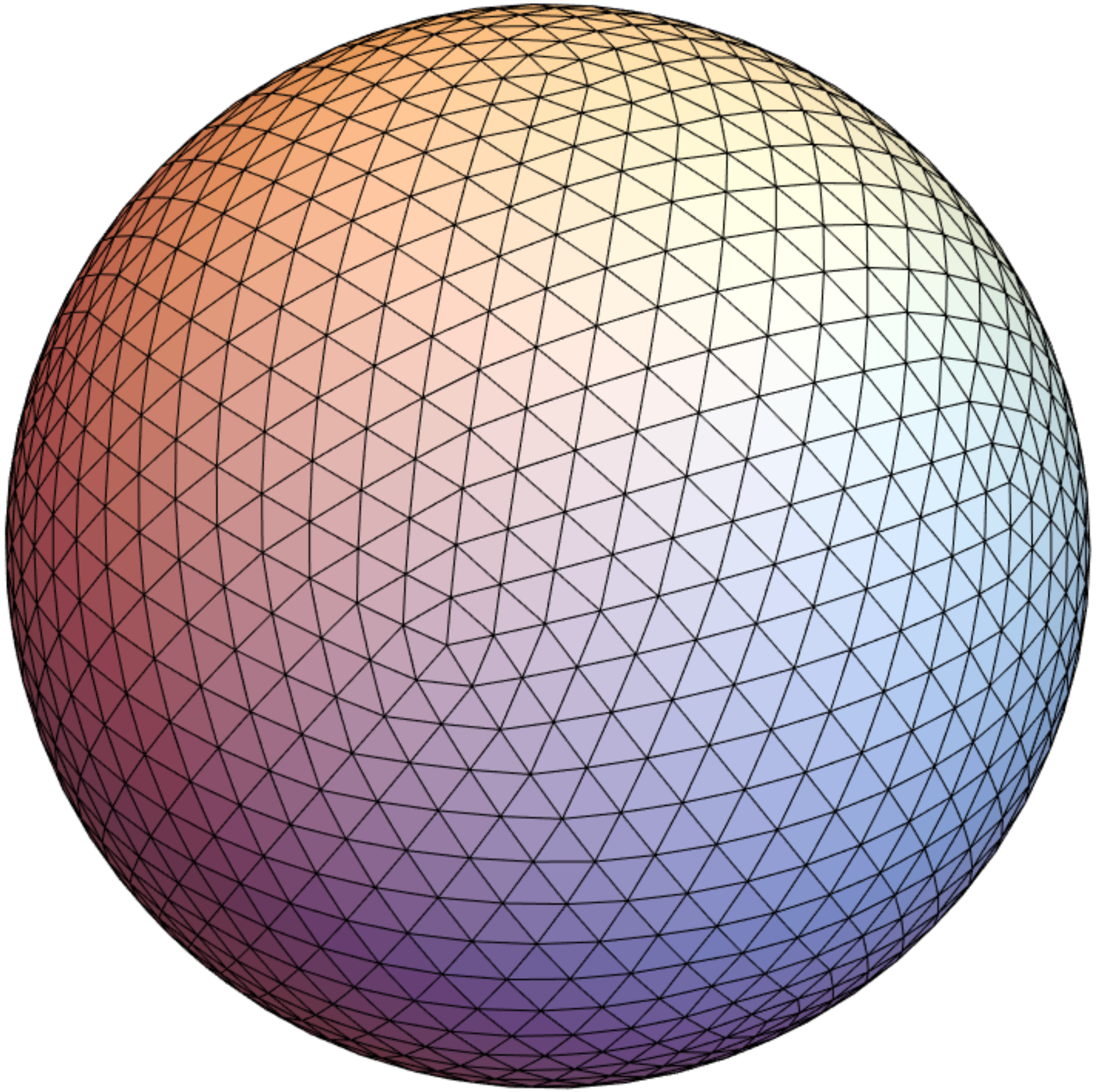
Z2 breaking primary

Marginal operator defines metric

Back to Putting critical 2d Ising on the sphere

- Now set all the circumradii equal to converge to a differential affine tangent planes.

- Give we believe the Exact Ising CFT in the continnum limit.



CFT Operator Algebra

$$S = \frac{1}{2}\int_{\mathcal{M}} d^d x \sqrt{g(x)}\left[g^{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x) + m^2\phi^2(x) + \lambda\phi^4(x)\right]$$

$$\psi^*\sigma_\mu\partial_\mu\psi + m^2\epsilon(x) + h(x)\sigma(x) + g^{\mu\nu}T_{\mu\nu}$$

Free Majorona CFT

mass deformed primary

Z2 breaking primary

Co-ordinate transform

Quantum Geometry* Tug of War

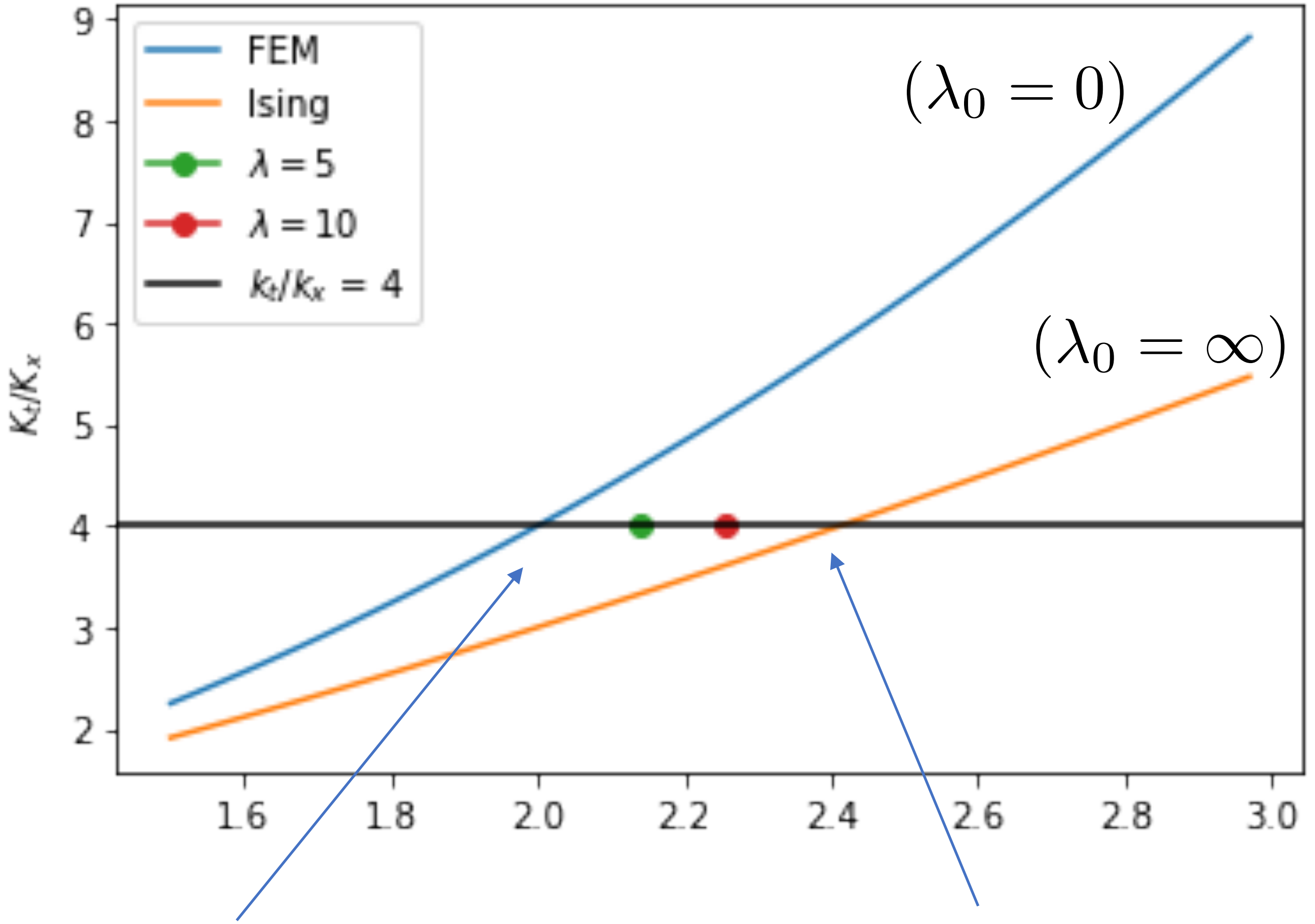
GRAVITY

* REGGE & FEM SOLVER CLASSICAL TUG OF WAR!

QUANTUM FIELD GEOMETRY

Using Radial Quantization to define Affine dependence on λ_0

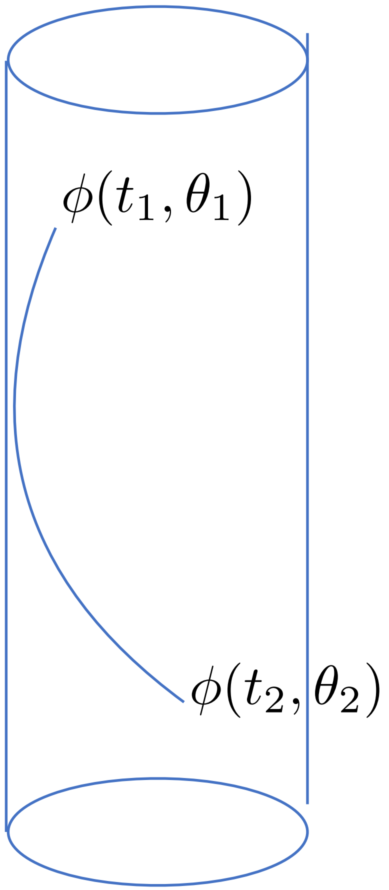
$$\langle\, \phi_2 \phi_1 \rangle \sim cos(l\theta_{21})e^{-l_t/l_x t_{21}(\Delta_\sigma + l)}$$



ℓ_x/ℓ_t

2.0 Fem

2.4126 Ising



$$\ell_x/\ell_t = \sqrt{K_t/K_x} \qquad \ell_x/\ell_t = \sqrt{\text{arcsinh}(K_x/K_t)}$$

$\mathbb{R} \times \mathbb{S}_1$

Conclusion — Need more Tests and Applications

- Find the EM Tensor and get geometry
  - Next Minimal CFT: Tricritical Ising Model (TIM) vs phi^6
  - 3d Ising on R x S2 — see next talk!
  - 3d Ising on S3
  - 3d QED with scalars or fermions
- Algorithm Development
  - “Machine Learning” of parameter geometry in flat Affine space.
  - Apply to A. Karsch coefficient for finite T QCD.
  - Simplicial Lattice extension of Grid (help Peter!)

