Operator mixing and non-perturbative running of $\Delta F = 2$ BSM four-fermion operators

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in collaboration with

Future goal: accurate evaluation of the CP-violation parameter $\delta$ in the CKM matrix

The most stringent limits on any generalisation beyond the Standard Model (BSM) are provided by the indirect investigation of BSM effects.

$K^0 - \bar{K}^0$ oscillations are sensitive to BSM loop effects that vanish in the SM at tree level.
Indirect investigation of CP violation: $\varepsilon$ parameter

$$
\varepsilon^{\text{theor}}(\delta) = \frac{A(K_L \to (\pi\pi)_{I=0})}{A(K_S \to (\pi\pi)_{I=0})} \propto \tilde{U}(\mu) \langle \bar{K}^0 | Q(\mu) | K^0 \rangle F(\delta)
$$

To be evaluated non-perturbatively

Comparing $\varepsilon^{\text{theor}}$ with its experimental estimate we obtain

- in the SM:
  1. a new estimate of the phase $\delta$
  2. non-perturbative uncertainties
- beyond the SM:
  1. $\delta$ is kept to the current estimate
  2. limits to any BSM contribution
Effective Hamiltonian for K oscillations:

**SM:** \( H_{\text{eff}}^{\Delta S=2} = \tilde{U}_1 Q_1 \)

**BSM:** \( H_{\text{eff}}^{\Delta S=2} = \sum_{i=1}^{5} \tilde{U}_i Q_i + \sum_{i=1}^{3} \tilde{U'}_i \tilde{Q}_i \)

Only one relevant operator

An operator basis \( Q_i \)

Transition amplitudes are calculated with

\[ \langle \bar{K}^0 | \mathcal{H}_{\text{eff}}^{\Delta S=2} | K^0 \rangle \]

The renormalisation procedure introduces an energy-scale in the matrix elements and in the Wilson coefficients:

\[ \langle \bar{K}^0 | \mathcal{H}_{\text{eff}}^{\Delta S=2} | K^0 \rangle = \tilde{U}_i(\mu) \langle \bar{K}^0 | Q_i(\mu) | K^0 \rangle \]

Wilson coefficients \[ \text{Renormalised Matrix elements} \]
New features

- Running evaluation for the four-fermion operators with 3 quark flavours in the sea;
- New theoretical formulation of the operator running and mixing in the perturbative regime for $N_f = 3$.

The difference at low energies ($\sim 4$ GeV) between the perturbative and non-perturbative theory can be relevant in the estimate of relevant quantities as $\varepsilon(\delta)$
Tensions!

\[ B_i \propto \langle Q_i(\mu = 3\text{GeV}) \rangle \]

<table>
<thead>
<tr>
<th>Collaboration</th>
<th>Ref.</th>
<th>Ref.</th>
<th>( N_f )</th>
<th>( B_2 )</th>
<th>( B_3 )</th>
<th>( B_4 )</th>
<th>( B_5 )</th>
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<tbody>
<tr>
<td>ETM 15</td>
<td>[55]</td>
<td>2+1+1</td>
<td>A ★ ○ ○ ★</td>
<td>0.46(1)(3)</td>
<td>0.79(2)(5)</td>
<td>0.78(2)(4)</td>
<td>0.49(3)(3)</td>
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<tr>
<td>RBC/UKQCD 16</td>
<td>[60]</td>
<td>2+1</td>
<td>A ○ ○ ○ ★</td>
<td>0.488(7)(17)</td>
<td>0.743(14)(65)</td>
<td>0.920(12)(16)</td>
<td>0.707(8)(44)</td>
</tr>
<tr>
<td>SWME 15A</td>
<td>[58]</td>
<td>2+1</td>
<td>A ★ ○ ★ ○ †</td>
<td>-0.525(1)(23)</td>
<td>0.773(6)(35)</td>
<td>0.981(3)(62)</td>
<td>0.751(7)(68)</td>
</tr>
<tr>
<td>SWME 14C</td>
<td>[508]</td>
<td>2+1</td>
<td>C ★ ○ ★ ○ †</td>
<td>-0.525(1)(23)</td>
<td>0.774(6)(64)</td>
<td>0.981(3)(61)</td>
<td>0.748(9)(79)</td>
</tr>
<tr>
<td>SWME 13A †</td>
<td>[495]</td>
<td>2+1</td>
<td>A ★ ○ ★ ○ †</td>
<td>-0.549(3)(28)</td>
<td>0.790(30)</td>
<td>1.033(6)(46)</td>
<td>0.855(6)(43)</td>
</tr>
<tr>
<td>RBC/UKQCD 12E</td>
<td>[502]</td>
<td>2+1</td>
<td>A ■ ○ ★ ★</td>
<td>0.43(1)(5)</td>
<td>0.75(2)(9)</td>
<td>0.69(1)(7)</td>
<td>0.47(1)(6)</td>
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<tr>
<td>ETM 12D</td>
<td>[59]</td>
<td>2</td>
<td>A ★ ○ ○ ★</td>
<td>0.47(2)(1)</td>
<td>0.78(4)(2)</td>
<td>0.76(2)(2)</td>
<td>0.58(2)(2)</td>
</tr>
</tbody>
</table>

