Searching for the QCD critical point using Lee-Yang edge singularities

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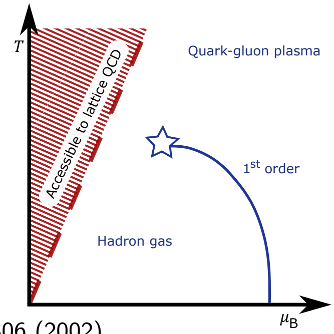
The infamous problem

Trick: μ_B pure imaginary avoids sign problem; can analytically continue to $\mu_B \in \mathbb{R}$[1,2].

Trick: Expand pressure P/T^4 in μ_B/T[3,4].

The latter is getting a bit too pricey. Popularity of resummation schemes[5,6,7,8].



[1] P. de Forcrand and O. Philipsen, Nuclear Physics B, 642.1-2, 290–306 (2002).
[2] M. D'Elia and M.-P. Lombardo, Phys. Rev. D, 67.1, 014505 (2003).
[3] C. R. Allton et al., Phys. Rev. D, 66.7, 074507 (2002).
[4] R. V. Gavai and S. Gupta, Phys. Rev. D, 68.3, 034506 (2003).
[5] S. Borsányi et al., Phys. Rev. Lett. 126.23, 232001 (2021).
[6] D. Bollweg et al., Phys. Rev. D, 105.7, 074511 (2022).
[7] S. Mitra, P. Hegde, and C. Schmidt, Phys. Rev. D, 106.3, 034504 (2022).
[8] S. Mondal, S. Mukherjee, and P. Hegde, Phys. Rev. Lett. 128.2, 022001 (2022).

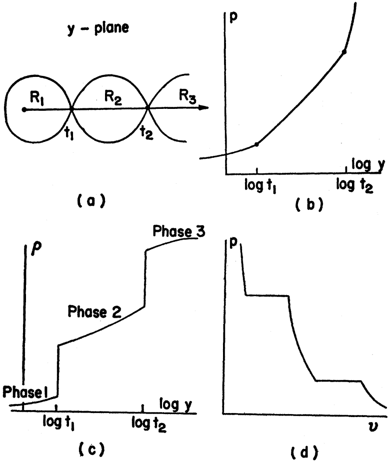
Lee-Yang theorem

Works where $\log \mathcal{Z}_{\text{QCD}}$ is free of singularities.

Lee-Yang theorem[9]: Zeroes of the partition function that approach the real axis as $V \to \infty$ correspond to phase transitions.

Intuition: Indications of non-analyticities in P

- may hint at phase transitions
- or singularities in $\mathbb{C}$
- constrain validity of Taylor series



[9] C. N. Yang and T. D. Lee, Phys. Rev. 87.3, 404–409 (1952).

Lee-Yang edges and extended analyticity

Ising: Generically have branch cuts on imaginary axis. (Pinch real axis at T_c.)

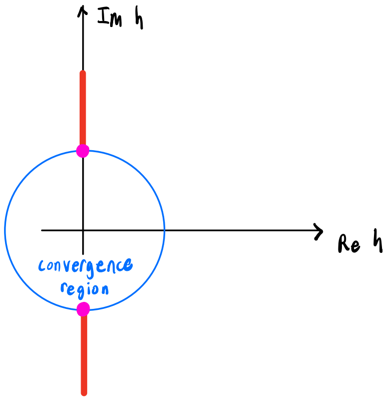
Lee-Yang edge (LYE): The singularities closest to real axis.

Extended analyticity conjecture[10]: LYE is the nearest singularity to the origin.

LYE position fixed at

$$z_c = |z_c| e^{\pm i\pi/2\beta\delta}$$

with $z \equiv th^{-1/\beta\delta}$ and critical exponents β, δ.



[10] P Fonseca and A Zamolodchikov, J. Stat. Phys. 110, 527–590 (2003).

Padé approximants

Want detailed information about singularities $\Rightarrow$ **rational functions**,

$$R_n^m(x) \equiv \frac{\sum_{i=0}^m a_i x^i}{1 + \sum_{j=1}^n b_j x^j}.$$

Singularities captured or mimicked by zeros in denominator.