Extra Slides

Thanks to my collaborators

- George T. Fleming, Yale University/FNAL
- Anna-Marie Gluck, Yale/Heidelberg University

- Venkitesh Ayyar, Boston University
- Evan Owen, Boston University/BNL
- Cameron Cogburn, Boston University/RPI
- Timothy G. Raben, Michigan State University

- Chung-I Tan, Brown University

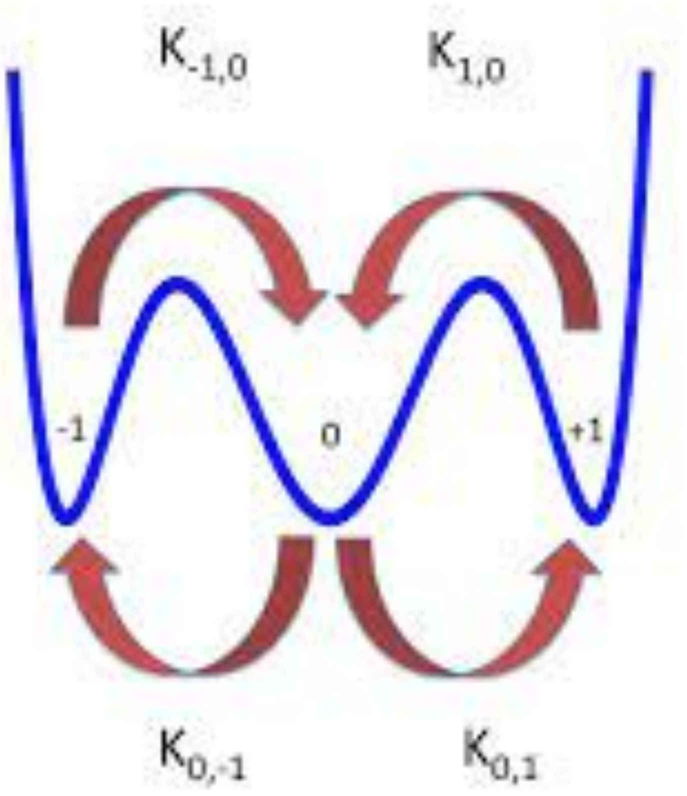
- Nobuyuki Matsumoto, BNL/BostonUniversity

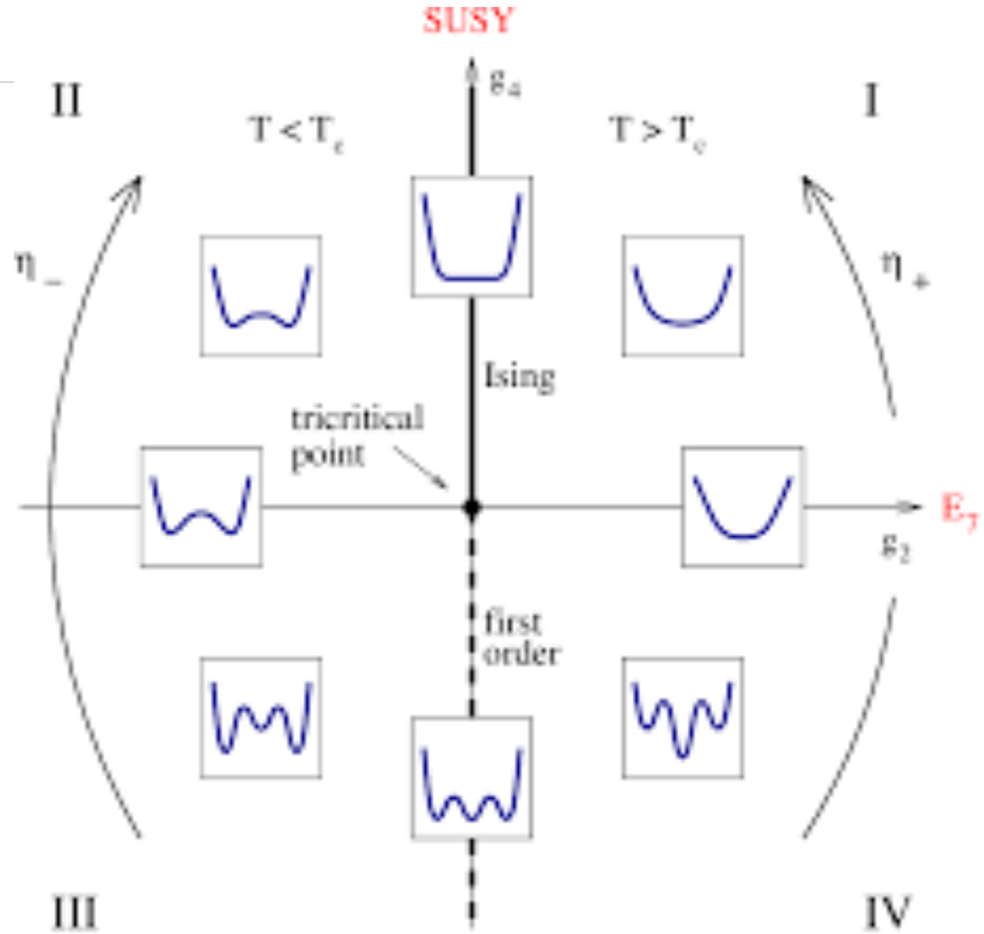
Making Steady Progress Going Backward!

- 2013: Lattice Radial Quantized: 3D Ising   (R x S2)
- 2017: Lattice Dirac on S2 Simplicial Riemann Manifold (S2:Free CFT)
- 2018: phi^4 test of 2-d Ising CFT on S2 (S2)
- 2019: Lattice Setup for Quantum Field Theory in AdS2
- 2021: Radial Lattice Quantization of 3D phi^4 Field Theory (R x S2)
- 2022: Lattice AdS3 for Scalar Field Theory (w. C. Cogburn, E. Owen)
- 2022: Ising Model on the Affine Plane (w. E. Owen) (2D Torus!)

See References in Back up Slides

Even more “super fun”—3 wells — tricritical Ising!

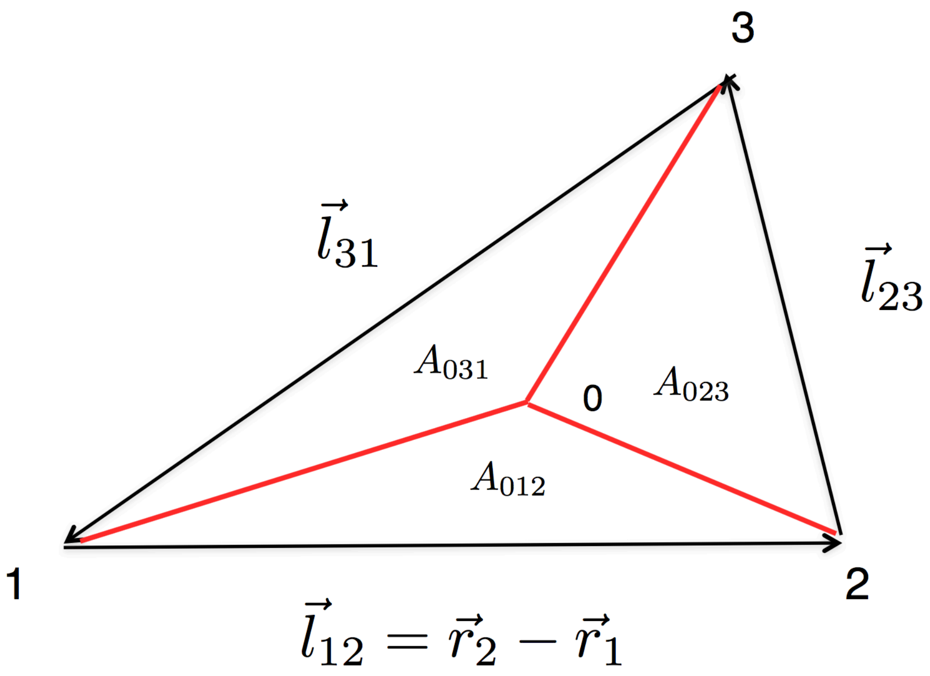


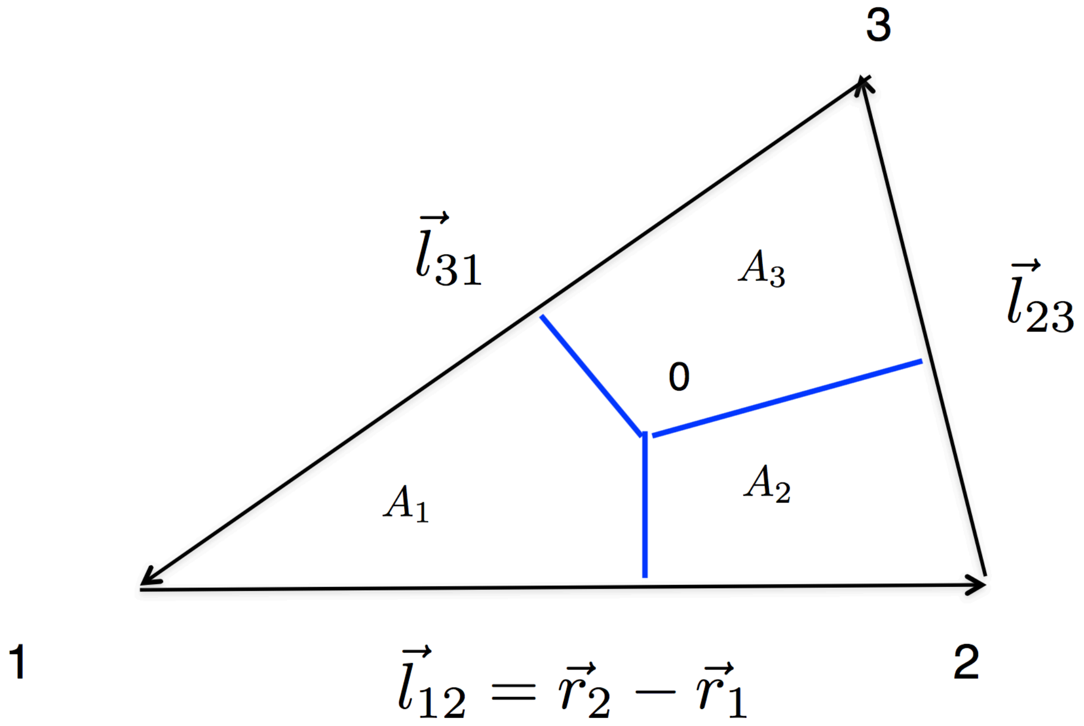


$$H_{TIM} = -K \sum_{\langle i,j \rangle} s_i s_j - \Delta \sum_i (1 - s_i s_i)$$

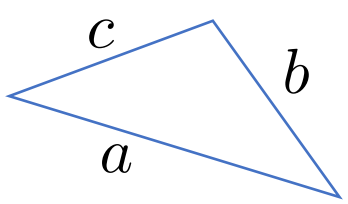
s = -1, 0, -1

Regge Calculus and FEM use simplex and the cricumcentric duals: The result necessary classical discrete Regge GR and Discrete exterior calculus. for FEM.