Inconsistencies between different estimates evaluating \( Z(\mu = 3\text{GeV}) \) in different ways (perturbatively or not)
In the continuum we map the SF into the \( \chi \text{SF} \) with a chiral rotation:

\[
q_f(x) \rightarrow \psi_f(x) = \exp \left( -i \frac{\pi}{4} r_f \gamma_5 \right) q_f(x)
\]

\[
\bar{q}_f(x) \rightarrow \bar{\psi}_f(x) = \bar{q}_f(x) \exp \left( -i \frac{\pi}{4} r_f \gamma_5 \right)
\]

\[
1 = r_1 = r_2 = r_3 = -r_4
\]

Correspondence between correlation functions in the SF and \( \chi \text{SF} \):

\[
\langle O \left[ \psi, \bar{\psi} \right] \rangle_{\text{SF}}^{\text{cont}} = \langle O \left[ R \left( \frac{\pi}{2} \right) \psi, \bar{\psi} R \left( \frac{\pi}{2} \right) \right] \rangle_{\chi \text{SF}}^{\text{cont}}
\]

The boundary rotation removes \( \mathcal{O}(a) \) effects!

\[
\langle O_{\text{even}} \rangle_c = \langle O_{\text{even}} \rangle_c^{\text{cont}} + \mathcal{O}(a^2)
\]
Four-Fermion Operators — Renormalisation

Parity-odd operators:

\[
Q_1^\pm = \mathcal{O}_{[\text{VA+AV}]}, \quad Q_3^\pm = \mathcal{O}_{[\text{PS-SP}]}, \quad Q_5^\pm = -2\mathcal{O}_{[T\bar{T}]},
\]

\[
Q_2^\pm = \mathcal{O}_{[\text{VA-AV}]}, \quad Q_4^\pm = \mathcal{O}_{[\text{PS+SP}]},
\]

with

\[
\mathcal{O}_{[\Gamma_1\Gamma_2\pm\Gamma_2\Gamma_1]} := \mathcal{O}_{[\Gamma_1\Gamma_2]} \pm \mathcal{O}_{[\Gamma_2\Gamma_1]},
\]

\[
\mathcal{O}_{[\Gamma_1\Gamma_2]} := \frac{1}{2} \left[ (\bar{\psi}_1\Gamma_1\psi_2) \left( \bar{\psi}_3\Gamma_2\psi_4 \right) \pm (\bar{\psi}_1\Gamma_1\psi_4) \left( \bar{\psi}_3\Gamma_2\psi_2 \right) \right]
\]

The parity-odd operators mix as in a regularisation with exact chiral symmetry:

\[
\begin{pmatrix}
    Q_1^\pm \\
    Q_2^\pm \\
    Q_3^\pm \\
    Q_4^\pm \\
    Q_5^\pm
\end{pmatrix} =
\begin{pmatrix}
    Z_{11} & 0 & 0 & 0 & 0 \\
    0 & Z_{22} & Z_{23} & 0 & 0 \\
    0 & Z_{32} & Z_{33} & 0 & 0 \\
    0 & 0 & 0 & Z_{44} & Z_{45} \\
    0 & 0 & 0 & Z_{54} & Z_{55}
\end{pmatrix}^\pm
\begin{pmatrix}
    Q_1^\pm \\
    Q_2^\pm \\
    Q_3^\pm \\
    Q_4^\pm \\
    Q_5^\pm
\end{pmatrix}
\]
Evolution matrices between to scales: \( \bar{Q}_i(\mu_2) = U_{ij}(\mu_2, \mu_1) \bar{Q}_j(\mu_1) \)

