Timelike pion form factor from lattice QCD

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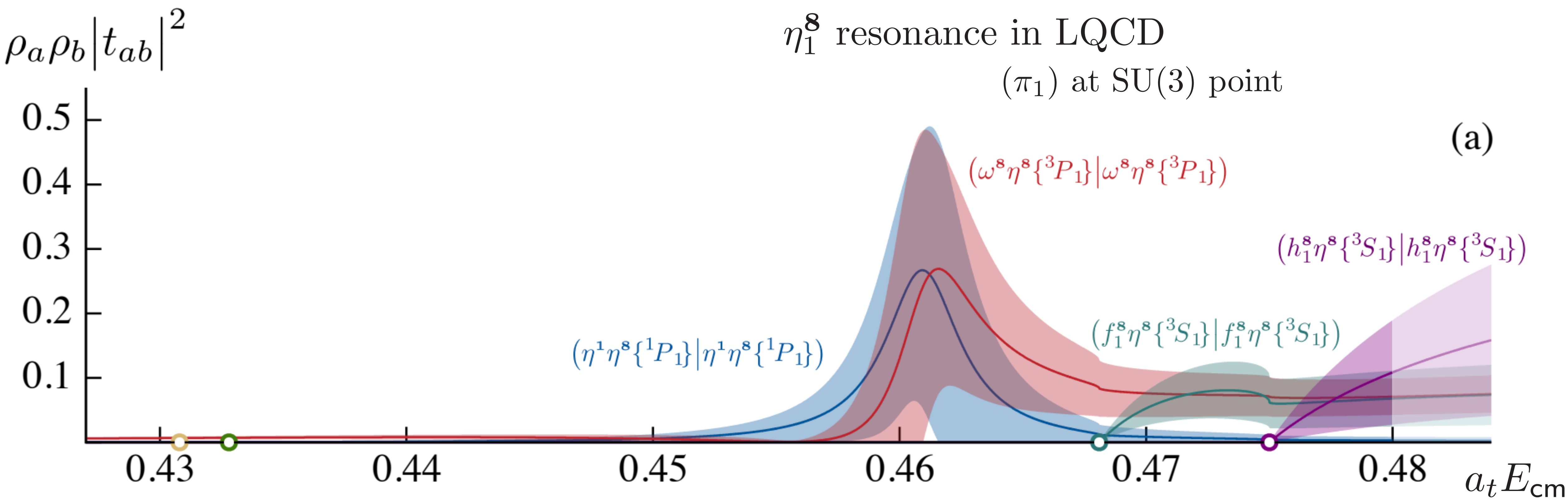
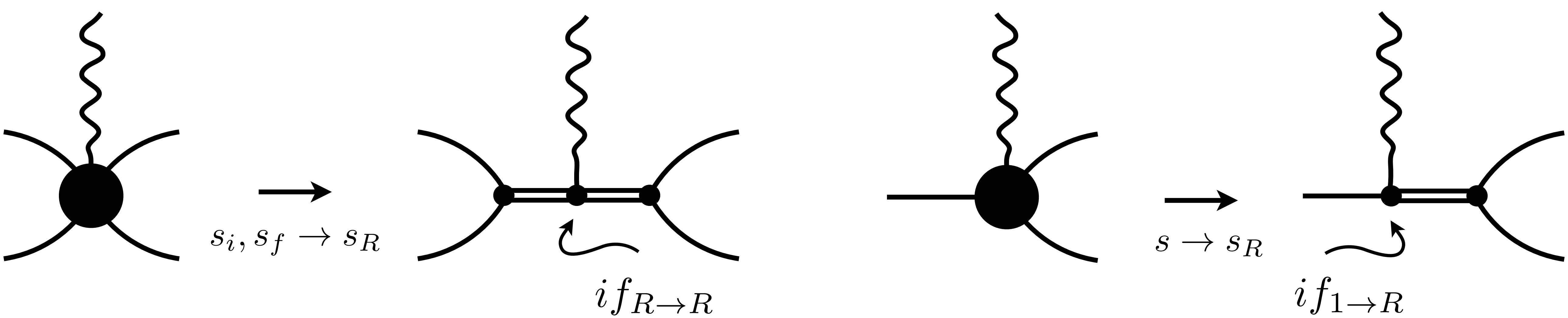
August - 4 - 2023