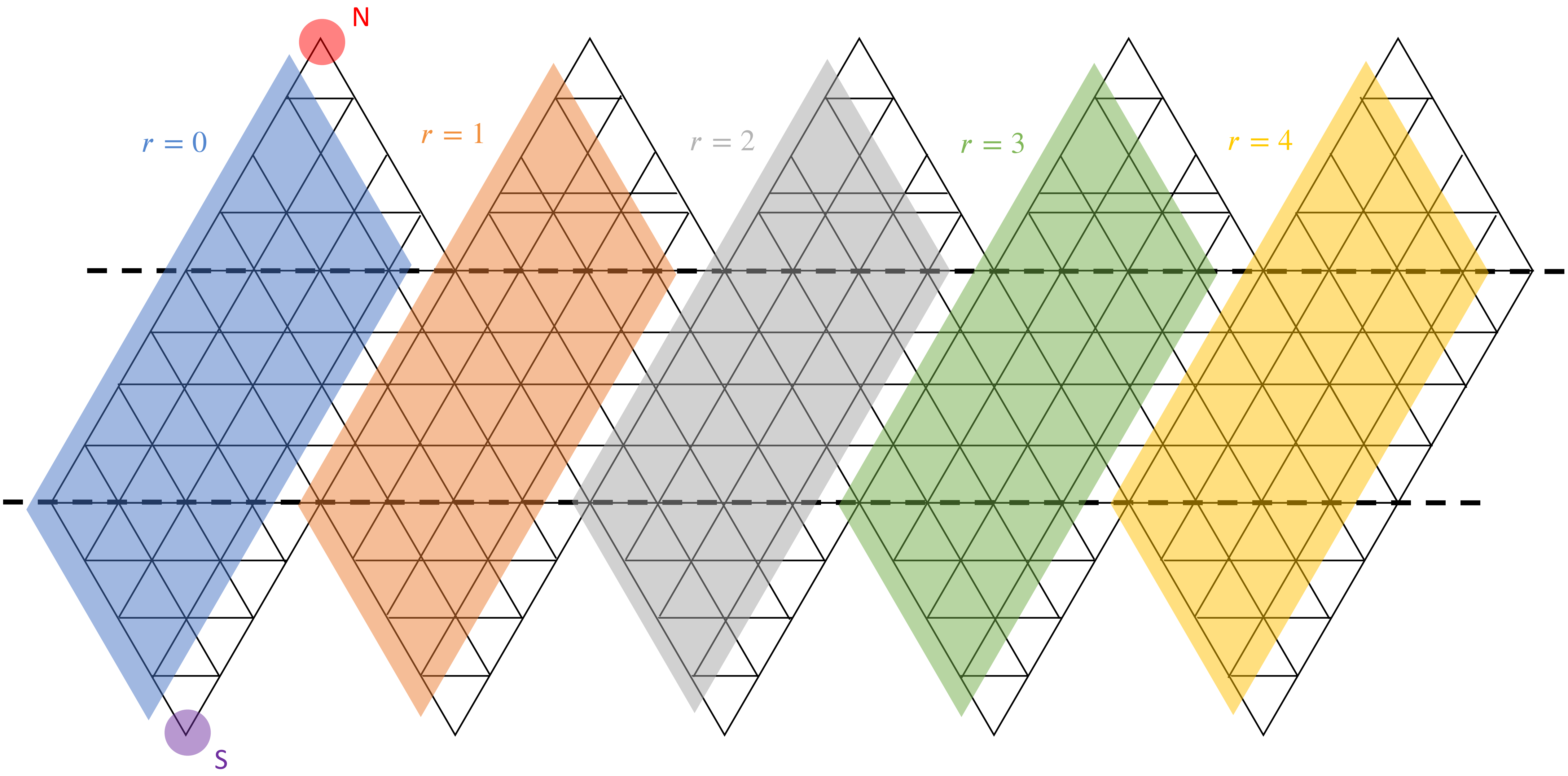


$$A_{\triangle(a,b,c)} = \sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}/4$$



$$R_{radius} = abc/(4A_{\triangle(a,b,c)})$$

Aiming to implement MPI, we divide the lattice into 5 patches + two pole points:

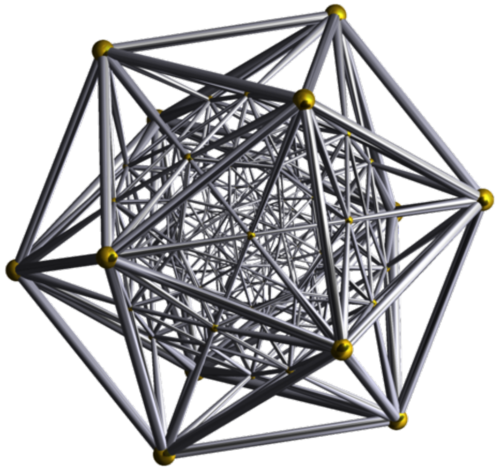


Uncolored points have an identical point somewhere else.


THE THEORIST EXPERIMENTAL LAB
FRONTIER
OAK RIDGE
National Laboratory
U.S. DEPARTMENT OF
ENERGY
Hewlett Packard
Enterprise
AMD
INDIRECT WASTE

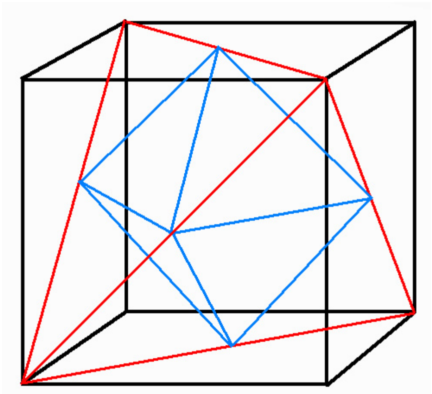
3 Spheres and 4D Radial Simplicial Lattices

$$\mathbb{S}^3 \quad \Longrightarrow \quad \mathbb{R} \times \mathbb{S}^3$$



Aristotle' s 2% Error!

$$(2\pi - 5ArcCos[1/3])/(2\pi) = 0.0204336$$



Fast Code Domains of
Regular 3D Grids on Refinement

600 cell: "Square of the icosahedron" –Symmetries 1440= 120 * 120 the 120 copies of icosahedron

$$O(4) \sim SU(2) \times SU(2)$$

The full symmetry group of the 600-cell is the Weyl group of H_4. This is a group of order 14400. It consists of 7200 rotations and 7200 rotation-reflections. The rotations form an invariant subgroup of the full symmetry group.

SPHERES AND CYLINDERS ARE NICE*

* MAXIMALLY SYMMETRIC (aka Einstein Metric) SPACES

- Conformal Field Theories are more easily studied on Sphere, Cylinders (Radial Quantization) and Hyperbolic Spaces (Gauge/Gravity Duality)

$\mathbb{S}^d$ $\quad$ $\mathbb{R}\times\mathbb{S}^{d-1}$ $\quad$ $\mathbb{A}d\mathbb{S}^{d+1}$

$\mathbb{R}^d \to \mathbb{S}^d$

$$ds^2_{flat} = \sum_{\mu=1}^{d} dx^\mu dx^\mu = e^{2\sigma(x)} d\Omega_d^2 \xrightarrow{Weyl} d\Omega_d^2 \, .$$

$\mathbb{R}^d \to \mathbb{R}\times\mathbb{S}^{d-1}$

$$ds^2_{flat} = \sum_{\mu=1}^{d} dx^\mu dx^\mu = e^{2\tau}(d\tau^2 + d\Omega_d^2) \xrightarrow{Weyl} (d\tau^2 + d\Omega_d^2) \, .$$

$\mathbb{R}^{d+1} \to \mathbb{A}d\mathbb{S}^{d+1}$

$$ds^2_{flat} = \sum_{\mu=1}^{d+1} dx^\mu dx^\mu \xrightarrow{Weyl} z^{-2}(dz^2 + d\vec{x}\cdot d\vec{x})$$

SIMPLICIAL EXTERIOR CALCULUS DOES ALMOST ALL FOR CLASSICAL

J = 0 $$S_{scalar} = \frac{1}{2} \sum_{\langle i,j \rangle} \frac{V_{ij}}{l_{ij}^2} (\phi_i - \phi_j)^2 \,,$$ $$l_{ij}^2 = |\sigma_1(ij)|^2$$

J = 1/2 $$S_{Wilson} = \frac{1}{2} \sum_{\langle i,j \rangle} \frac{V_{ij}}{l_{ij}} (\bar\psi_i \hat{e}_a^{j(i)} \gamma^a \Omega_{ij} \psi_j - \bar\psi_j \Omega_{ji} \hat{e}_a^{i(j)} \gamma^a \psi_i)$$

J = 1 $$S_{gauge} = \frac{1}{2g^2 N_c} \sum_{\triangle_{ijk}} \frac{V_{ijk}}{A_{ijk}^2} Tr[2 - U_{\triangle_{ijk}} - U^\dagger_{\triangle_{ijk}}]$$

FFdual $$\epsilon^{ijkl} Tr[U_{\triangle_{0ij}} U_{\triangle_{0kl}}] \simeq V_{ijkl} \epsilon^{\mu\nu\rho\sigma} Tr[F_{\mu\nu}(0) F_{\rho\sigma}(0)]$$

$$U_{\triangle_{ijk}} = U_{ij} U_{jk} U_{ki} \quad A_{ijk} = |\sigma_2(ijk)| \quad V_{ijk} = |\sigma_2(ijk) \wedge \sigma_2^*(ijk)|$$

$$U_{0ij} = U_{0i} U_{ij} U_{j0} \quad , \quad U^\dagger_{0ij} = U_{0j} U_{ji} U_{i0} \quad V_{ij} = |\sigma_1(ij) \wedge \sigma_1^*(ij)|$$

But Dirac needs Spin Connection (Kahler Dirac doesn't)

