Sphaleron rate computation via inverse problem resolution

Talk based on:

- C. Bonanno, F. D'Angelo, M. D'Elia, L. Maio, MN **[arXiv:2305.17120] ;**
- C. Bonanno, F. D'Angelo, M. D'Elia, L. Maio, MN **[arXiv:2308.01287] ;**

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The sphaleron rate is defined as

$$\Gamma_{sphal} = \lim_{V_s \to \infty} \frac{1}{V_s t_M} \left\langle \left[\int_0^{t_M} dt'_M \int_{V_s} d^3x q(t'_M, \vec{x}) \right]^2 \right\rangle$$

$$= \int dt_M d^3x \langle q(t_M, \vec{x}) q(0,0) \rangle,$$

where $q(x) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} Tr\{G^{\mu\nu}(x) G^{\rho\sigma}(x)\}$ is the topological charge density.

Its importance in phenomenology:

- It can be one of the reasons of the presence of Chiral Magnetic Effect (e.g., during heavy collisions; **[Fukushima et al., 2008, 0808.3382]**
- It describes the rate of creation/annihilation of axions in the early Universe **[Notari et al., 2022, 2211.03799]**.

So far very few attempts to compute this quantity, limited to the quenched theory:

[Kotov, 2018] [Altenkort et al., 2021, 2012.08279] [Mancha & Moore, 2022, 2210.05507].

The sphaleron rate can be computed using the Kubo formulas

$$\Gamma_{sphal} = 2T \lim_{\omega \to 0} \frac{\rho(\omega)}{\omega}$$

The spectral function can be extracted from

$$G(t) = -\int_0^\infty \frac{d\omega}{\pi} \rho(\omega) \frac{\cosh\left[\frac{\omega}{2T} - \omega T\right]}{\sinh\left[\frac{\omega}{2T}\right]} = -\int_0^\infty \frac{d\omega}{\pi} \frac{\rho(\omega)}{\omega} K_t(\omega) \quad \text{where } K_t(\omega) = \omega \frac{\cosh\left[\frac{\omega}{2T} - \omega T\right]}{\sinh\left[\frac{\omega}{2T}\right]}.$$

$G(t)$ is simply given by $G(t) = \int d^3x \langle q(t,\vec{x}) q(0,0) \rangle$.

We have to solve an inverse problem

$$G(t) \to \rho(\omega)$$

Goal: to extract $\frac{\rho(\omega)}{\omega}$ *for* $\omega \to 0$

We can write an approximate function $\bar{\rho}$ of the real spectral function ρ as

$$\frac{\bar{\rho}(\bar{\omega})}{\bar{\omega}} = \int_0^\infty d\omega \Delta(\omega, \bar{\omega}) \frac{\rho(\omega)}{\omega}.$$

The more $\Delta(\bar{\omega}, \omega)$ is peaked around $\bar{\omega}$, the better $\bar{\rho}$ approximates ρ.

The idea is that $\bar{\rho}$ can be found as

$$\frac{\bar{\rho}(\bar{\omega})}{f(\bar{\omega})} = -\pi \sum_{t=0}^{1/T} g_t(\bar{\omega}) G(t), \quad f(\omega) = \omega$$

if we look for the function $\Delta(\omega, \bar{\omega})$ in the space of the basis K_t as

$$\Delta(\omega, \bar{\omega}) = \sum_{t=0}^{1/T} g_t(\bar{\omega}) K_t'(\omega)$$

We only need to find the coefficients g_t such that $\Delta(\omega,0)$ is sufficiently peaked.

We fix a sufficiently peaked "simil-gaussian" target function $\delta_\sigma(\omega)$ (σ is related to the width of the function) and we fix g_t requiring that $\Delta(\omega,0)$ is as close as possible to it.

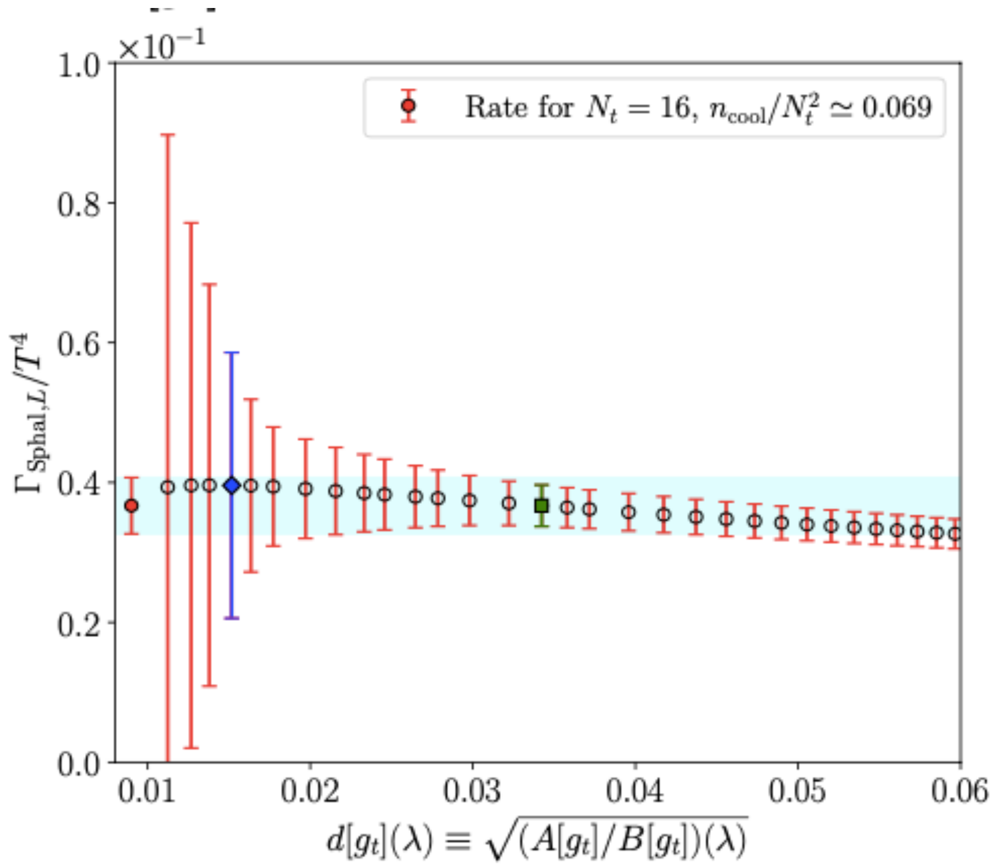
To do this, we minimize the functional

$$F[g_t] = (1-\lambda)A[g_t] + \frac{\lambda}{\mathscr{C}}B[g_t]$$

[arxiv:1903.06476]

where

$$A[g_t] = \int_0^\infty d\omega[\Delta(\omega) - \delta_\sigma(\omega)]^2 e^{2\omega}, \qquad B[g_t] = \sum_{t=0}^{1/T} Cov_{t,t'} g_t g_{t'}.$$



λ is a trade-off between two limits:

- $\lambda \to 0$: we only minimize the distance $|\Delta(\omega) - \delta_\sigma(\omega)|$ but the computation is dominated by statistical errors;
- $\lambda \to 1$: only systematic effects

We consider an intermediate regime where statistical error dominates over systematic and we take an error band taking into account the systematic variations in that region.