In collaboration with J. Dudek

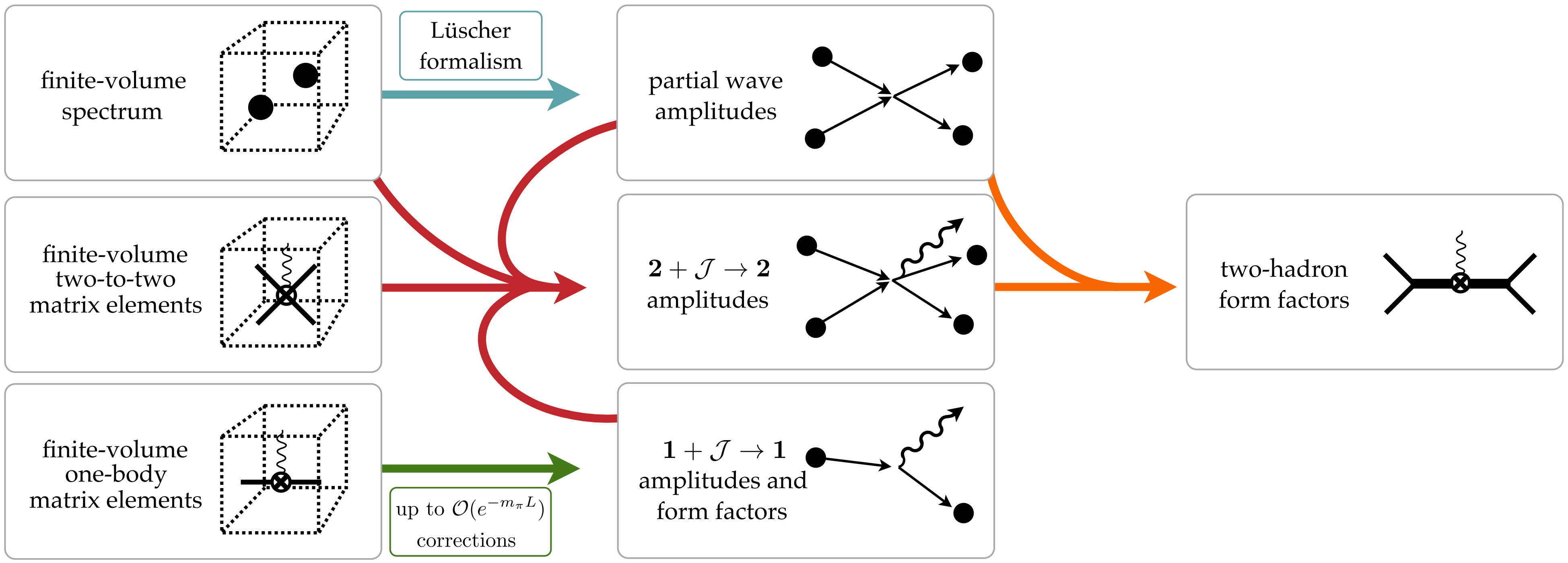
Resonance structure and production

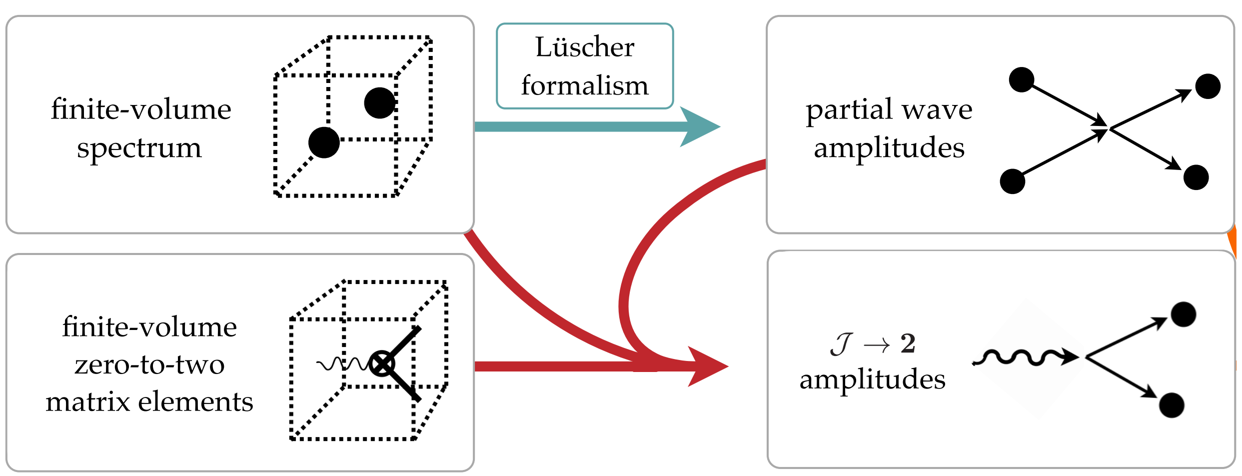


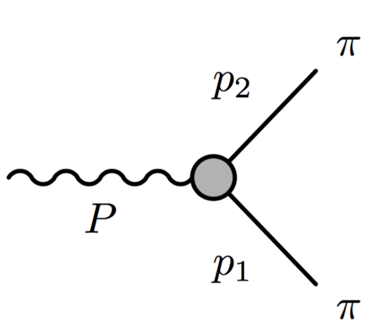
Diagonal elements of the scattering amplitude from [2009.10034 Woss et al.]