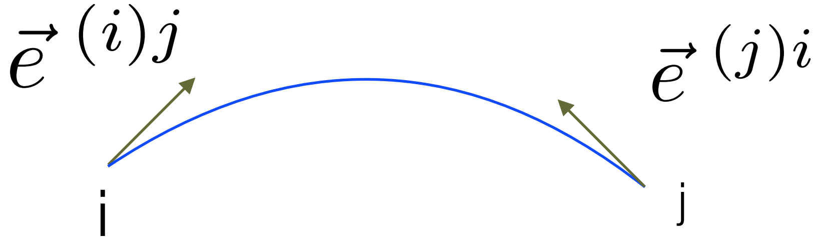
$$S = \frac{1}{2}\int d^D x \sqrt{g}\bar{\psi}[\mathbf{e}^{\mu}(\partial_\mu - \frac{i}{4}\boldsymbol{\omega}_\mu(x)) + m]\psi(x)$$

$$\mathbf{e}^{\mu}(x) \equiv e^{\mu}_{a}(x)\gamma^a$$

Verbein & Spin connection*

$$\boldsymbol{\omega}_\mu(x) \equiv \omega^{ab}_{\mu}(x)\sigma_{ab} \quad , \quad \sigma_{ab} = i[\gamma_a, \gamma_a]/2$$

$$S_{naive} = \frac{1}{2}\sum_{\langle i,j\rangle}\frac{V_{ij}}{l_{ij}}[\bar{\psi}_i \vec{e}^{\,(i)j}\cdot\vec{\gamma}\Omega_{ij}\psi_j - \bar{\psi}_j\Omega_{ji}\vec{e}^{\,(i)j}\cdot\vec{\gamma}\psi_i] + \frac{1}{2}mV_i\bar{\psi}_i\psi_i$$

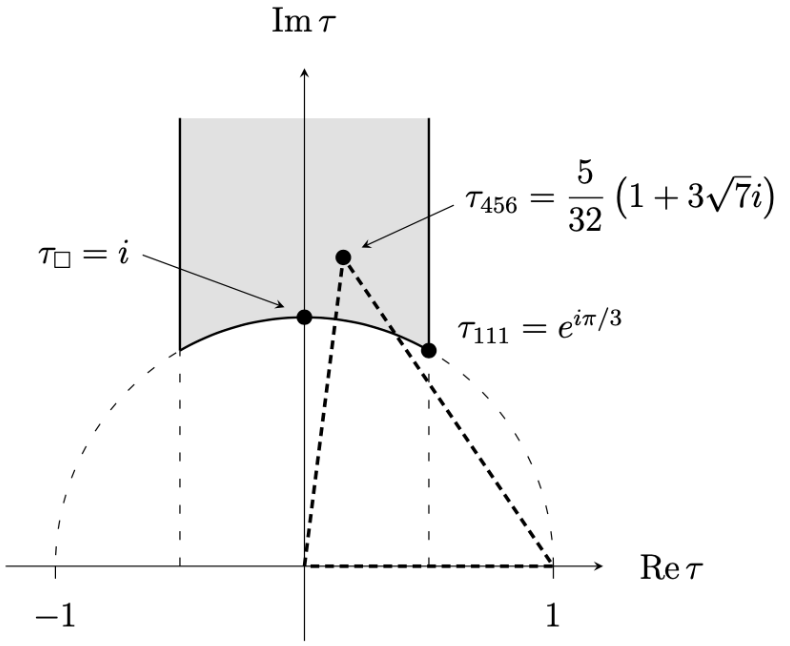


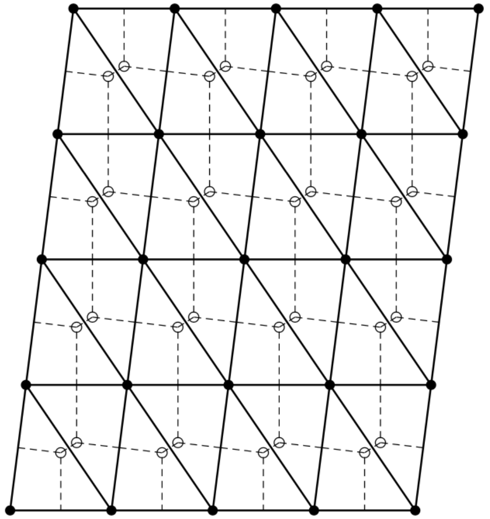
Simplicial Tetrad Hypothesis

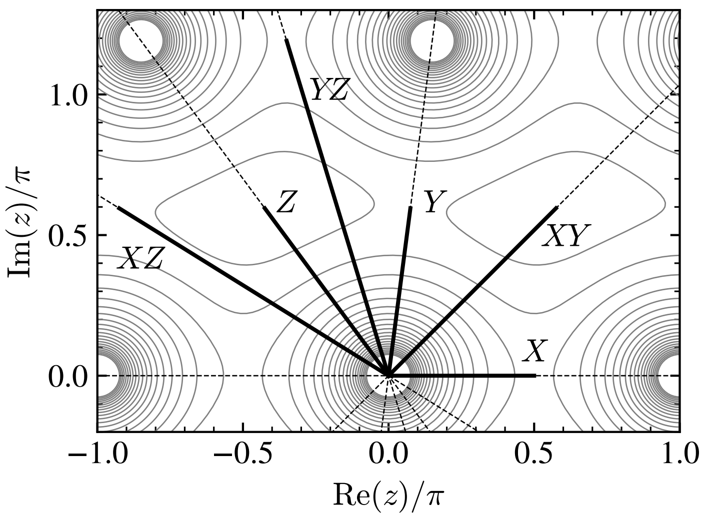
$$e^{(i)j}_{a}\gamma^a\Omega_{ij} + \Omega_{ij}e^{(j)i}_{a}\gamma^a = 0$$

See next talk + 3 by Shoto Aoki

Calculation Modular dependent on the torus





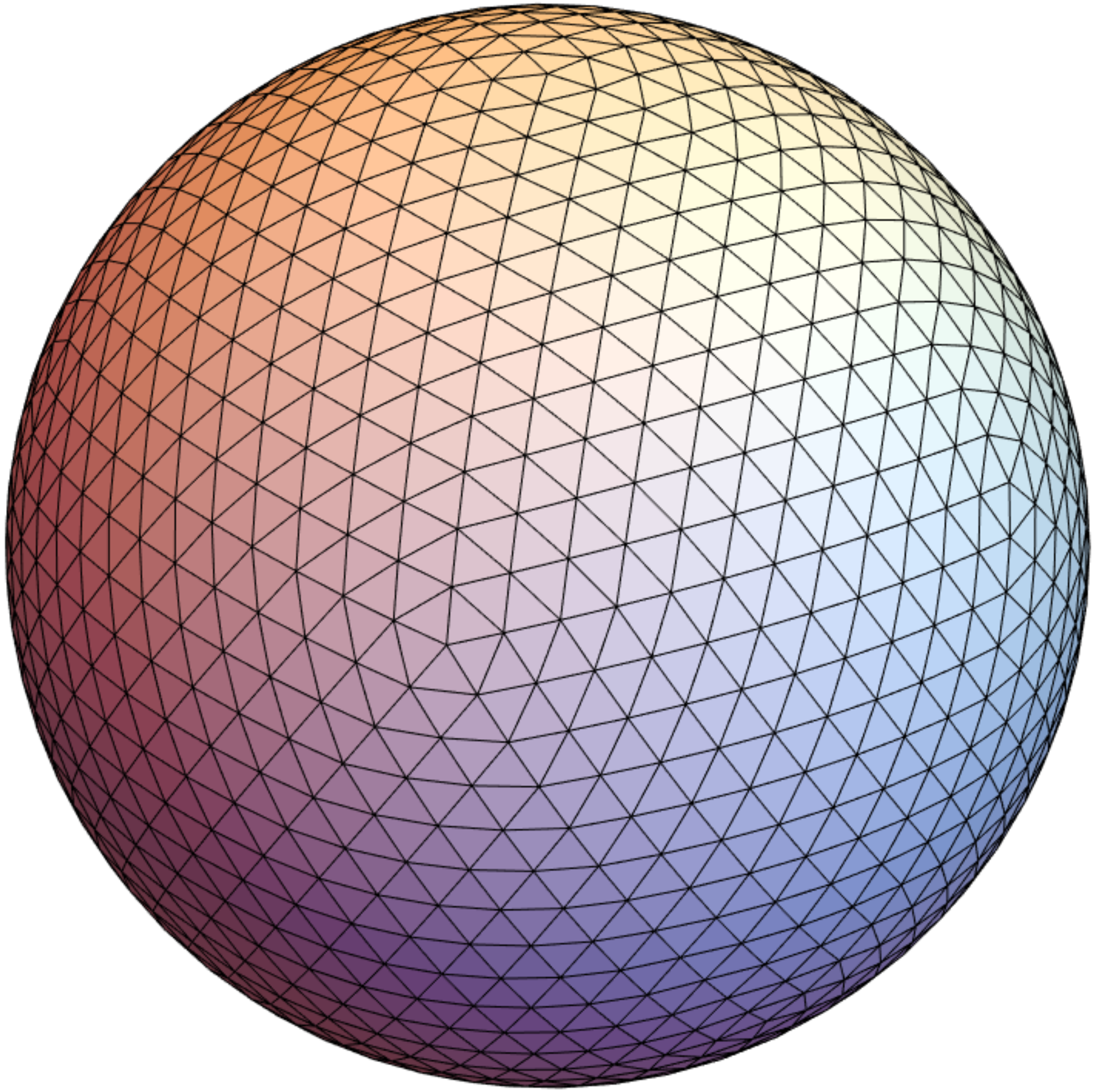


$$\langle \sigma(0)\sigma(z)\rangle = \left|\frac{\vartheta_1'(0|\tau)}{\vartheta_1(z|\tau)}\right|^{1/4} \frac{\sum_{\nu=1}^{4} |\vartheta_\nu(z/2|\tau)|}{\sum_{\nu=2}^{4} |\vartheta_\nu(0|\tau)|}$$

Back to Putting critical 2d Ising on the sphere

- Fix the UV cut off dual areas equal Result is uniform

- Result is critical Ising mode But not a good co-oridnate system (I blelieve!) curvature density (a uniform UV cut-off