Our strategy

Our method in **[arxiv:2305.17120]**:

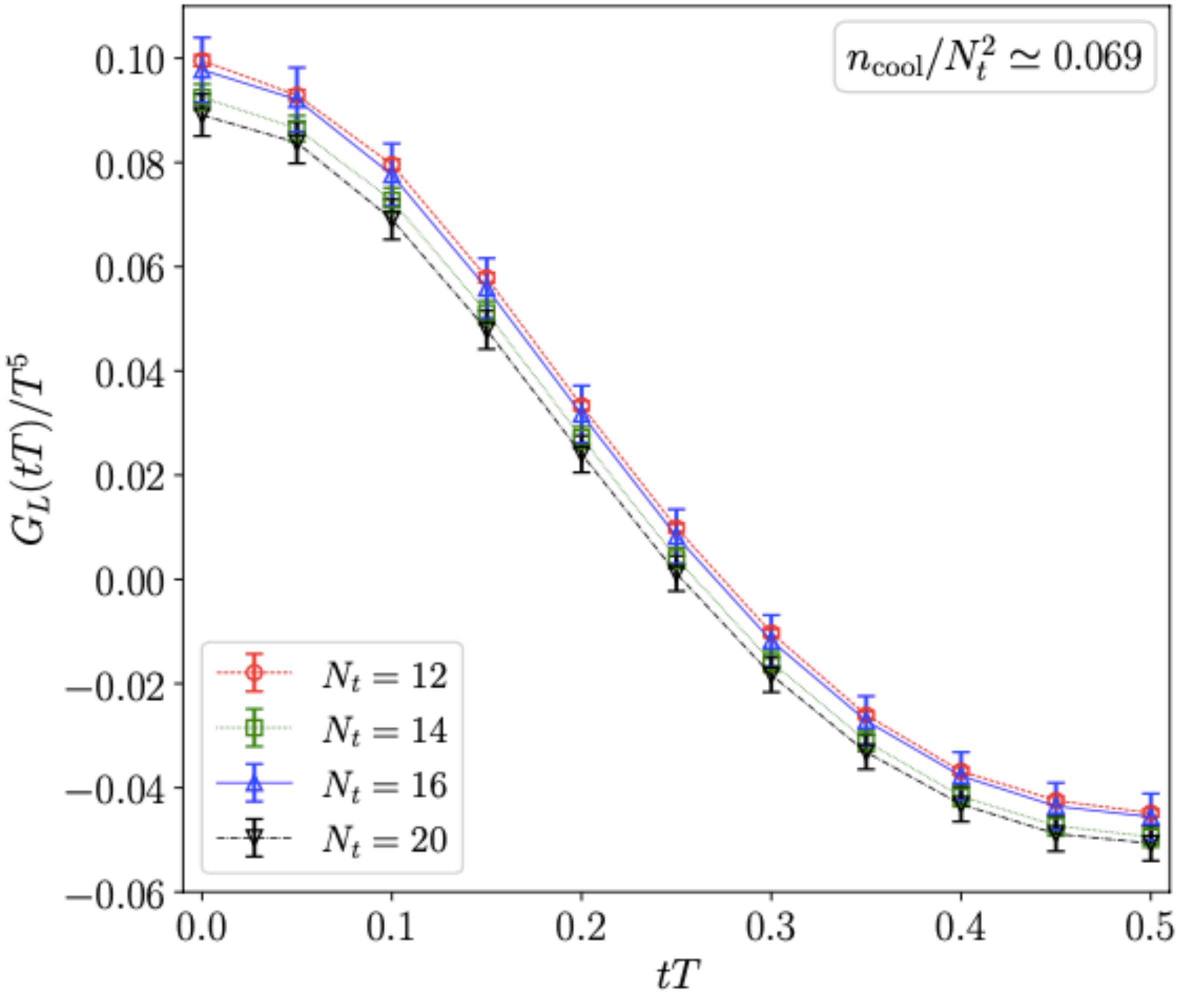
- Computation of the Euclidean time correlation functions $G(t)$ of $q(t,\vec{x})$ at finite-lattice spacing and finite-smoothing radius;
- Extraction of the sphaleron rate $\Gamma_{sphal}(a, r_s)$ from the resolution of the inverse problem using a modified version of the Backus-Gilbert method **[arxiv:1903.06476]**;
- Perform the double limit, i.e., $a \to 0$ at fixed r_s followed by $r_s \to 0$, directly on $\Gamma_{sphal,L}(a, r_s)$.

We have applied this strategy both to the quenched and full QCD case.

We performed simulations on 4 lattices with different lattice spacing a and temporal extents N_t = 12, 14, 16, 20 and $N_s = 3N_t$.

The bare gauge coupling β was tuned so that the physical spatial extent $aN_s \sim 1.65\ fm$ and $T = 1/(aN_t) \simeq 1.24T_c \simeq 357\ MeV$ was constant.

- **[arXiv:2305.17120]**

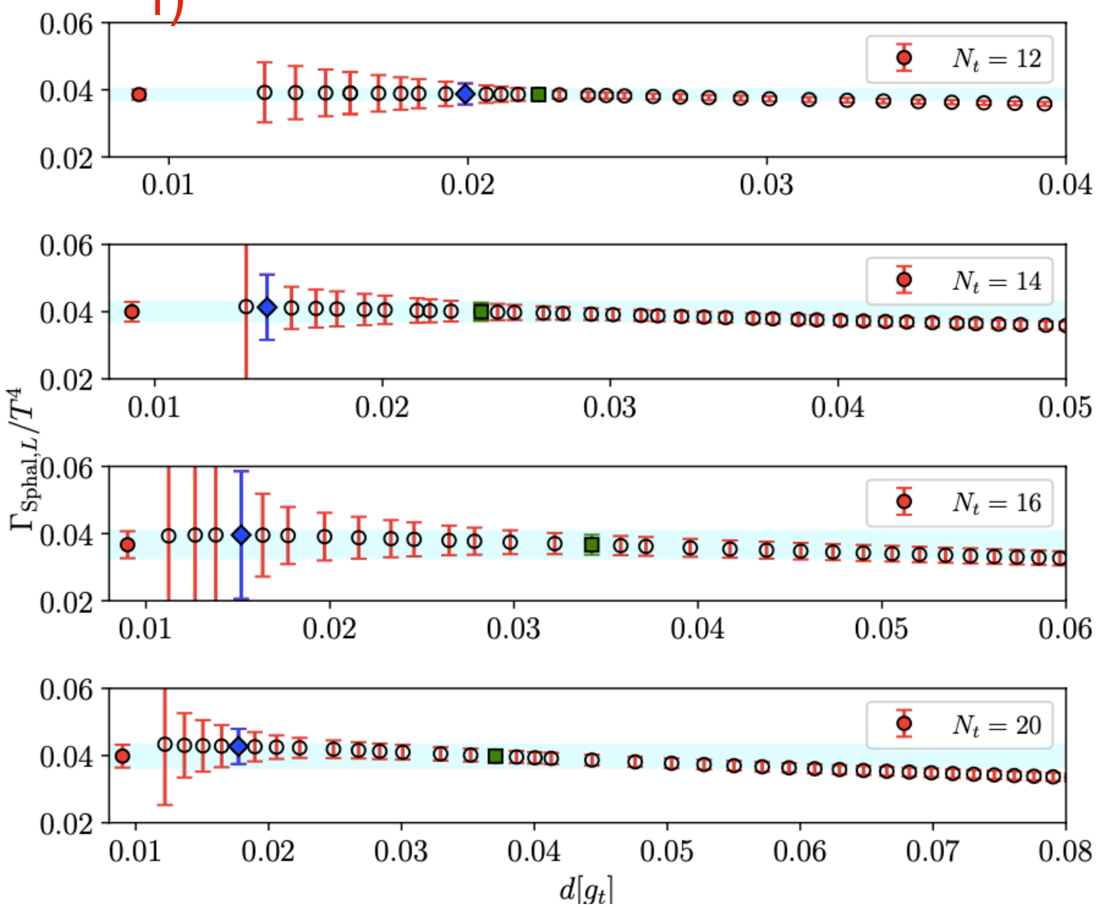


In our work we compute $G(t)$ after n_{cool} cooling steps.