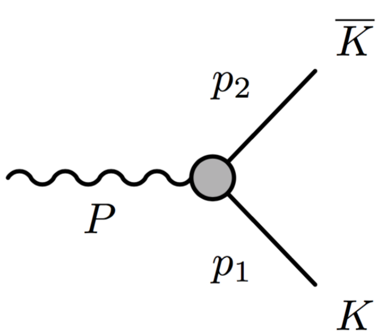


Finite-volume roadmap









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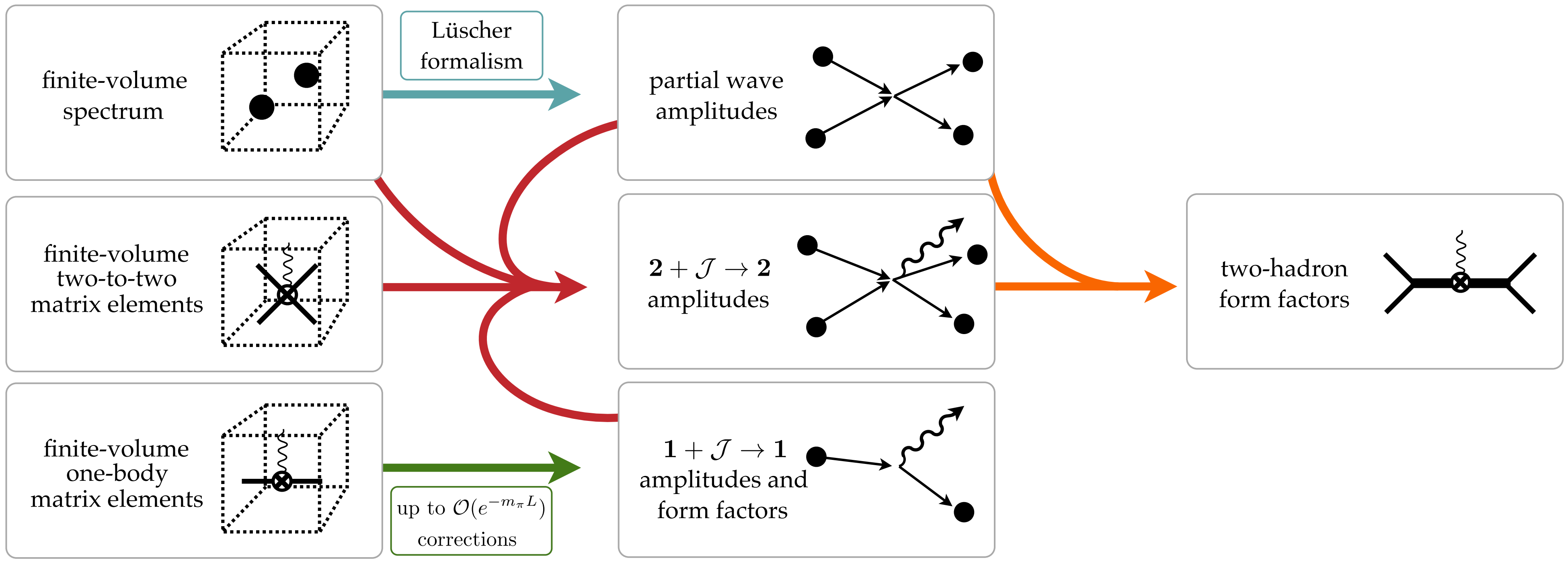