Evolution matrices down to a scale: \( \bar{U}(\mu_2, \mu_1) =: \left[ \hat{U}(\mu_2) \right]^{-1} \hat{U}(\mu_1) \)

**Problem**: for \( N_f = 3 \) the operator basis is resonant and we cannot adopt the usual definition

\[
\bar{U}(\mu) = \left[ \frac{g^2(\mu)}{4\pi} \right]^{-\frac{\gamma_0}{2b_0}} \bar{W}(\mu)
\]

\( W(\mu) \) not well-defined for \( N_f = 3 \)
Poincaré-Dulac Theorem [2013.16220v3]:
there exists an operator basis where \( A(g) := \frac{\gamma(g)}{\beta(g)} \) takes the form

\[
A_{\text{can}}(g) = \frac{1}{g}\left(\Lambda + g^2N_2\right)
\]

through a transformation \( S(g) \approx \left(1 + \sum_{k=1}^{n} H_{2k}g^{2k}\right)S_{D} \equiv s_n(g)S_{D} \)

**Solution**: in this basis it is well-defined a perturbative running:

\[
\hat{U}(u) = S_{D}^{-1} \exp\left(-\frac{1}{2}\Lambda \ln(u)\right) \exp\left(-\frac{1}{2}N_2 \ln(u)\right)s_n(u)S_{D}
\]
Step-scaling functions — Definitions

Non-perturbative evolution from the step-scaling functions:

\[ \sigma(u) := \mathbf{U}(\mu/2, \mu) \bigg|_{g^2(\mu) = u} \quad \Rightarrow \quad \mathbf{U}(u_{\text{had}}, u_{\text{pt}}) = \sigma(u_1) \cdots \sigma(u_N) \]

Discrete step-scaling functions:

\[ \Sigma(g_0^2, a/L) := \mathcal{Z}(g_0^2, a/2L) [\mathcal{Z}(g_0^2, a/L)]^{-1} \]

\( O(g^2) \) lattice artefacts are removed adopting subtracted step-scaling functions [2112.10606]:

\[ \tilde{\Sigma}(u, a/L) := \Sigma(u, a/L) [1 + u \log(2) \delta_k(a/L) \gamma_0]^{-1} \]
We perform global fits with ansatz

\[
\begin{align*}
\tilde{\Sigma}(u_n, \frac{a}{L})_{ij} &= \sigma(u_n)_{ij} + \left(\frac{a}{L}\right)^2 \sum_{m=0}^{2} [\rho_m]_{ij} u_n^m
\end{align*}
\]

The parameters are found minimising the \(\chi^2\)

\[
\chi^2(\tilde{\sigma}, \tilde{\rho}) := \sum_{n=1}^{7} \sum_{r=1}^{2(3)} \frac{1}{\Delta \Sigma^2_{r,n}} \left[ \tilde{\Sigma}_{n,r} - \sigma_n - x_r \sum_{m=0}^{2} \rho_m u_n^m \right]^2
\]
The continuum extrapolations are fitted as power series in the coupling

\[ \sigma(u) = 1 + r_1 u + r_2 u^2 + r_3 u^3 + r_4 u^4 \]

Fixed coefficients:

\[ r_1 = \gamma_0 \ln 2, \]

\[ r_2 = \gamma_1 \ln 2 + b_0 \gamma_0 \ln^2 2 + \frac{1}{2} (\gamma_0)^2 \ln^2 2 \]

N coupling evaluated as: \[ \sigma^{-1}(u_{n-1}) = u_n \]

Evolution operator between \( u_1 \) and \( u_N \):

\[ U(u_{\text{had}}, u_{\text{pt}}) = \sigma(u_1) \cdots \sigma(u_N) \]
Non-perturbative running — Errors

Non-perturbative running:

\[
\hat{U}(u) = S_D^{-1} \exp\left(-\frac{\Lambda}{2} \ln \upsilon_{pt}\right) \exp\left(-\frac{N_2}{2} \ln \upsilon_{pt}\right) s_n(g) S_D [U(u, \upsilon_{pt})]^{-1}
\]

- **Statistical errors**: propagation of the errors from the fits
- **Systematic errors (guess)**: lack of knowledge on the anomalous dimension matrix at higher orders
Non-perturbative running — BSM $2|3$ indices
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Future developments

We computed non-perturbatively the running down to a scale \( \mathcal{O}(4 \text{ GeV}) \) incorporating the NLO in the perturbative part of the study and solving the problem that appears for \( N_f = 3 \).