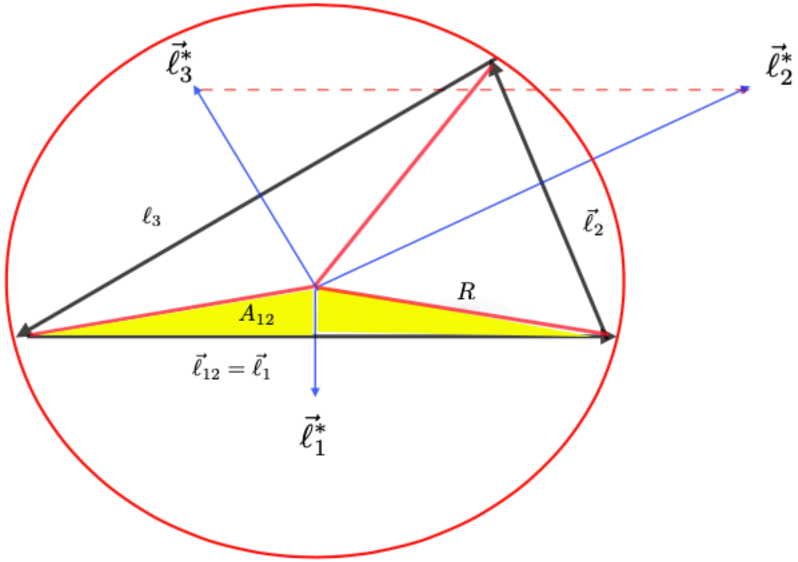
1985: Cardy’s Radial Quantization Challenge

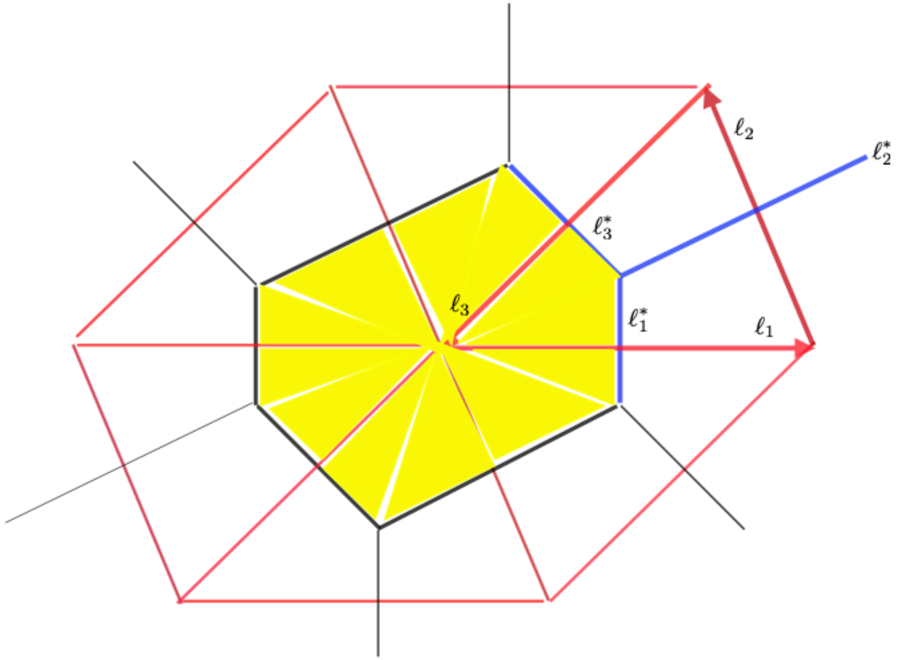
“It would therefore be very useful to generalize this result (in 2D) to dimensionality D > 2” “Unfortunately the result appears to be difficult to utilize for numerical work”

Last Sentence in 3 page article says

“Whether this will provide a useful numerical approach to critical exponents remains to be seen”

YES INDEED





$$S_\triangle = \frac{\ell_{23}^2 + \ell_{31}^2 - \ell_{12}^2}{8A_\triangle}(\phi_1 - \phi_2)^2 + (23) + (31) = \frac{\ell_{12}^*}{4\ell_{12}}(\phi_1 - \phi_2)^2 + (23) + (31)$$

PIECE WISE LINEAR FEM

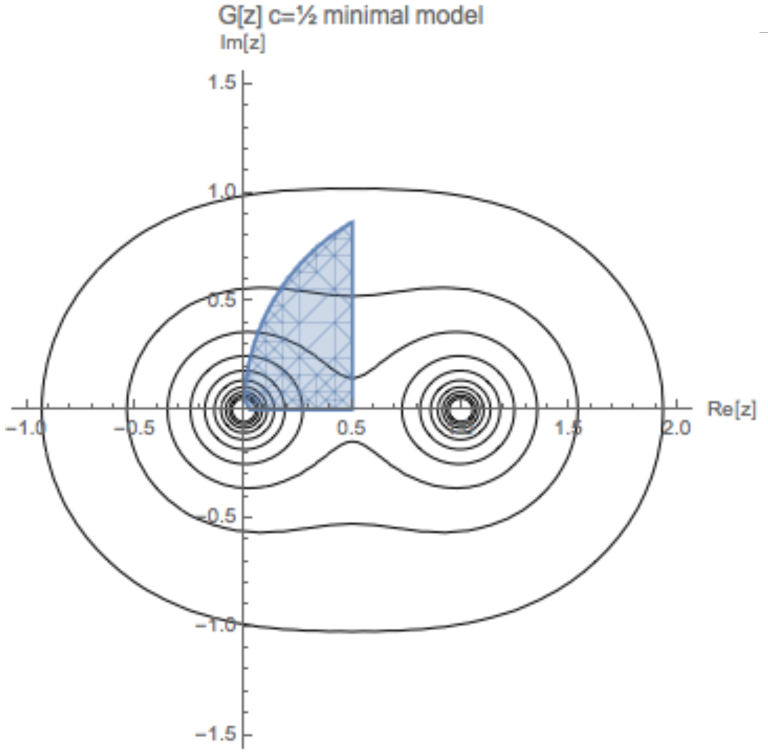
(Negative sign is Not problem in spectrum)

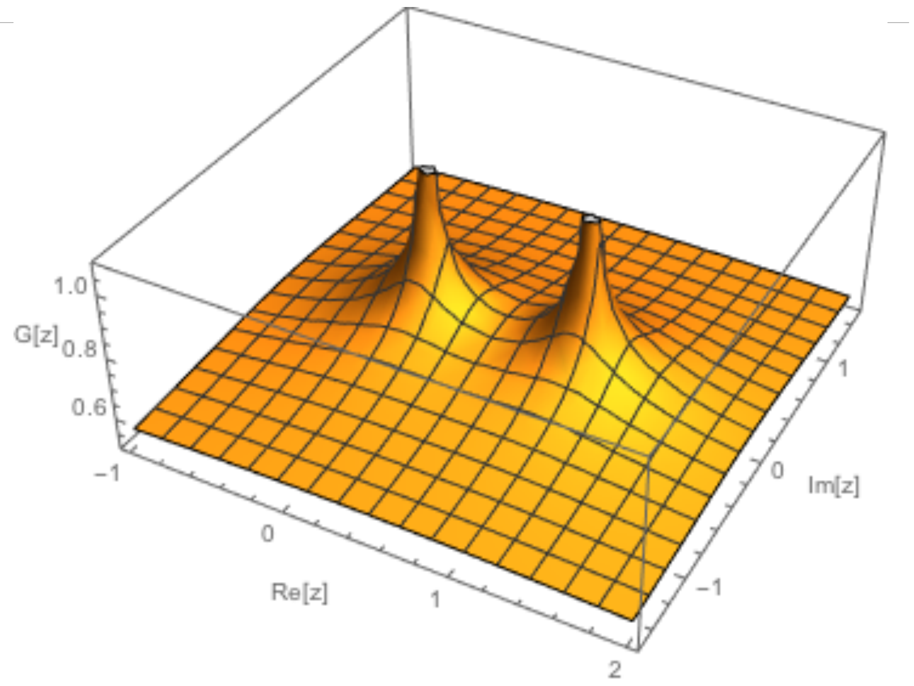
Discrete Exterior Calculus (DE)

$$\langle\, \sigma_n | d\omega \rangle = \langle\, \partial\sigma_n | \omega \rangle$$

$$*d*d\phi_i = *\frac{1}{|\sigma_0^*(i)|}\int_{\sigma_0^*} d[*(\phi_i - \phi_j)/l_{ij}] = \frac{1}{\sqrt{g_i}} \sum_{j \in \langle\, i,j \rangle} \frac{V_{ij}}{l_{ij}} \frac{\phi_i - \phi_j}{l_{ij}}$$

OPE Expansion: $\phi \times \phi = \mathbf{1} + \phi^2$ or $\sigma \times \sigma = \mathbf{1} + \epsilon$