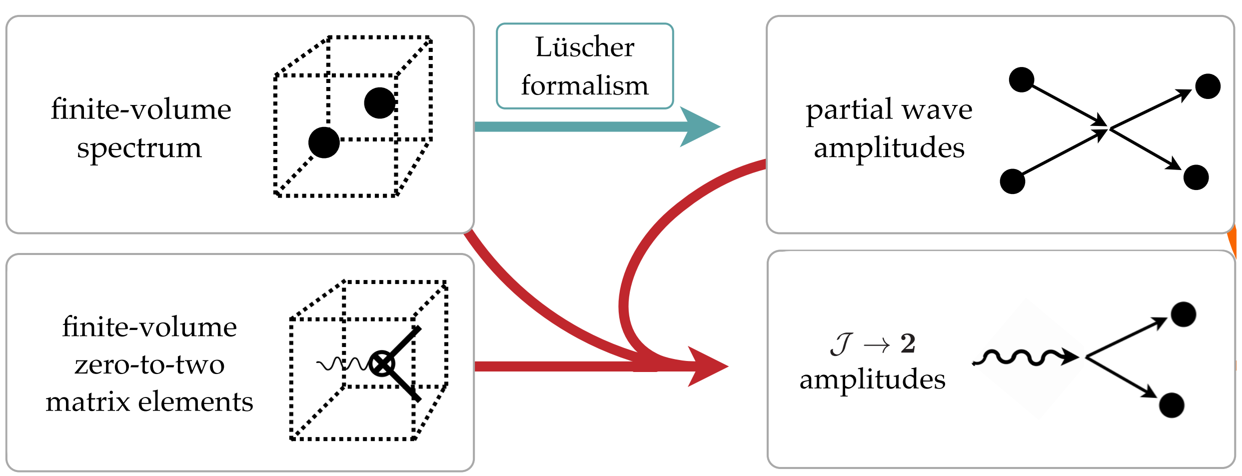
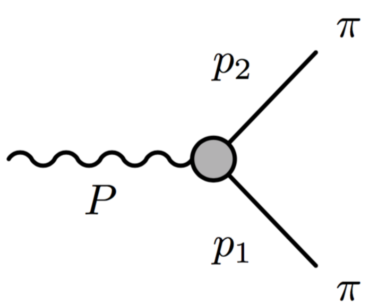
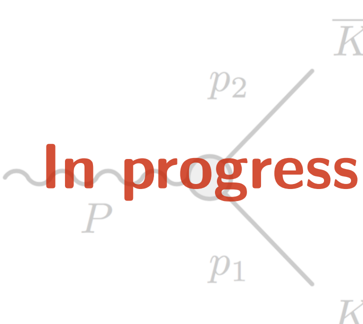


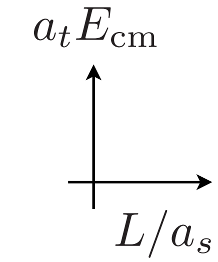
This talk:

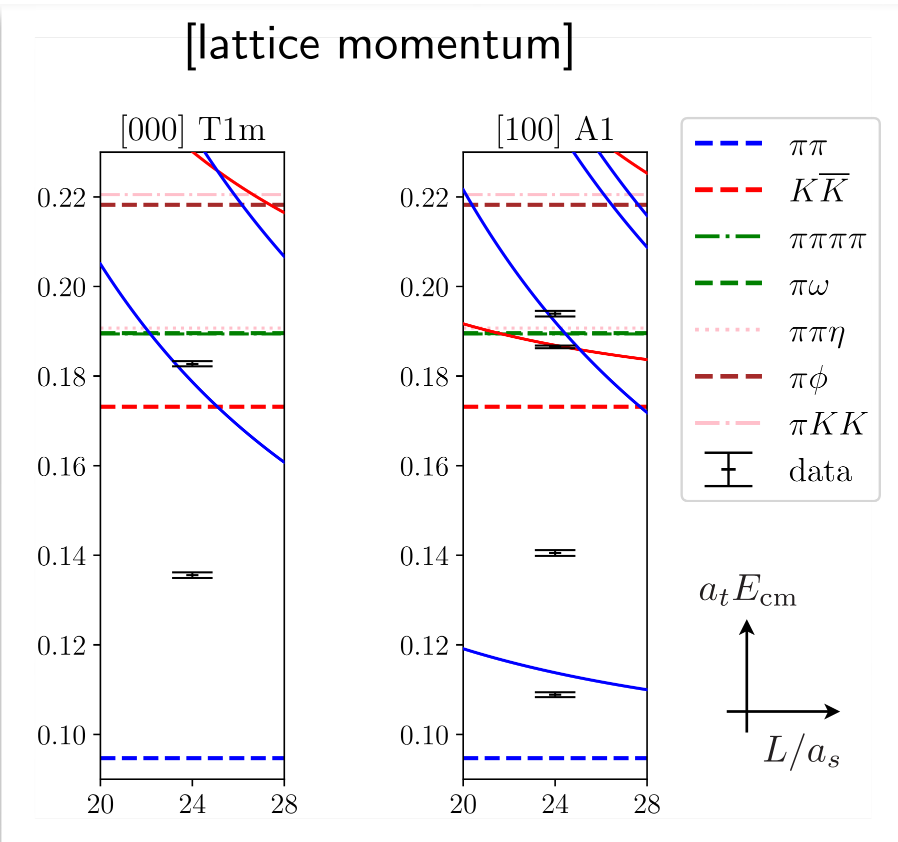


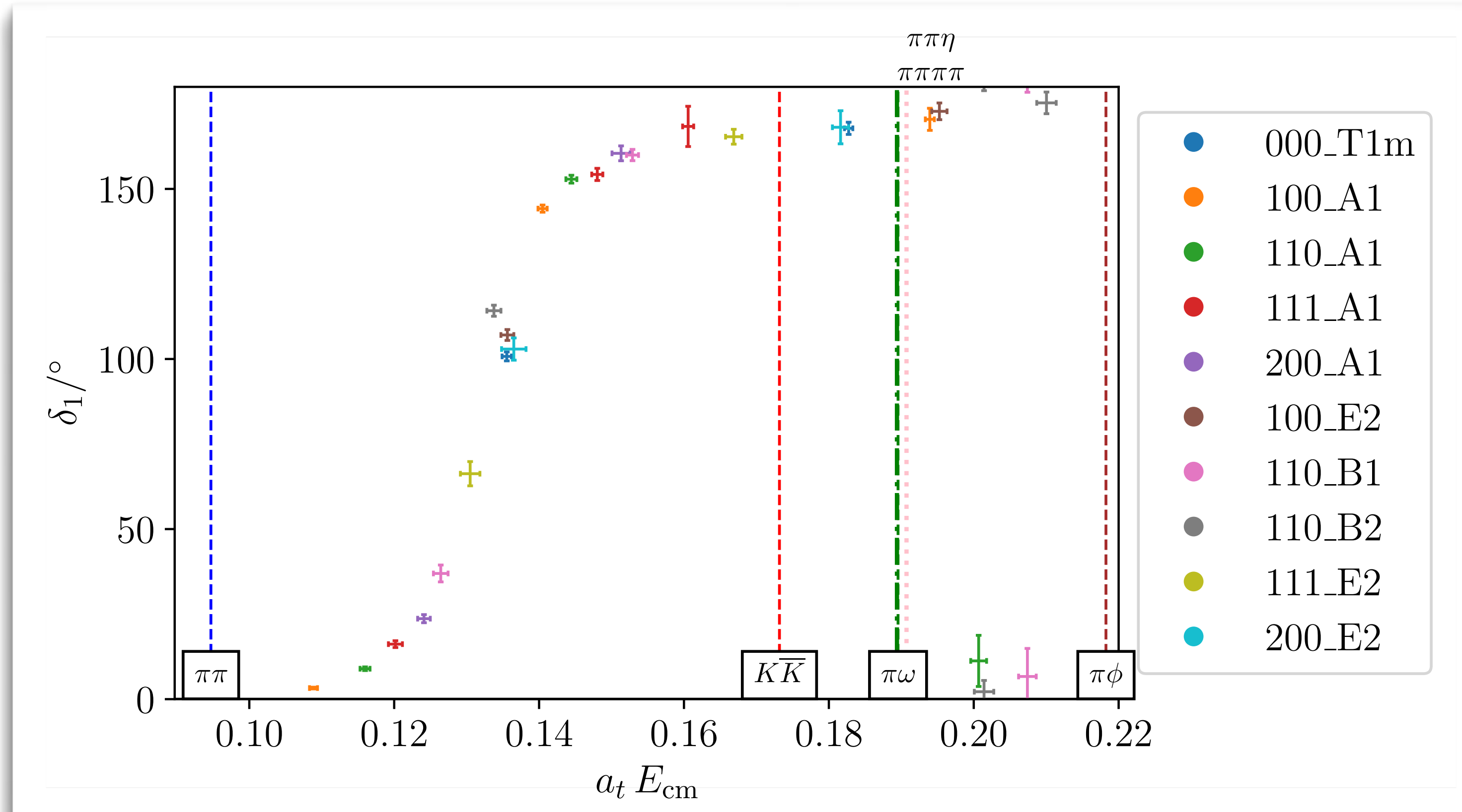




Scattering in the finite volume



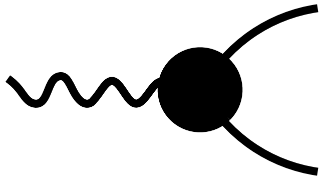




$$\det\left(F^{-1}(E_n, L) + \mathcal{M}(E_n)\right) = 0$$

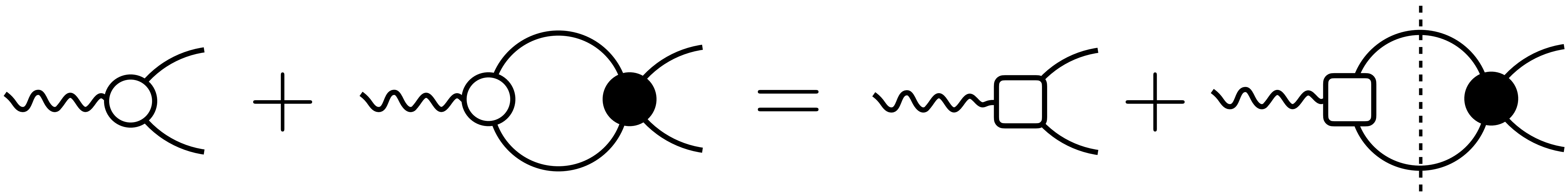