The smoothing radius is given by $r_s/a \simeq \sqrt{8n_{cool}/3}$, thus, $n_{cool}/N_t^2 \propto (r_sT)^2$.

When $\Delta t > r_s$, the lattice correlator $G_L(t) = (1/N_s^3)\sum_{\vec{x}}\sum_{t_1-t_1=t}\langle q_L(t_1,\vec{x})q_L(t_2,\vec{x})\rangle$ is negative as expected from reflection positivity, otherwise it will be positive.

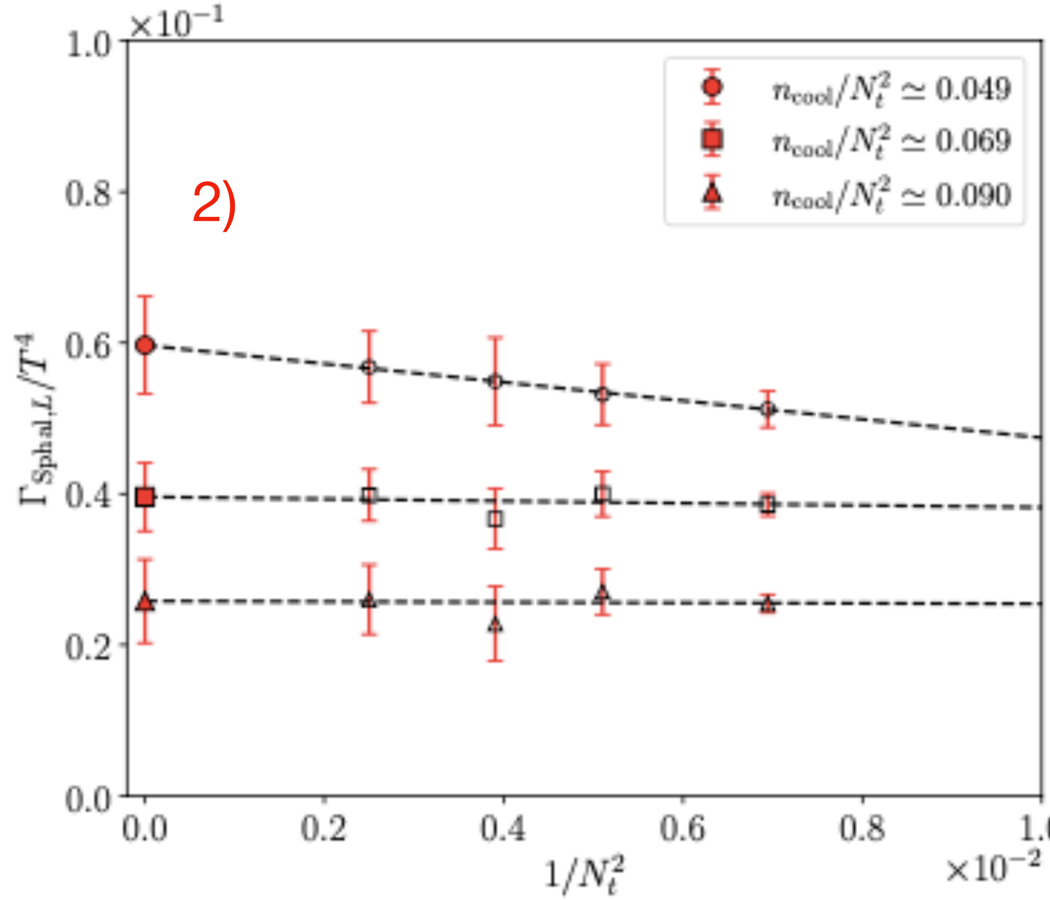
1)



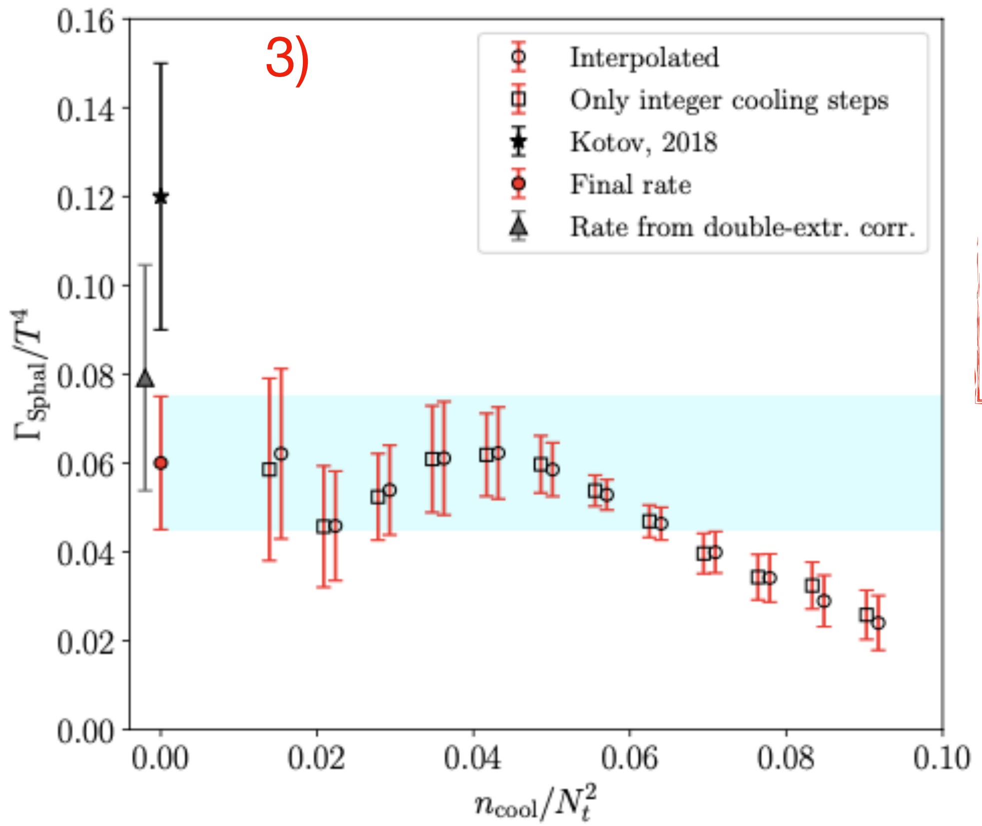
1) After we compute the correlation function we solve the inverse problem resolution method to extract $\Gamma_{sphal,L}(a, r_s)$

2) Then, we we extrapolate it towards the continuum limit at fixed r_s.