Let f have a formal Taylor series

$$f(x) = \sum_{k=0}^{\infty} c_k x^k.$$

Padé approximant of order $[m, n]$: R_n^m with coefficients so that it equals the Taylor series up to order $m + n$. Gives relationship between coefficients a_i, b_j, c_k.

Padé approximants

Things to think about with Padé:

- Theorem: Unique when it exists
- Theorem: $[m, n]$ converges to f exactly as $m \to \infty$ when f has pole of order n
- Other properties deduced from numerical experiments
- Limited by number of known Taylor coefficients
- Only have up to 8^{th} order[11,12] for $\log \mathcal{Z}_{\text{QCD}}$; difficultly far greater for higher orders[13]

[11] S. Borsanyi et al., J. High Energ. Phys. 2018.10, 205 (2018).
[12] D. Bollweg et al., Phys. Rev. D, 108.1, 014510 (2023).
[13] Computational requirements of HotQCD EoS exceed 2000 GPU-years and 2.4 PB.

Multi-point Padé approximants

Padé approximants you get by demanding[14]

$$R_n^m(x) = f^{m+n}(x) \equiv \sum_{i=0}^{m+n} c_k x^k .$$

Multi-point Padé: The R_n^m satisfying

$$R_n^m(x_1) = f^{m+n}(x_1), \quad R_n^m(x_2) = f^{m+n}(x_2), \quad \dots , \quad R_n^m(x_N) = f^{m+n}(x_N)$$

for N points x_ℓ. Some pros/cons:

- Need fewer Taylor coefficients!
- Less seems to be known about them...

[14] One expects corresponding relationships among derivatives of R and f.

The strategy

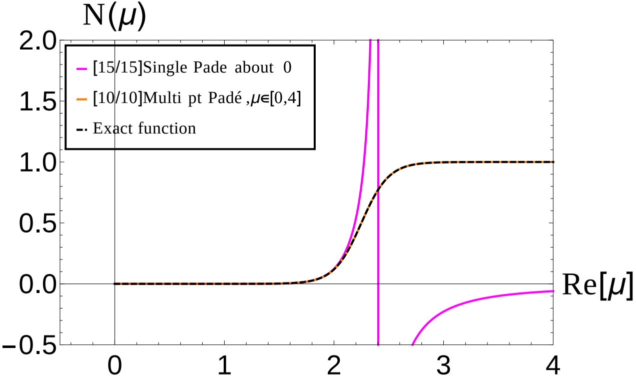
Roughly follow this procedure:

1. What transition are you interested in?
2. How should the singularities scale?
3. Lattice calculations at multiple, pure imaginary μ_B.
4. Estimate singularities with multi-point Padé.
5. Does scaling match expectation?
6. Analytically continue results to $\mu_B \in \mathbb{R}$.

Next: Why we trust it. (Francesco's Monday talk.)

Test: 1-d Thirring model[15,16]

Number density $N(\mu)$ can be worked out exactly.

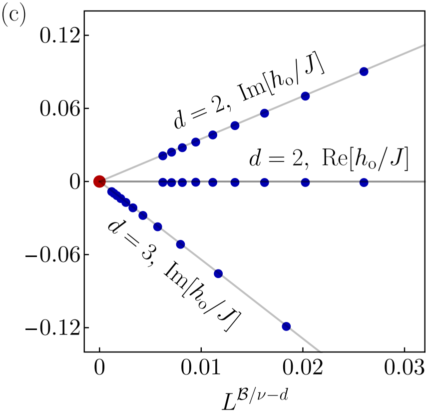


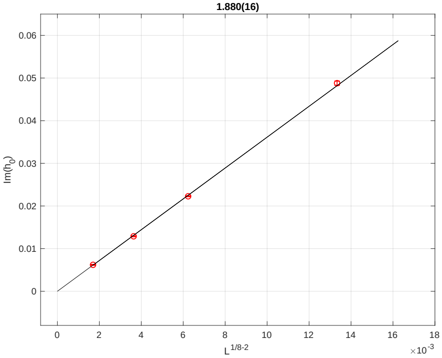
Multi-point captures the exact $N(\mu)$ well, outperforms single point.