Finite-volume correlation function poles in momentum space

Known geometric function

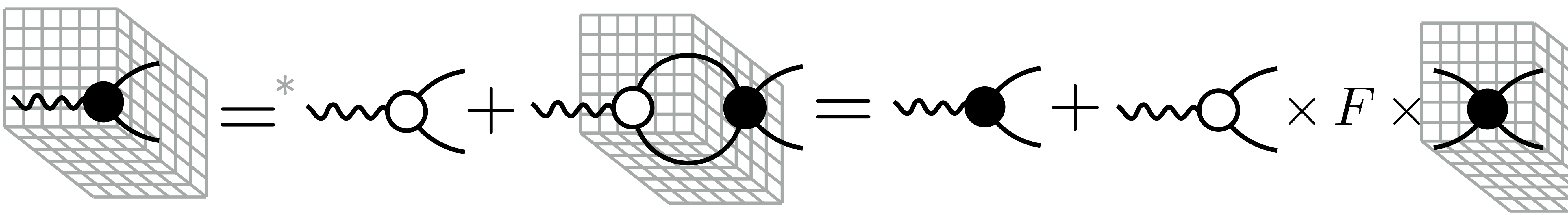


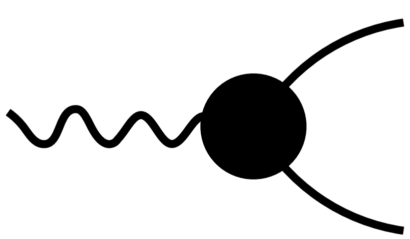
Finite volume corrections

Infinite volume: Watson's theorem. $\mathcal{H}(s) = \mathcal{A}_{02}(s)\,\mathcal{M}(s)$