In order to evaluate the value of \( \varepsilon^{\text{theor}} \) we are planning to perform the following computations:

- a non-perturbative evaluation of the running also in the region from \( \Lambda_{\text{QCD}} \) to \( \mathcal{O}(4 \text{ GeV}) \);
- renormalised tM-QCD matrix elements estimate at the scale \( \Lambda_{\text{QCD}} \) on Wilson gauge configurations (CLS);
- renormalisation constants at \( \Lambda_{\text{QCD}} \) in the \( \chi\text{SF} \).
Thank you!

If you need to contact me: riccardomarinelli1999@gmail.com
Backup — Results at the most hadronic scale

<table>
<thead>
<tr>
<th>Operator</th>
<th>$N = 5$</th>
<th>$N = 7$</th>
<th>$N = 9$</th>
<th>$N = 11$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{U}_{11}(\mu_0)$</td>
<td>$0.71 \pm 0.05$</td>
<td>$0.71 \pm 0.06$</td>
<td>$0.72 \pm 0.06$</td>
<td>$0.72 \pm 0.06$</td>
</tr>
<tr>
<td>$\hat{U}_{22}(\mu_0)$</td>
<td>$1.23 \pm 0.07$</td>
<td>$1.20 \pm 0.08$</td>
<td>$1.18 \pm 0.08$</td>
<td>$1.17 \pm 0.08$</td>
</tr>
<tr>
<td>$\hat{U}_{23}(\mu_0)$</td>
<td>$-0.76 \pm 0.02$</td>
<td>$-0.76 \pm 0.02$</td>
<td>$-0.76 \pm 0.02$</td>
<td>$-0.76 \pm 0.02$</td>
</tr>
<tr>
<td>$\hat{U}_{32}(\mu_0)$</td>
<td>$-0.43 \pm 0.05$</td>
<td>$-0.41 \pm 0.06$</td>
<td>$-0.40 \pm 0.06$</td>
<td>$-0.40 \pm 0.06$</td>
</tr>
<tr>
<td>$\hat{U}_{33}(\mu_0)$</td>
<td>$2.22 \pm 0.07$</td>
<td>$2.21 \pm 0.08$</td>
<td>$2.21 \pm 0.08$</td>
<td>$2.21 \pm 0.09$</td>
</tr>
<tr>
<td>$\hat{U}_{44}(\mu_0)$</td>
<td>$1.78 \pm 0.07$</td>
<td>$1.77 \pm 0.08$</td>
<td>$1.77 \pm 0.08$</td>
<td>$1.76 \pm 0.09$</td>
</tr>
<tr>
<td>$\hat{U}_{45}(\mu_0)$</td>
<td>$-0.050 \pm 0.003$</td>
<td>$-0.049 \pm 0.004$</td>
<td>$-0.048 \pm 0.004$</td>
<td>$-0.047 \pm 0.004$</td>
</tr>
<tr>
<td>$\hat{U}_{54}(\mu_0)$</td>
<td>$1.7 \pm 0.4$</td>
<td>$1.7 \pm 0.4$</td>
<td>$1.7 \pm 0.5$</td>
<td>$1.7 \pm 0.5$</td>
</tr>
<tr>
<td>$\hat{U}_{55}(\mu_0)$</td>
<td>$0.48 \pm 0.03$</td>
<td>$0.48 \pm 0.03$</td>
<td>$0.48 \pm 0.03$</td>
<td>$0.48 \pm 0.03$</td>
</tr>
</tbody>
</table>
In general, \( W(\mu) \) is the solution of the equation

\[
\mu \frac{d}{d\mu} W(\mu) = [\gamma[\bar{g}(\mu)], W(\mu)] - \beta[\bar{g}(\mu)] \left( \frac{\gamma[\bar{g}(\mu)]}{\beta[\bar{g}(\mu)]} - \frac{\gamma^{(0)}}{\bar{g}(\mu)b_0} \right) W(\mu)
\]

and admits the perturbative expansion

\[
W(\mu) = 1 + \bar{g}^2(\mu)J_1 + \bar{g}^4(\mu)J_2 + \bar{g}^6(\mu)J_3 + \ldots
\]

implying

\[
2J_1 - \left[ \frac{\gamma_0}{b_0}, J_1 \right] = \frac{b_1}{b_0} \frac{\gamma_0}{b_0} - \frac{\gamma_1}{b_0}
\]

Non-invertible system of equations if \( N_f = 3! \)