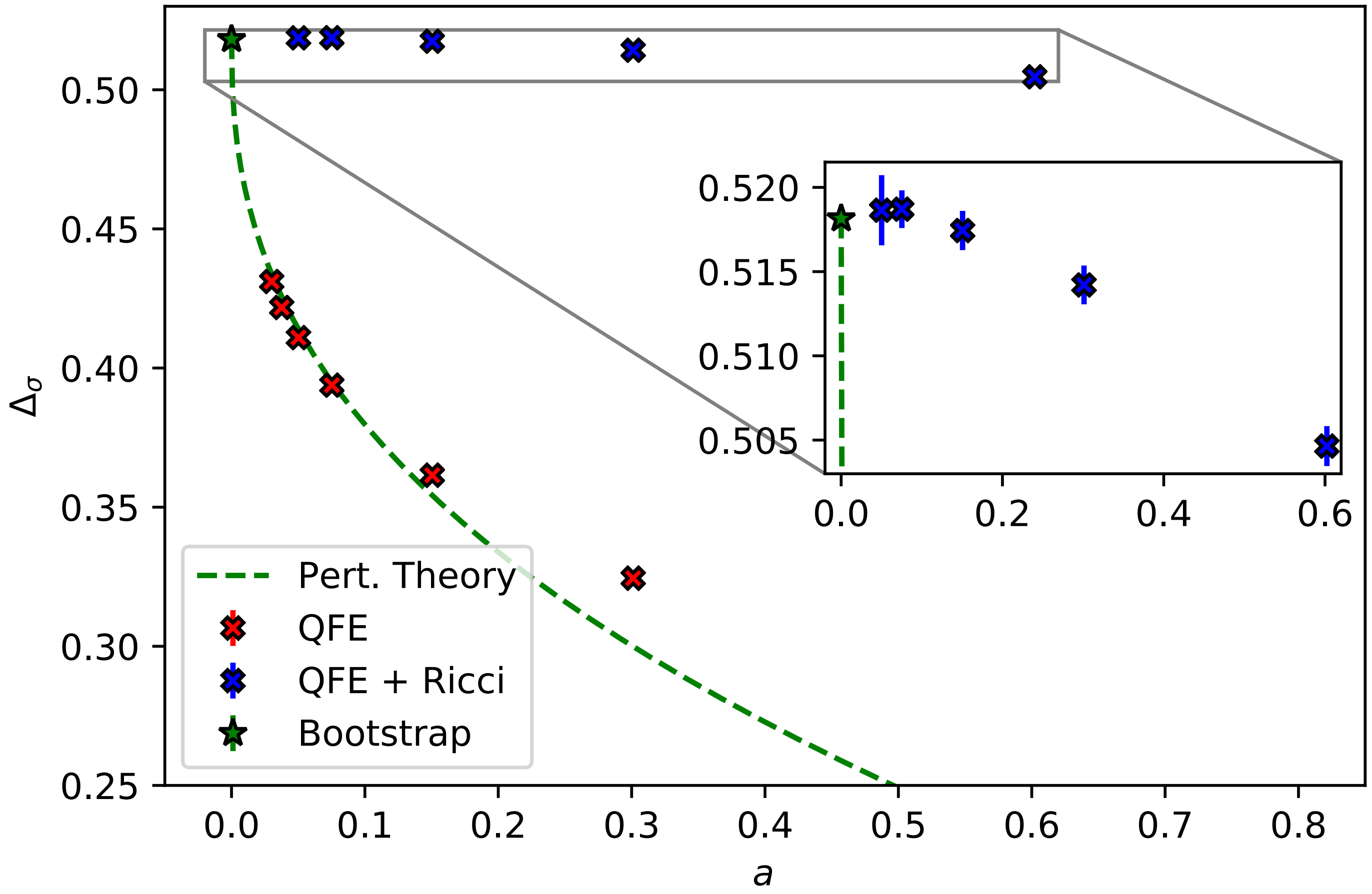




$$G_s(r,\theta) \propto 1 + \lambda_\epsilon^2\, g_{\epsilon,0}(r,\theta) + \lambda_T^2\, g_{T,2}(r,\theta)$$

$$\lambda_T^2 = \frac{\Delta_\sigma^2 d^2 |z|^{d-2}}{C_T(d-1)^2} \to \frac{1}{16 C_T} \quad \text{for } d = 2\,, \qquad g_{T,2}(z) = -3\left(1 + \frac{1}{z}\left(1 - \frac{z}{2}\right)\log(1-z)\right) + \text{c.c.}$$

Lattice Test against very precise CFT Bootstrap constraint



$$S_{FEM} = \frac{a_t}{2}\Big[\sum_{y\in\langle x,y\rangle} \frac{l^*_{xy}}{l_{xy}}(\phi_{t,x}-\phi_{t,y})^2 + \frac{\sqrt{g_x}}{4R^2}\phi^2_{t,x}$$

$$+\sqrt{g_x}\Big[\frac{(\phi_{t,x}-\phi_{t+1,x})^2}{a_t^2} + m^2\phi^2_{t,x} + \lambda\phi^4_{t,x}\Big]\Big]$$

Exact Ricci term
at lambda = 0

Need for Ricci Term
Improvement Scheme

Step II: Map Hexagonal Loop Expansion is easy to map to free Ising to Free Wilson-Majorana Fermion*

$$Z_N^{\psi} = \prod_n \iint d\psi_n^1 d\psi_n^2 \, e^{-S[\bar{\psi},\psi]} = \prod_n \int d^2\psi_n e^{-\frac{1}{2}\sum_n \bar{\psi}_n \psi_n} \prod_{n,i} [1 + \kappa_i \bar{\psi}_n P(\hat{e}_i) \psi_{n+\hat{i}}]$$

$$S[\psi] = \frac{1}{2}\sum_n \bar{\psi}_n \psi_n - \frac{1}{2}\sum_{n,i} \kappa_i \bar{\psi}_n (1 + \hat{e}_i \cdot \vec{\sigma}) \psi_{n+\hat{i}} \; .$$

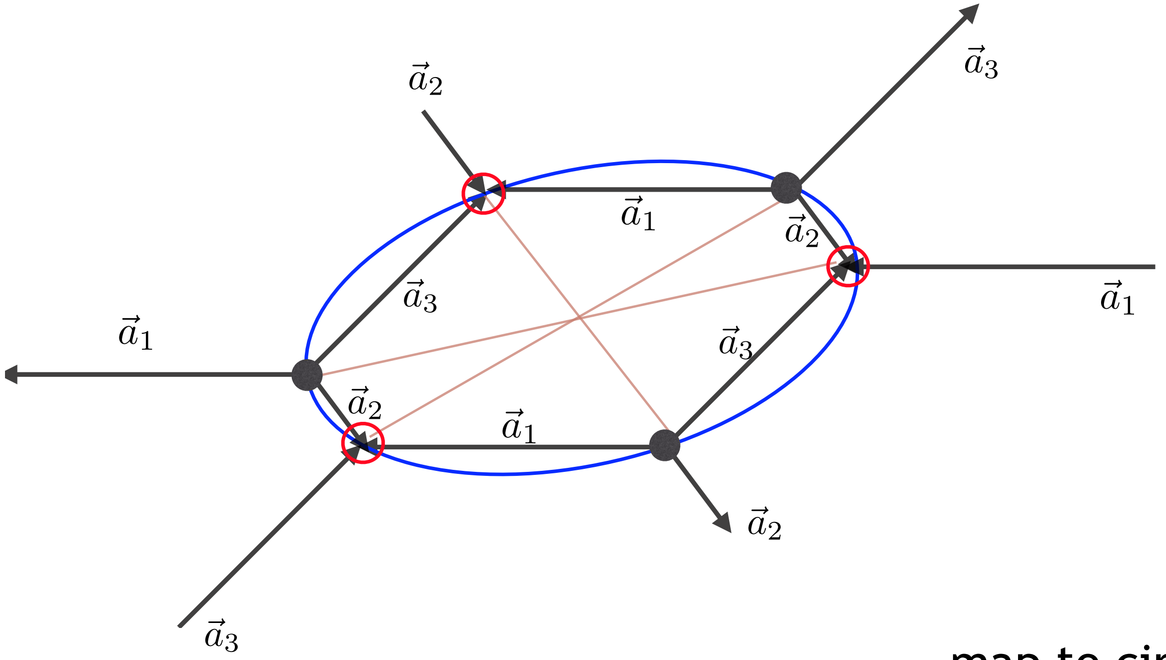
Horrible algebra (unless you are Baxter?)
but beautiful Geometry in spirit of Pascal's theorem**

***Generalizing very nice paper by Ulli Wolff.**

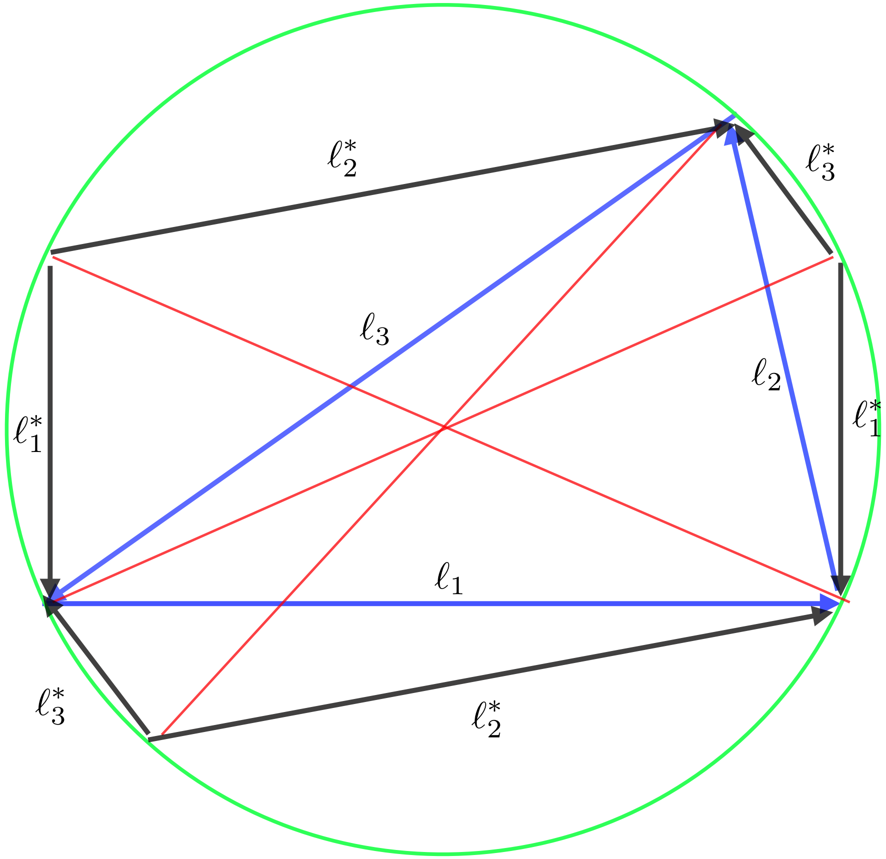
Ising model as Wilson-Majorana Fermions. Nucl. Phys. B, 955:115061, 2020.

**Blaise Pascal. Essay pour les conique (1640).

Elliptical Hexagon to a Circular Hexagon



map to circle ==>



Basic algebra of Projective Geometry going back to Pascal in 1640!