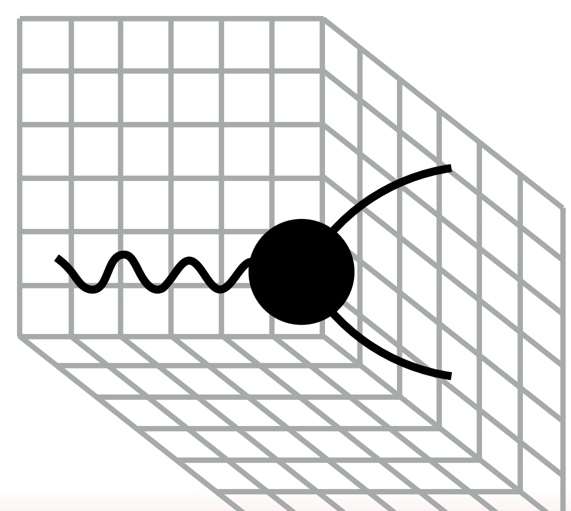


Finite volume: Lellouch-Lüscher factor.



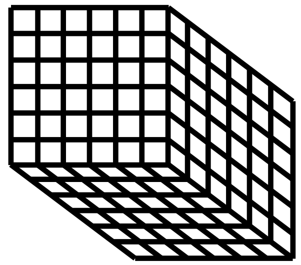


$$\times \lim_{E\to E_n} \frac{E - E_n}{F^{-1} + \mathcal{M}} =$$



*Up to exponentially small corrections

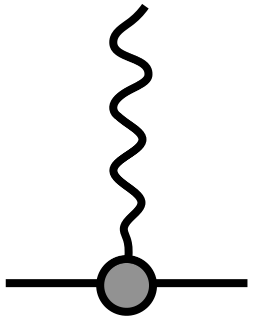
Scattering and pair-production $J^P(I^G) = 1^-(1^+)$



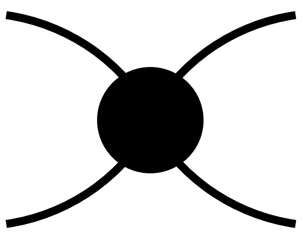
$L^3 \times T = 24^3 \times 256$

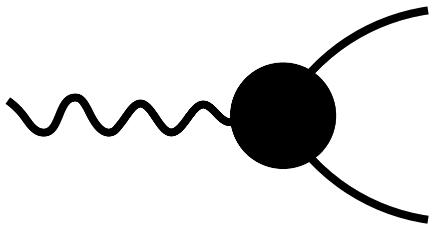
$L = 2.7$ fm

	$a_t m$	m/MeV
π	0.0474	284
K	0.0866	519



For renormalization of the current





$$C_{ab}(t - t_s) = \left\langle O_a(t) O_b^\dagger(t_s) \right\rangle$$

- Two-meson operators
- $q\bar{q}$ operators

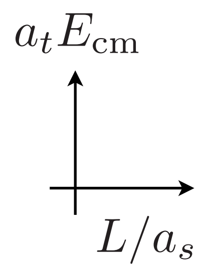
$$\lim_{t\to\infty} \frac{\langle \mathcal{J}(t)\Omega_n^\dagger(0)\rangle}{\sqrt{\left\langle \Omega_n(t)\Omega_n^\dagger(0)\right\rangle}} = \mathcal{H}_L Z_n^{-1/2}$$

- Optimized operators

1. GEVP $\quad C_{ab}(t) v_b^n = C_{ab}(t_0) v_b^n \lambda_n(t - t_0)$
2. Fit eigenvalues $\quad \lambda_n(t - t_0) = e^{-E_n(t-t_0)}$