2)



3)

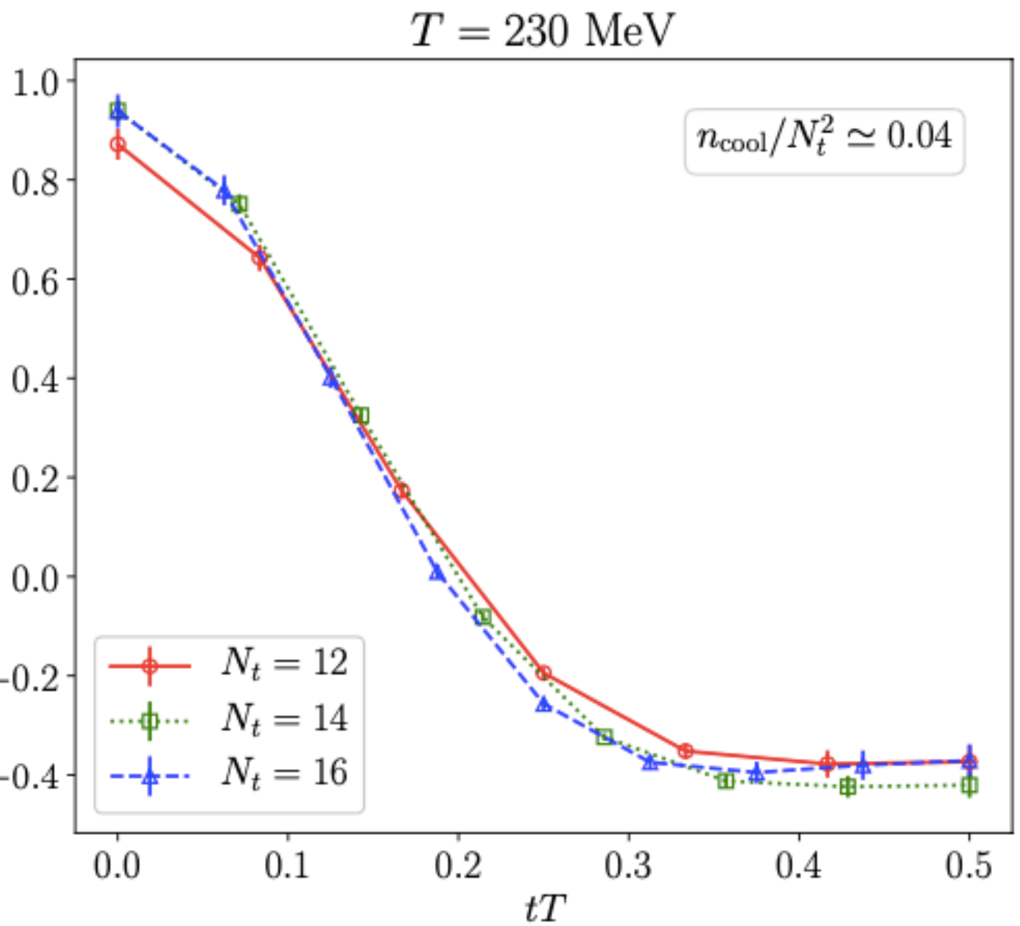


3) Finally, $n_{cool} \to 0$ extrapolation is needed, as results turn out to be independent of ncool for sufficiently small r_s.

The final result is $\Gamma_{sphal} = 0.60(15)$.

- It is smaller than **[Kotov, 2018]** at the same temperature;
- It is in excellent agreement with the result obtained in **[Mancha & Moore, 2022]** (for $T = 1.30T_c, \Gamma_{sphal} = 0.61(2)$).

Results: the full QCD case

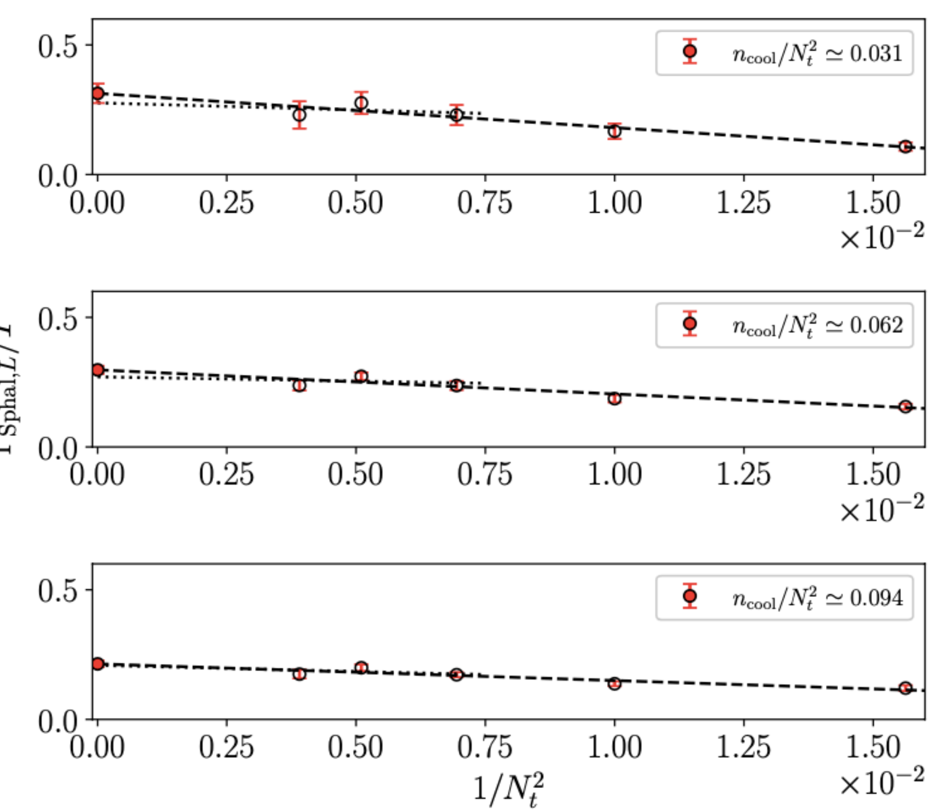


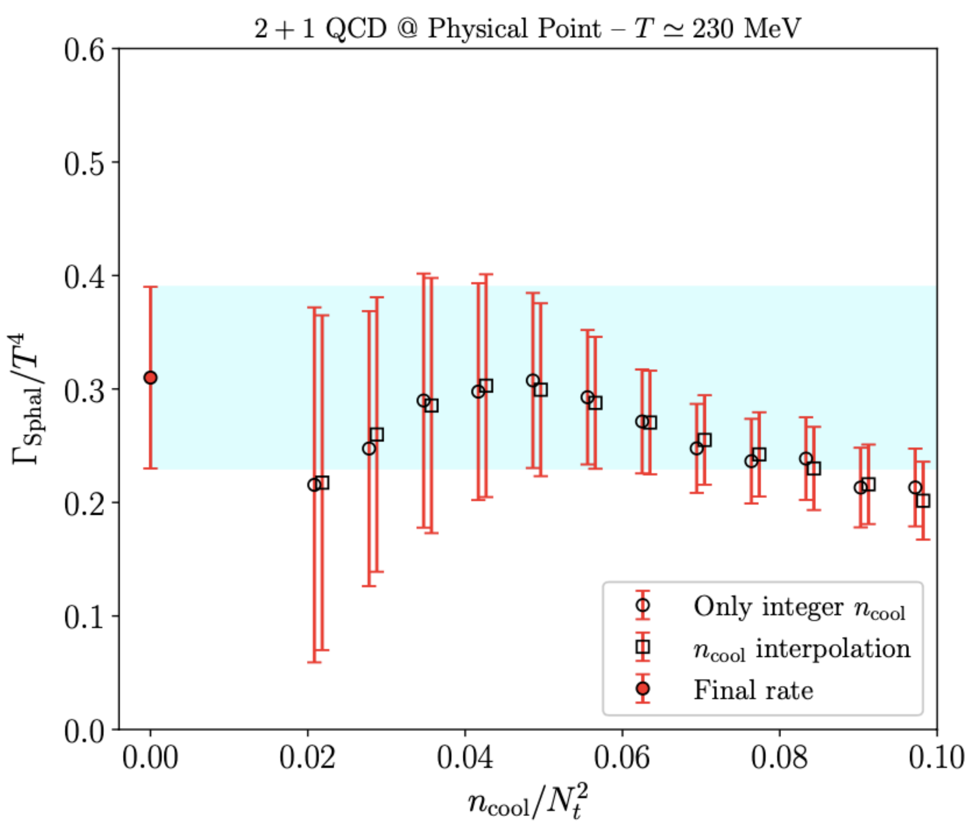
We computed the sphaleron rate of $N_f = 2+1$ **for the first time** (see also poster by J. Weber) at the physical point for a range of temperatures $200\ \mathrm{MeV} \lesssim T \lesssim 600\ \mathrm{MeV}$.