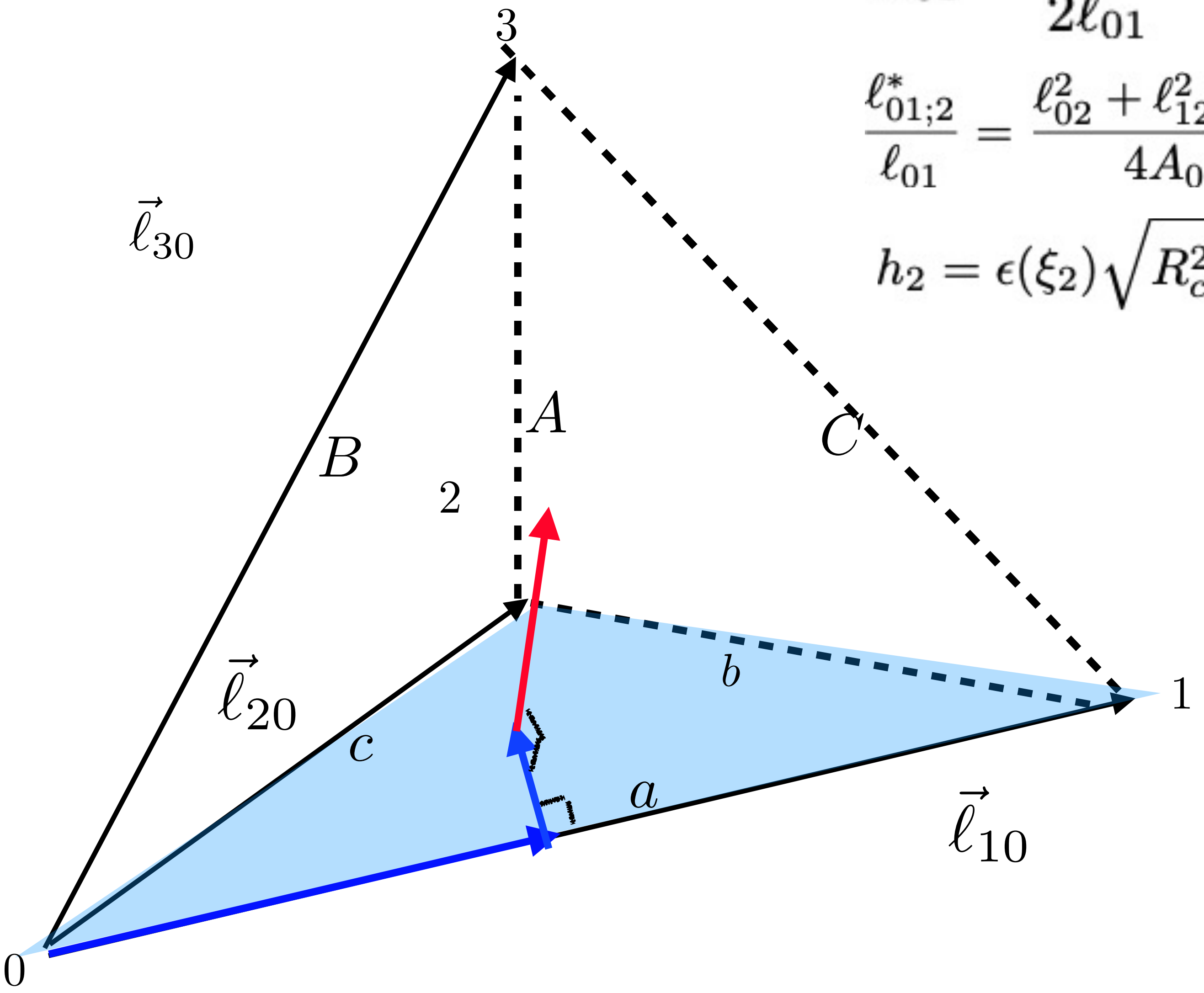
- Blaise Pascal. Essay pour les conique. (facsimile) Nieders¨achsiche Landesbibliothek, Gottfried Wilhelm Leibniz Bibliothek, 1640.

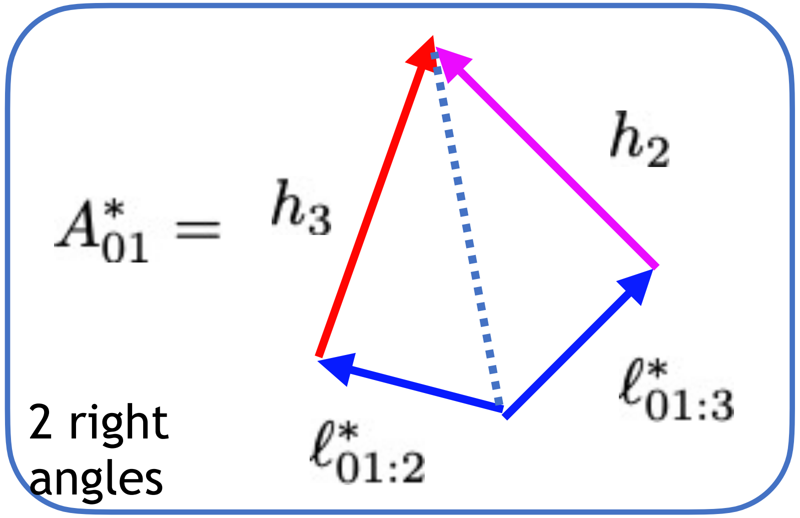
DEC contribution form the 01 edge of Tetrahedron

$$K_{01} = \frac{A_{01}^*}{2\ell_{01}} = \frac{h_3 \ell_{01:2}^* + h_2 \ell_{01:3}^*}{2\ell_{01}}$$

$$\frac{\ell_{01;2}^*}{\ell_{01}} = \frac{\ell_{02}^2 + \ell_{12}^2 - \ell_{01}^2}{4A_{012}} \quad , \quad \frac{\ell_{01;3}^*}{\ell_{01}} = \frac{\ell_{03}^2 + \ell_{13}^2 - \ell_{01}^2}{4A_{013}}$$

$$h_2 = \epsilon(\xi_2)\sqrt{R_{cc}^2 - R_{013}^2} \quad , \quad h_3 = \epsilon(\xi_3)\sqrt{R_{cc}^2 - R_{012}^2}$$



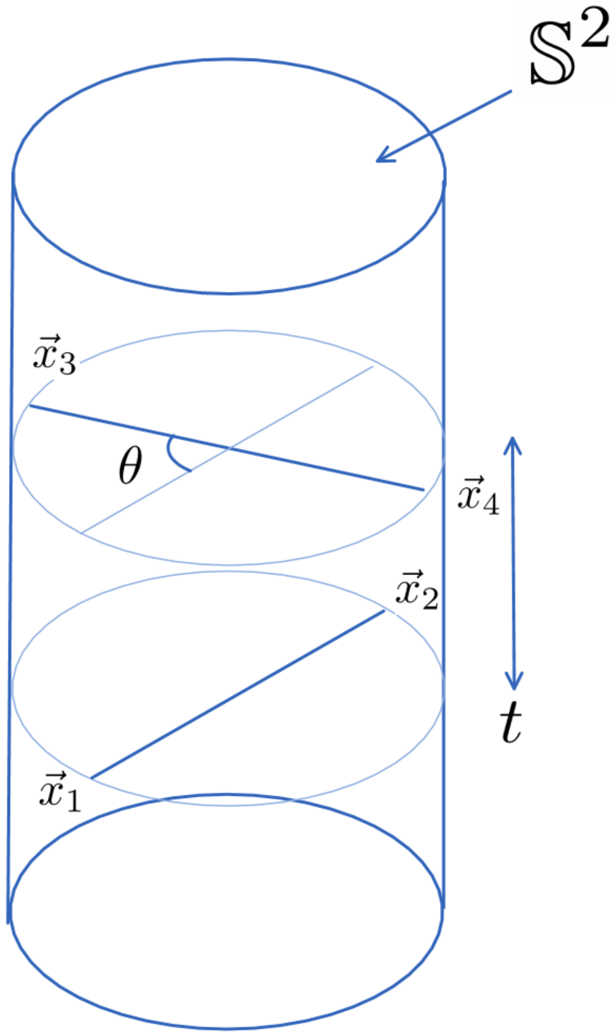


$$R_{cc} = A_{\triangle(aA,bB,cC)}/(6V_T)$$

Antipodal 4-point function on $\mathbb{R} \times \mathbb{S}^2 \ni (t, \vec{x})$

$$\frac{\langle\sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle}{\langle\sigma(x_1)\sigma(x_2)\rangle\langle\sigma(x_3)\sigma(x_4)\rangle} = 1 + \sum_{\mathcal{O}} f^2_{\sigma\sigma\mathcal{O}} G_{\mathcal{O}}(\Delta_{\mathcal{O}}; x_1, x_2, x_3, x_4)$$

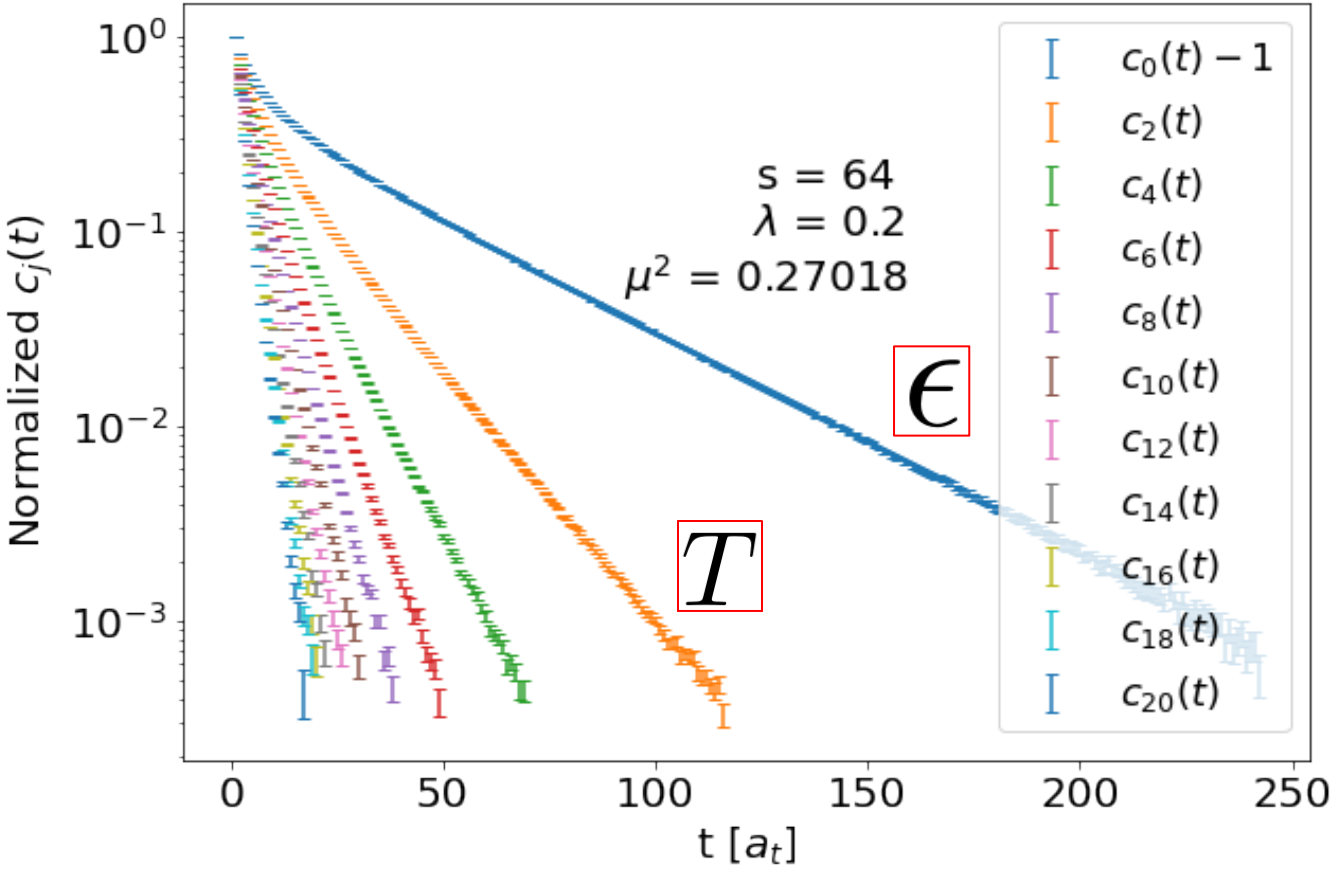
$$G_{\mathcal{O},l} = \sum_{n=0,2,4,\dots} \sum_j e^{-(\Delta_{\mathcal{O}}+n)t} B_{n,j}(\Delta_{\mathcal{O}}) P_j(\cos(\theta))$$