[15] P. Dimopoulos et al., Phys. Rev. D, 105.3, 034513 (2022).

[16] F. Di Renzo, S. Singh, and K. Zambello, Phys. Rev. D, 103.3, 034513 (2021).

Test: 2-d Ising model[17,18]





Reproduces correct scaling and critical exponents extremely well.

[17] A. Deger and C. Flindt, Phys. Rev. Research, 1.2, 023004 (2019).
[18] F. Di Renzo and S. Singh, PoS(LATTICE2022)148, (2023).

Test: The Roberge-Weiss transition[19,20]

Lattice setup:

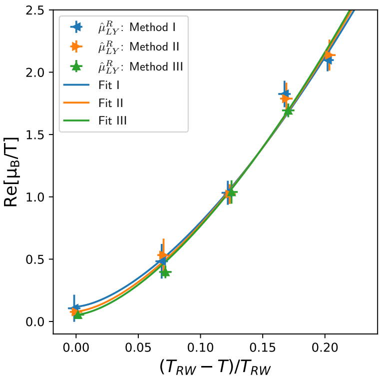
- 2+1 dynamical HISQ quarks
- m_s/m_l fixed to physical value
- $N_\tau = 4$, 6 with $N_s/N_\tau = 6$

$$h \sim \hat{\mu}_B - i\pi \qquad t \sim T - T_{\rm RW}$$

$$z = th^{-1/\beta\delta} \qquad z_c = |z_c| e^{\pm i\pi/2\beta\delta}$$

$$\Rightarrow \quad \mathrm{Re}\,\hat{\mu}_{\mathsf{LY}} = \pm\pi \left(\frac{z_0}{|z_c|}\right)^{\beta\delta}$$

Taking $|z_c| = 2.43$ yields $9.1 \lesssim z_0 \lesssim 9.4$.

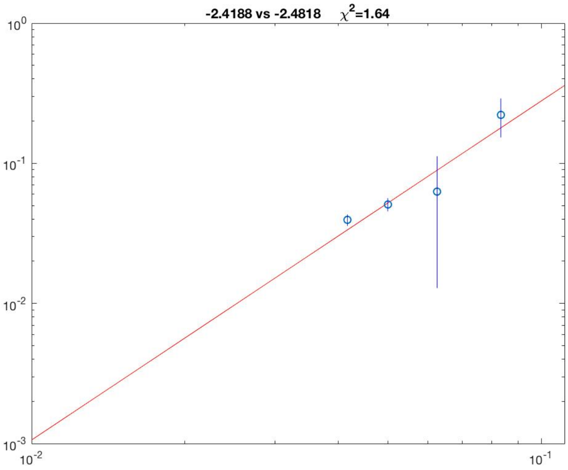


Taking $T_{\rm RW}^{N_\tau=4} = 201.4$ MeV yields $\beta\delta \approx 1.5635$, compare 1.563495(15).

Prelim: $T_{\rm RW} = 211.1(3.1)$ MeV, compare 208(5) MeV.

[19] C. Bonati et al., Phys. Rev. D, 93.7, 074504 (2016).
[20] G. Johnson et al., Phys. Rev. D, 107.11, 116013 (2023).

Test: Roberge-Weiss FSS



FSS scaling of Re near RW transition reasonably captured.

Toward the CEP

Assuming multi-point Padé reliable, turn attention to CEP. Also in 3-d, $\mathbb{Z}_2$ universality class, so $\beta\delta \approx 1.5$. Exact mapping to Ising not yet known. Linear ansatz:

$$t = \alpha_t \Delta T + \beta_t \Delta\mu_B$$
$$h = \alpha_h \Delta T + \beta_h \Delta\mu_B,$$