We followed the same steps as in the quenched case.

Here we show an example for $T = 230\ \mathrm{MeV}$.

- **[arXiv:2308.01287]**





Final results

T [MeV]	$\Gamma_{\rm Sphal}/T^4$
230	0.310(80)
300	0.165(40)
365	0.115(30)
430	0.065(20)
570	0.045(12)

We present two ansatz that could describe the behaviour of our results.

These are valid in two different limits of energy scales.

1) In the limit in which it holds the relation

$$m_\ell / T \ll \alpha_s^2,$$

In presence of light dynamical fermions, the semiclassical picture **[Phys.Rev.D104,083520(2021)]** predicts a chiral suppression

$$\frac{\Gamma_{sphal}}{T^4} \sim \alpha_s \frac{m_l^2}{T^2}.$$

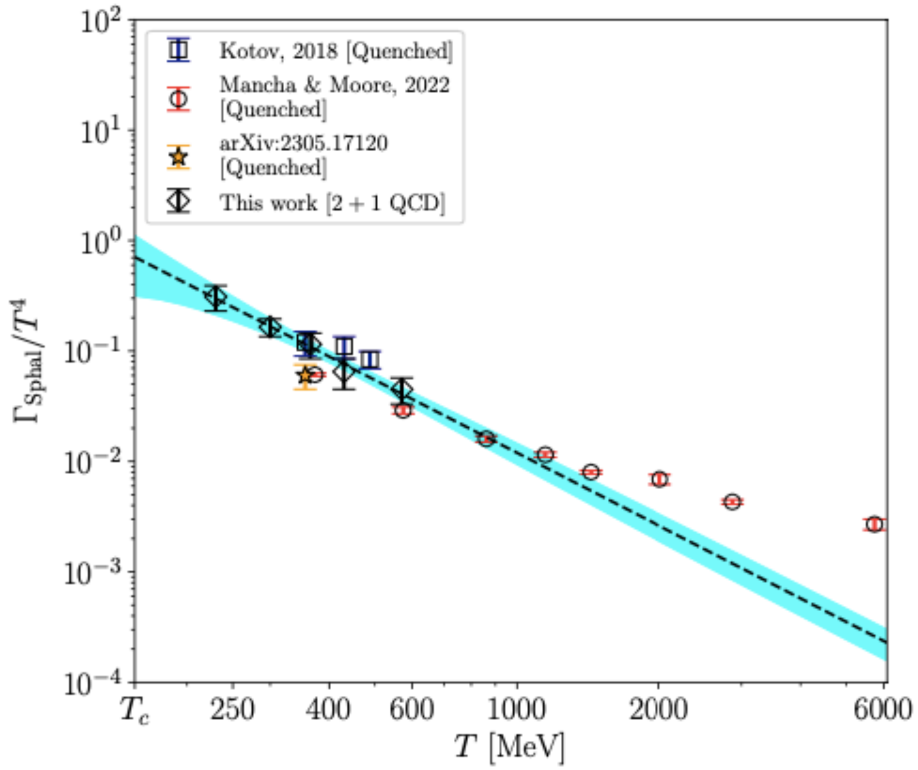
Using the ansatz

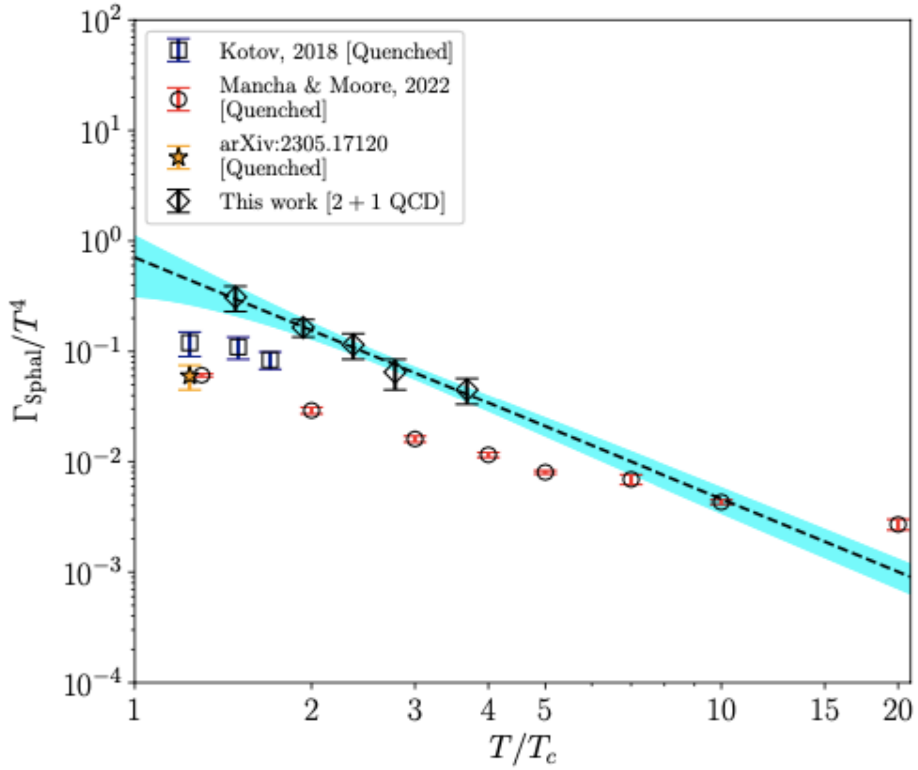
$$\frac{\Gamma_{sphal}}{T^4} = \tilde{A}\left(\frac{T}{T_c}\right)^{-b}$$

where we neglect the T dependence contained in α_s and also m_ℓ.

The fit perfectly describes data with a reduced chi-squared of $0.48/3$ and

$$\tilde{A} = 0.71(23), \quad b = 2.19(38)$$





2) In the opposite limit

$$m_\ell / T \gg \alpha_s^2,$$

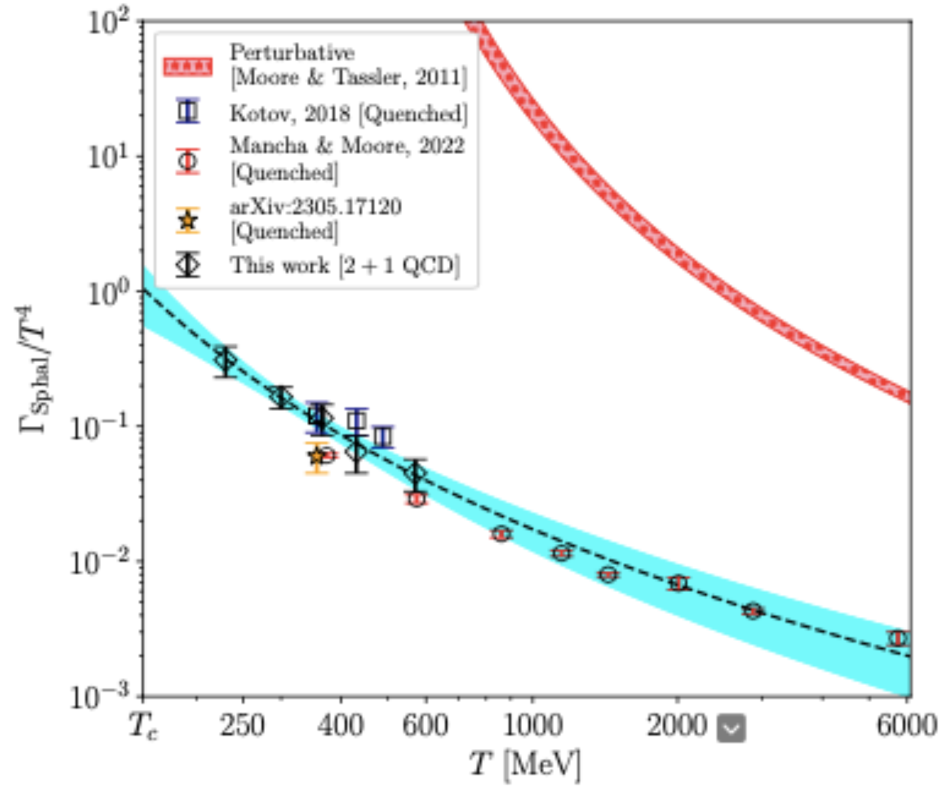
the sphaleron rate is expected to approach the value of the Yang-Mills case.