Numerical results

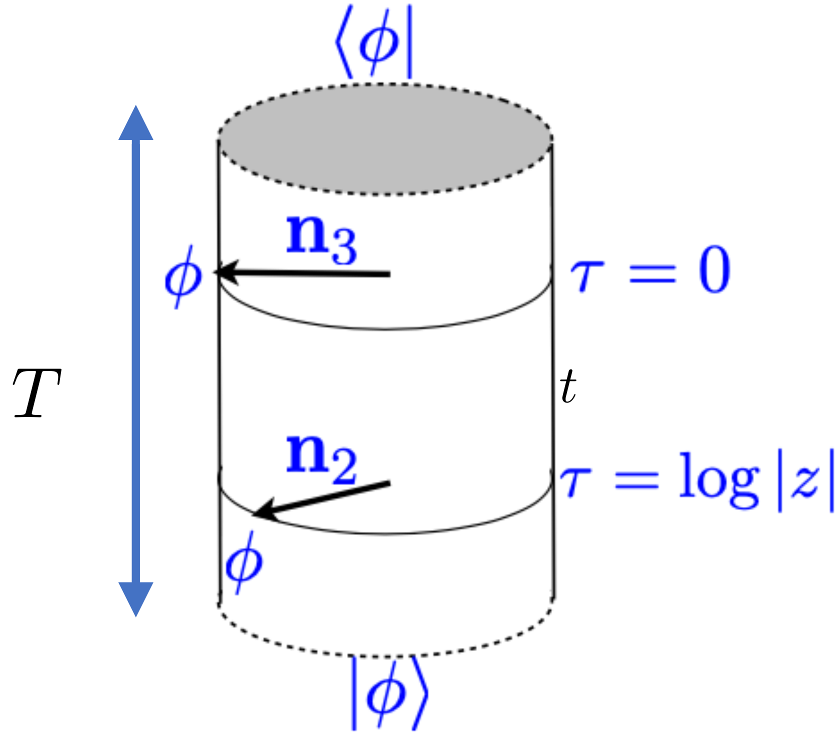
$$j \in \{\max(0, l-n), ..., l+n-2, l+n\}$$

$$\frac{\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle}{\langle \sigma(x_1)\sigma(x_2)\rangle\langle \sigma(x_3)\sigma(x_4)\rangle} = \sum_{\text{even } j} c_j(\Delta t) P_j(\cos(\theta)) = 1 + \sum_{\mathcal{O}} f^2_{\sigma\sigma\mathcal{O}} \sum_{n=0,2,4,\dots} \sum_j e^{-(\Delta_{\mathcal{O}}+n)c_R g t} B_{n,j}(\Delta_{\mathcal{O}}) P_j(\cos(\theta))$$



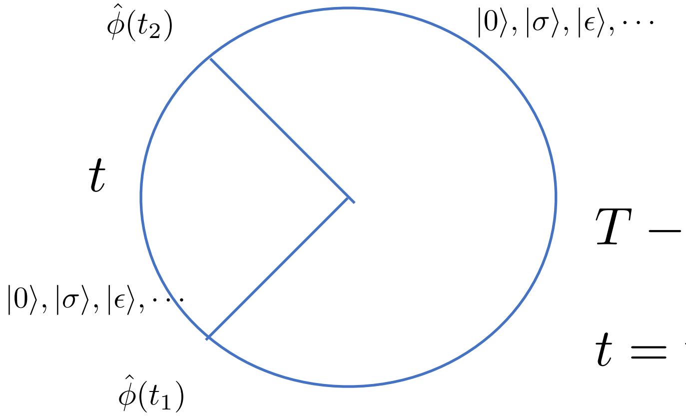
Simultaneous fits of c_0(t) and c_2(t) using primaries $\epsilon, \;\; T, \;\; \epsilon', \;\; T'$ up to n=20

Finite Volume/Temperature Measurements



$$Tr[e^{-\beta\hat{H}+\hat{h}_x\hat{\phi}_x}] = e^{-F(\beta, h_x)}$$

Central charge enters finite temperature free energy and amplitudes:
Can trace from UV to IR



$T - t$

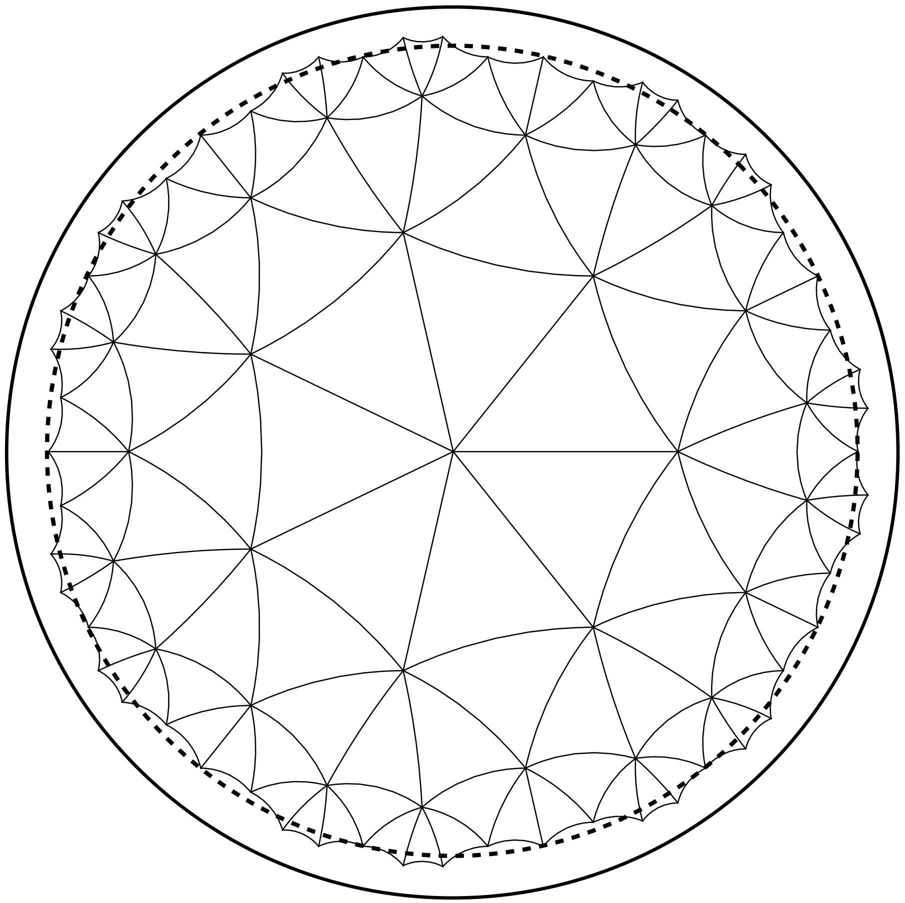
$t = t_2 - t_1$

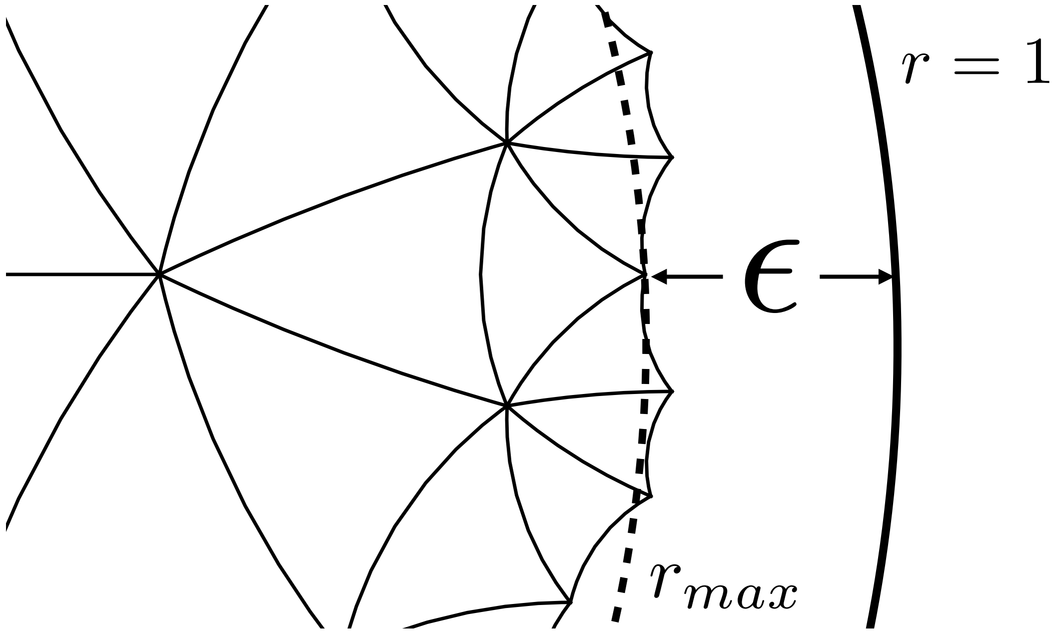
$$\langle \phi_\ell(t_2)\phi_{\ell_1}(t_1)\rangle_T = Tr[\,\hat{\phi}_\ell(0)e^{-t\hat{H}}\hat{\phi}_\ell(0)e^{-(T-t)\hat{H}}\,]$$
$$\equiv \sum_{\mathcal{O}} e^{-T\Delta_{\mathcal{O}}}\langle\mathcal{O}|\hat{\phi}_\ell(0)e^{-t(\hat{H}-\Delta_{\mathcal{O}})}\hat{\phi}_\ell(0)|\mathcal{O}\rangle$$
$$\simeq e^{-t\Delta_{\sigma,\ell}} + e^{-(T-t)\Delta_{\sigma,\ell}}$$
$$+ \quad f^2_{\epsilon\phi_\ell\sigma}e^{-\Delta_\sigma T}[e^{-t(\Delta_\epsilon-\Delta_\sigma)} + e^{-(T-t)(\Delta_\epsilon-\Delta_\sigma)}] + \cdots$$

$0 \le t \le T$ and periodic $t \to t + nT$

$\hat{H}|0\rangle = 0$ and $\hat{\phi}|0\rangle = |\sigma\rangle$

UV cut off problem





Bulk to Boundary Critical Phenomena