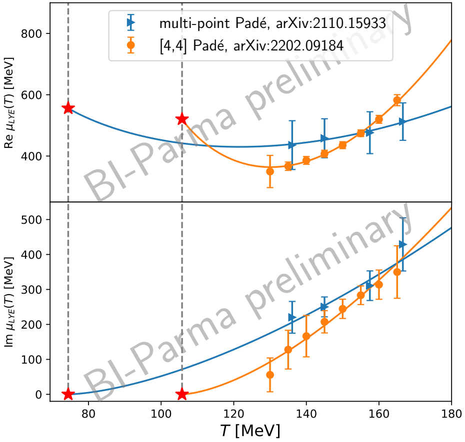
where $\Delta T \equiv T - T^{\text{CEP}}$ and $\Delta\mu_B \equiv \mu_B - \mu_B^{\text{CEP}}$, which leads to[21]

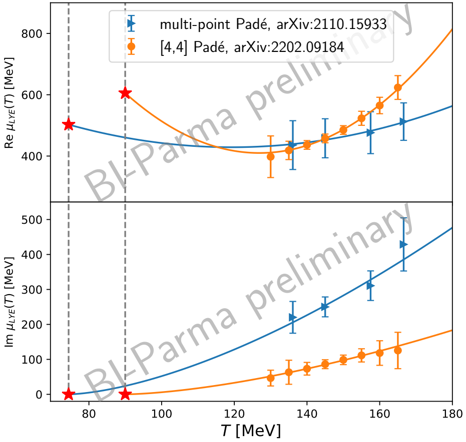
$$\mu_{\text{LY}} = \mu_B^{\text{CEP}} - c_1 \Delta T + i c_2 |z_c|^{-\beta\delta} \Delta T^{\beta\delta} + \mathcal{O}\left(\Delta T^2\right).$$

Expectation from lattice[22]: $\mu_B^{\text{CEP}}/T^{\text{CEP}} \gtrsim 3$.

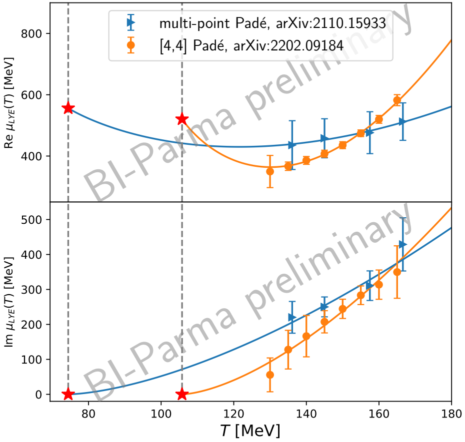
[21] M. A. Stephanov, Phys. Rev. D, 73.9, 094508 (2006).
[22] D. Bollweg et al., Phys. Rev. D, 105.7, 074511 (2022).

Toward the CEP: Single-point and multi-point, erratum





Toward the CEP: Single-point and multi-point



Some comments:

- Must propagate fit uncertainties
- Led to discovery of a bug
- Orange: smaller N_s/N_τ
- Orange: $N_\tau = 8$
- Orange: error estimates correct?
- Blue: $N_\tau = 6$
- Blue: Need lower T

Rough suggestion of CEP:
$T \sim 90$ MeV $\quad \mu_B \sim 600$ MeV

Toward the CEP: Evaluation of rough estimate

$$T \sim 90 \text{ MeV} \quad \mu_B \sim 600 \text{ MeV}$$

- $T < T_c \approx 130$ MeV[23]
- $\mu_B/T \sim 6$ is well outside apparent convergence radius
- Functional renormalization group[24] $\mu_B \sim 600$ MeV, $T \sim 100$ MeV
- Dyson-Schwinger[25] $\mu_B \sim 500$ MeV, $T \sim 125$ MeV

[23] H.-T. Ding et al., Phys. Rev. Lett. 123.6, 062002 (2019).
[24] W.-j. Fu, J. M. Pawlowski, and F. Rennecke, Phys. Rev. D, 101.5, 054032 (2020).
[25] J. Bernhardt et al., Phys. Rev. D, 104.7, 074035 (2021).

Summary and Outlook

- Multi-point Padé tested in a variety of situations
- Possible indication of CEP around $T \sim 90$ MeV, $\mu_B \sim 600$ MeV
- In progress: Refinement of CEP estimate strategy
- In progress: Continuum limit extrapolation
- In progress: Examine chiral transition

Thanks for your attention.