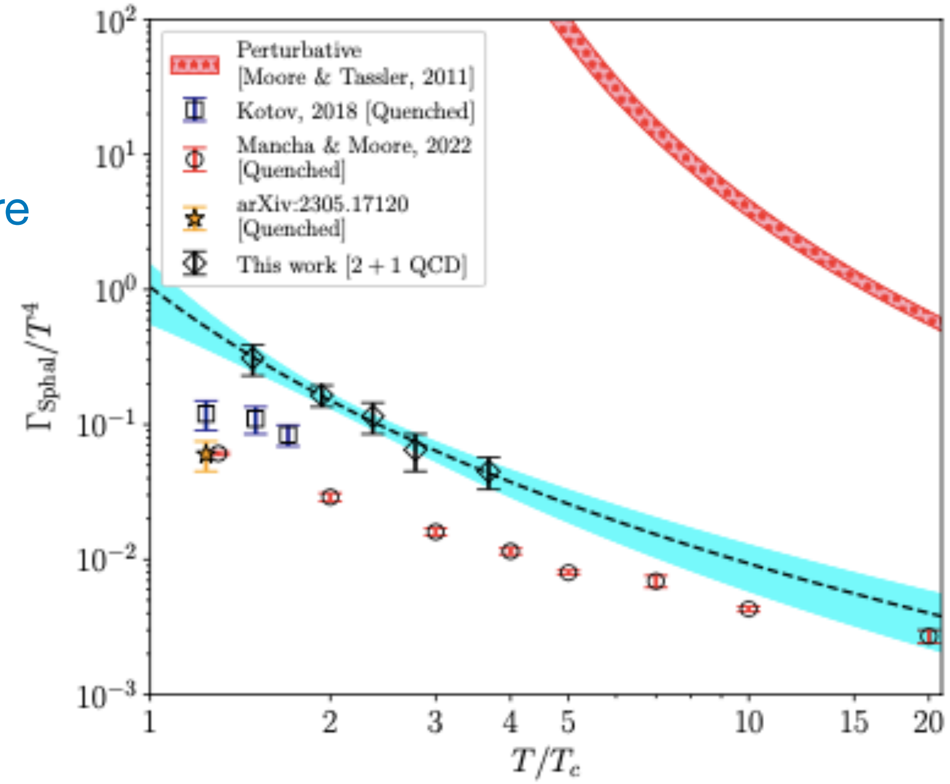
From **[PRD104,083520 (2021] [JHEP02,105(2011)]**, the following semiclassical estimate for the quenched theory is reported

$$\frac{\Gamma_{sphal}}{T^4} \simeq C_1 \alpha_s^5 = \left[\frac{A_0}{\log(T^2/T_C^2) + \log(B_0^2)} \right]^5 .$$

where we used the 1-loop result of $\alpha_s(T)$ of **[J. Phys 36,582(2006)]** and where $B_0 = T_c/\Lambda_{QCD} \simeq 0.46(2)$ from FLAG value for 3 flavours ($\Lambda_{QCD}^{(\bar{M}S)}(\mu = 2\ GeV) \simeq 338(12)\ MeV$) while $A_0 \simeq 3.08(2)$.

Using this ansatz we obtain on our data $A = 2.96(51)$ and $B = 4.3(1.7)$ with a reduced chi-squared of 0.36/3.





Take home messages

➡We computed the sphaleron rate from the inversion of finite lattice spacing and finite smoothing radius Euclidean topological charge density correlators;

➡Computation made in pure gauge and **for the first time** in 2+1 full QCD with physical quark masses as a function of the temperature;

➡ (Guess of) study of the results on the basis of **two ansatz** coming from existent Semiclassical predictions:

- Polinomial fit using semiclassical ansatz valid in the limit $m_\ell/T \ll \alpha_s^2$, it gives good results but **we don't observe the predicted chiral suppression;**
- Logarithmic fit valid for $m_\ell/T \gg \alpha_s^2$, it gives good results, as well.
  These ansatz come from perturbative arguments that we are not sure are valid in this limit.

Future prospects

➡Computation with different discretization;

➡Higher temperatures or different pion mass to check chiral suppression predicted by semiclassical arguments (topological freezing problem);

➡Extend the computation at non-zero spatial momentum $\vec{k} \neq 0$.

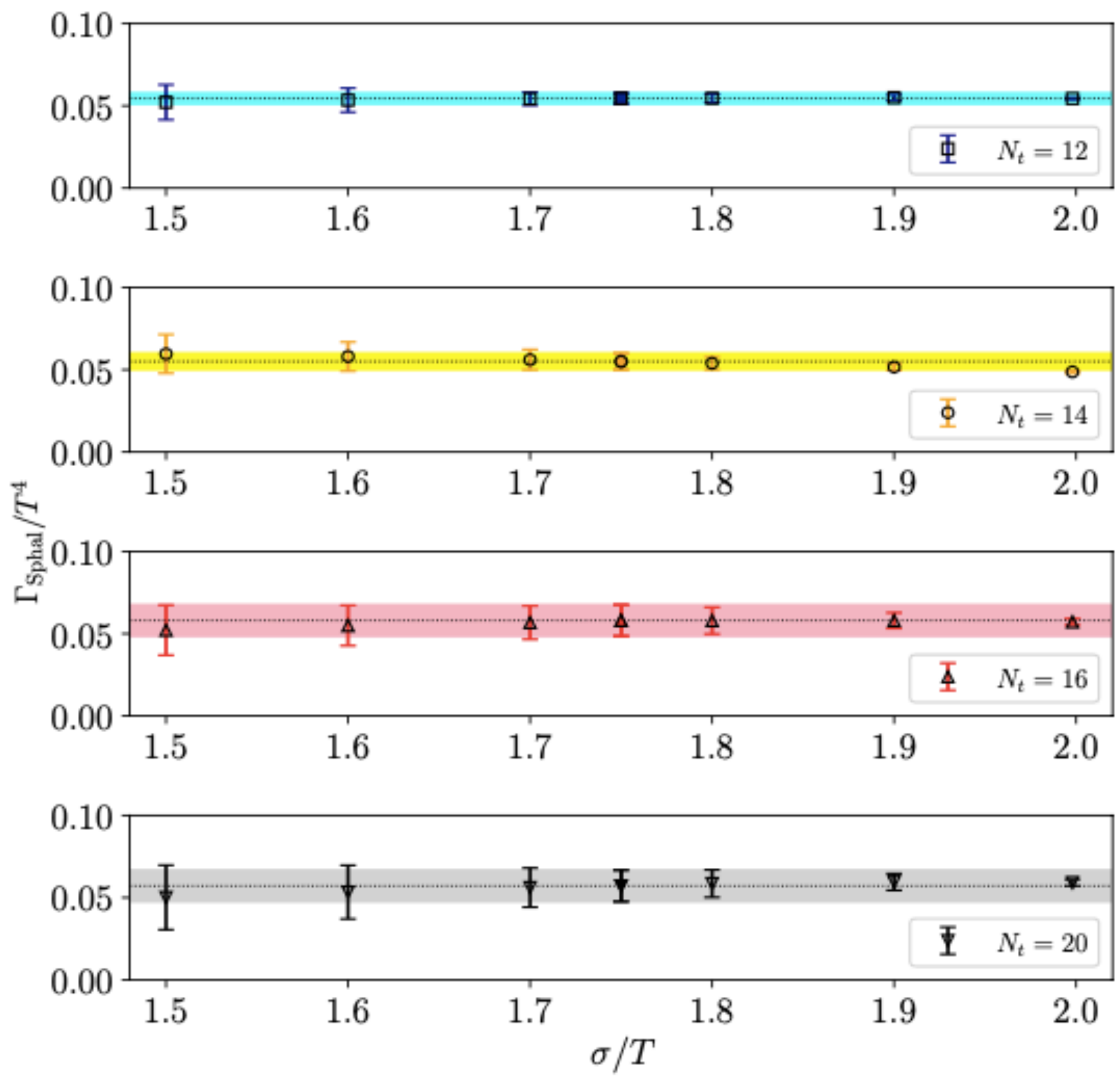
Thanks for your attention!

Back-up slides

We chose the target function

$$\delta_\sigma(\omega) = \left(\frac{2}{\sigma\pi}\right)^2 \frac{\omega}{\sinh(\omega/\sigma)}, \qquad \delta_\sigma \to \delta(\omega) \;\; for \;\; \sigma \to 0$$

Our results were obtained choosing σ/T=1.75. We observed that such determination (shaded area) is in perfect agreement with those obtained for $1.5 \leq \sigma \leq 2$.

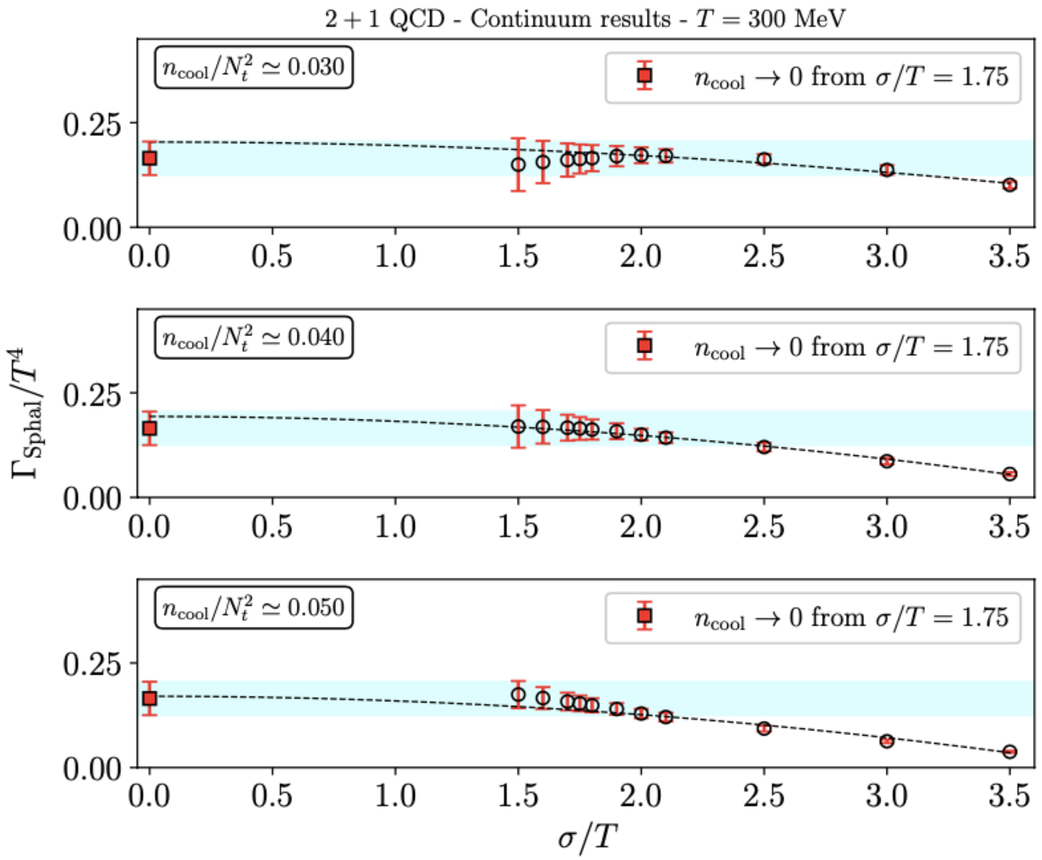


From general theoretical arguments, it can be extracted that for our choice of the target function the dependence of sphaleron rate as function of σ starts quadratically

$$\Gamma_{sphal}(\sigma) = \Gamma_{sphal}(0) + c\left(\frac{\sigma}{T}\right)^2 + \mathcal{O}\left[\left(\frac{\sigma}{T}\right)^5\right]$$

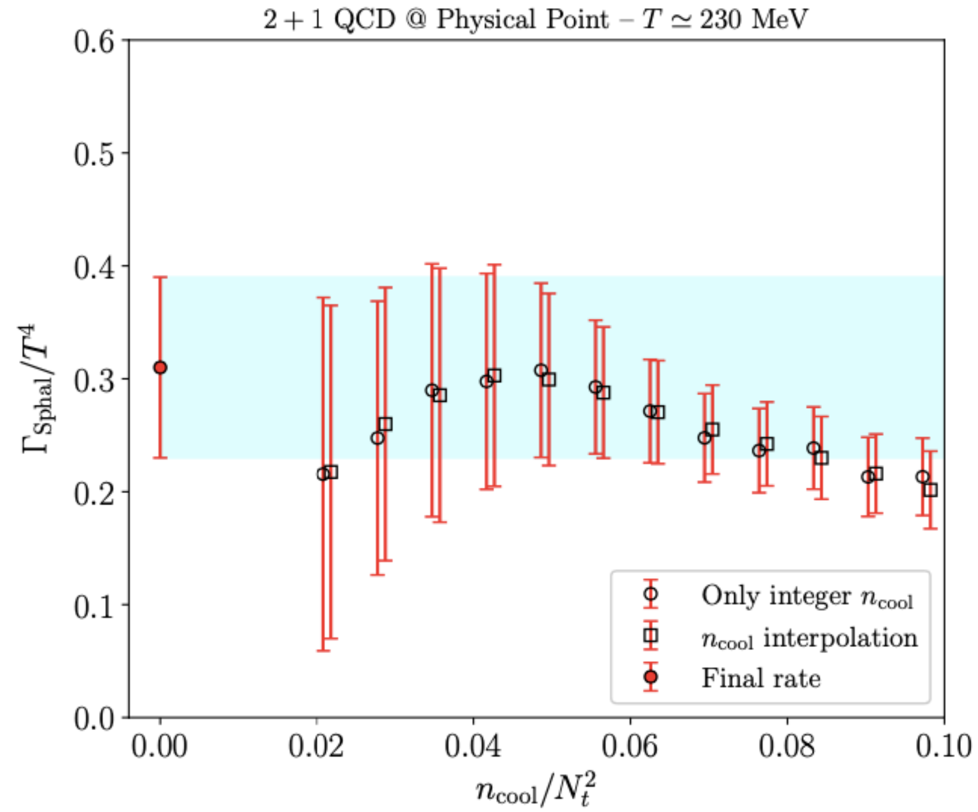
Thus, since this expansion starts from $\mathcal{O}(\sigma^2)$, we expect to observe a mild dependence on σ, if it is small enough.

We checked that $\sigma/T = 1.75$ is a good choice to take into account this dependence. In all the cases, we find that the extrapolation agrees with this choice of σ/T.

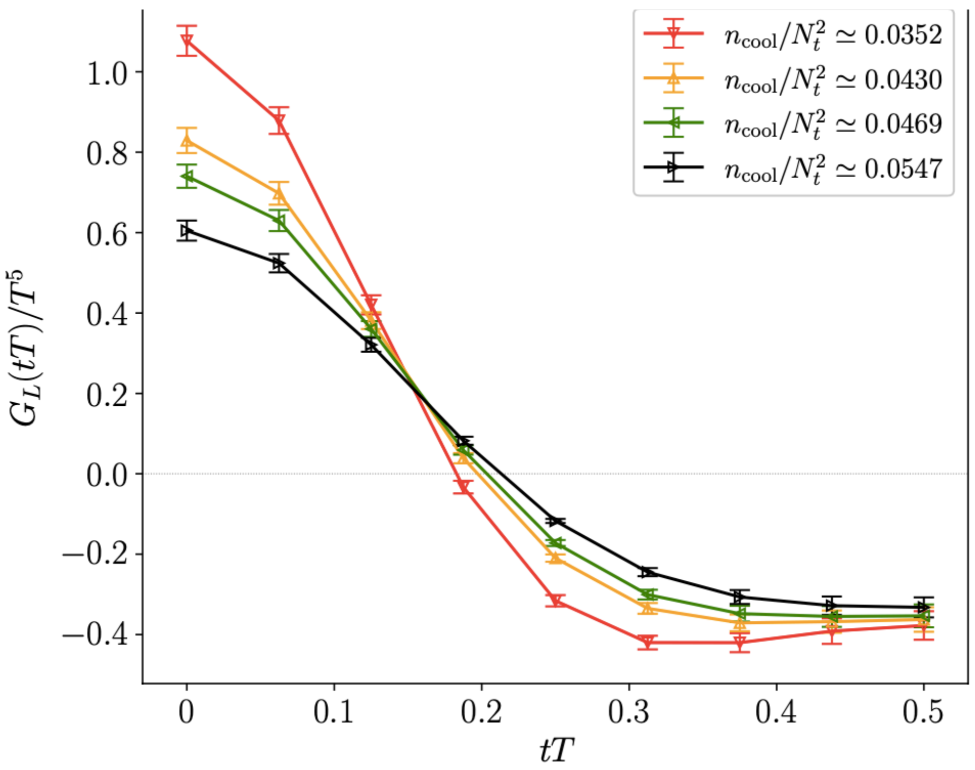


Correlation functions at different n_{cool}

It could be guessed that we don't observe the chiral suppression because of the smearing on the correlation function, since it produces the positiveness of the correlation function at small values of the Euclidean time.



If we look at the correlation functions corresponding to the values of the n_{cool} in the plateau, we observe a non negligible variation in the small t region that, however, does not affect the plateau itself.

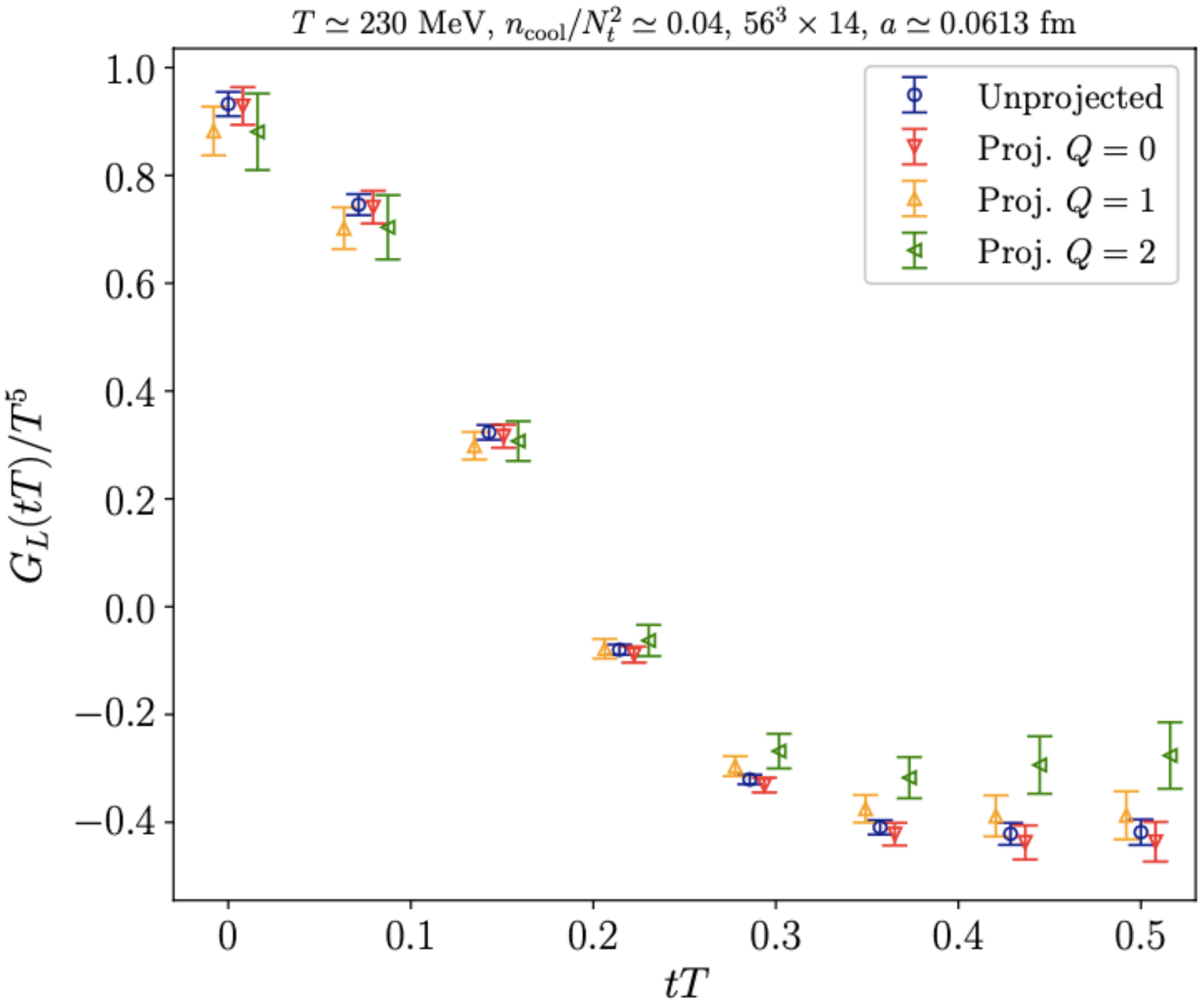


This is probably related to the fact that we are taking the limit $\omega \to 0$, corresponding $\tau \to \infty$, and thus we expect that the points of the correlators at small values of τ do not affect the final estimate of the sphaleron rate.

Correlation functions at different topological sectors

One could guess that the absence of chiral suppression is a consequence of the nature of the staggered discretization which breaks the chiral symmetry.

In the case of the topological susceptibility, this is indeed an issue. In that case, the different topological sectors are wrongly weighted and this produces a non-expected increase of the susceptibility.



However, in our case the correlation function is not dependent on the topological sector within the errors. Thus, how we weight the different contributions coming from different sectors is not relevant.

Furthermore, these are the same ensembles already used in **[JHEP 10,197 (2022)]** to compute the topological susceptibility at finite temperature. In that case, the continuum limit was under control.

Correlation functions at different topological sectors