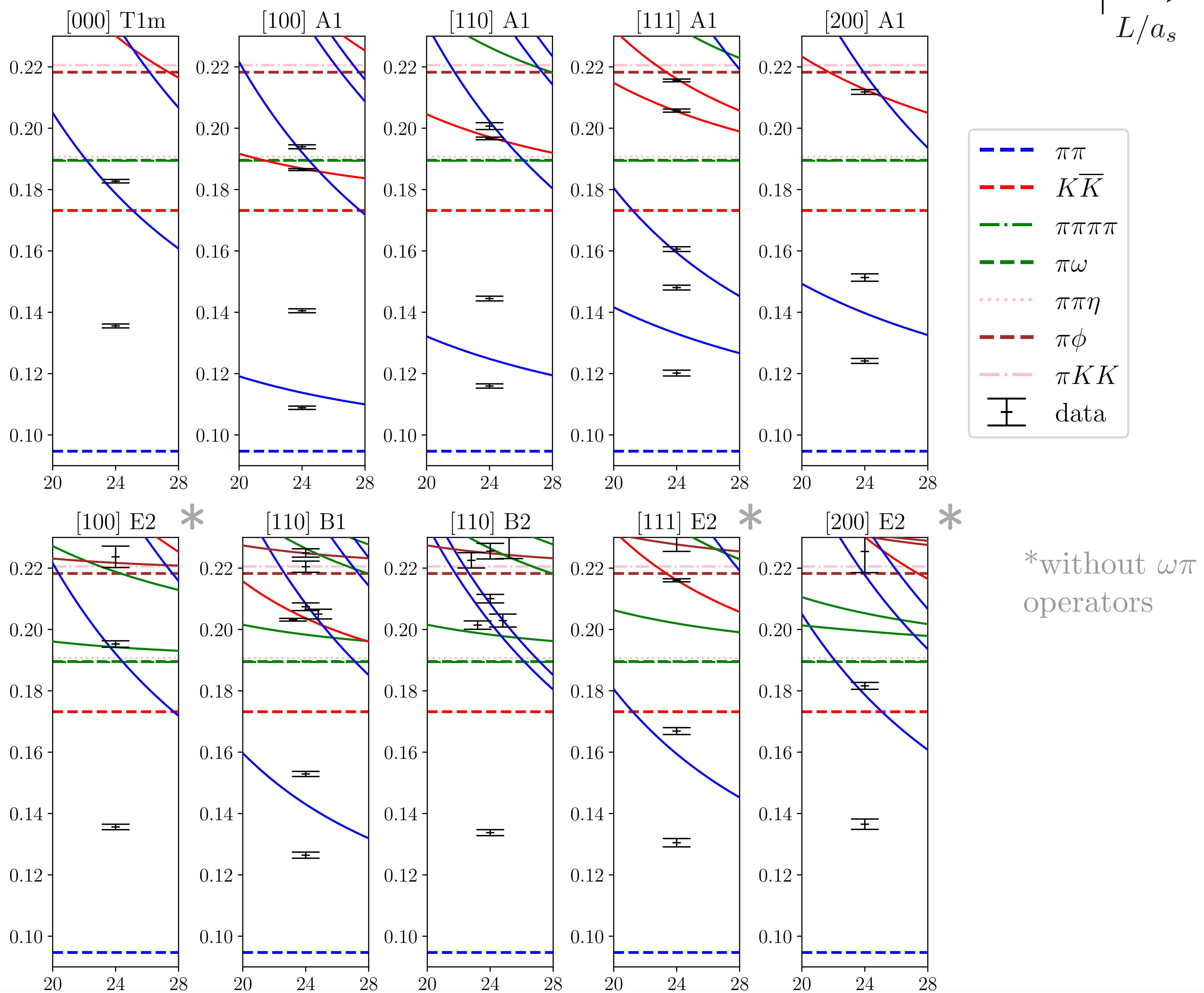
1. Extract invariant $\quad \mathcal{H}_L^\mu = K^\mu \mathcal{F}_L$
2. FV correction $\quad \mu_0'^\star = \frac{\partial}{\partial E^\star}(\mathcal{M}^{-1} + F)$

Finite volume spectrum



- 17 elastic levels
- 7 above KK thr

- ✦ 25 $\pi\pi$-like levels
- ✦ 7 KK-like levels



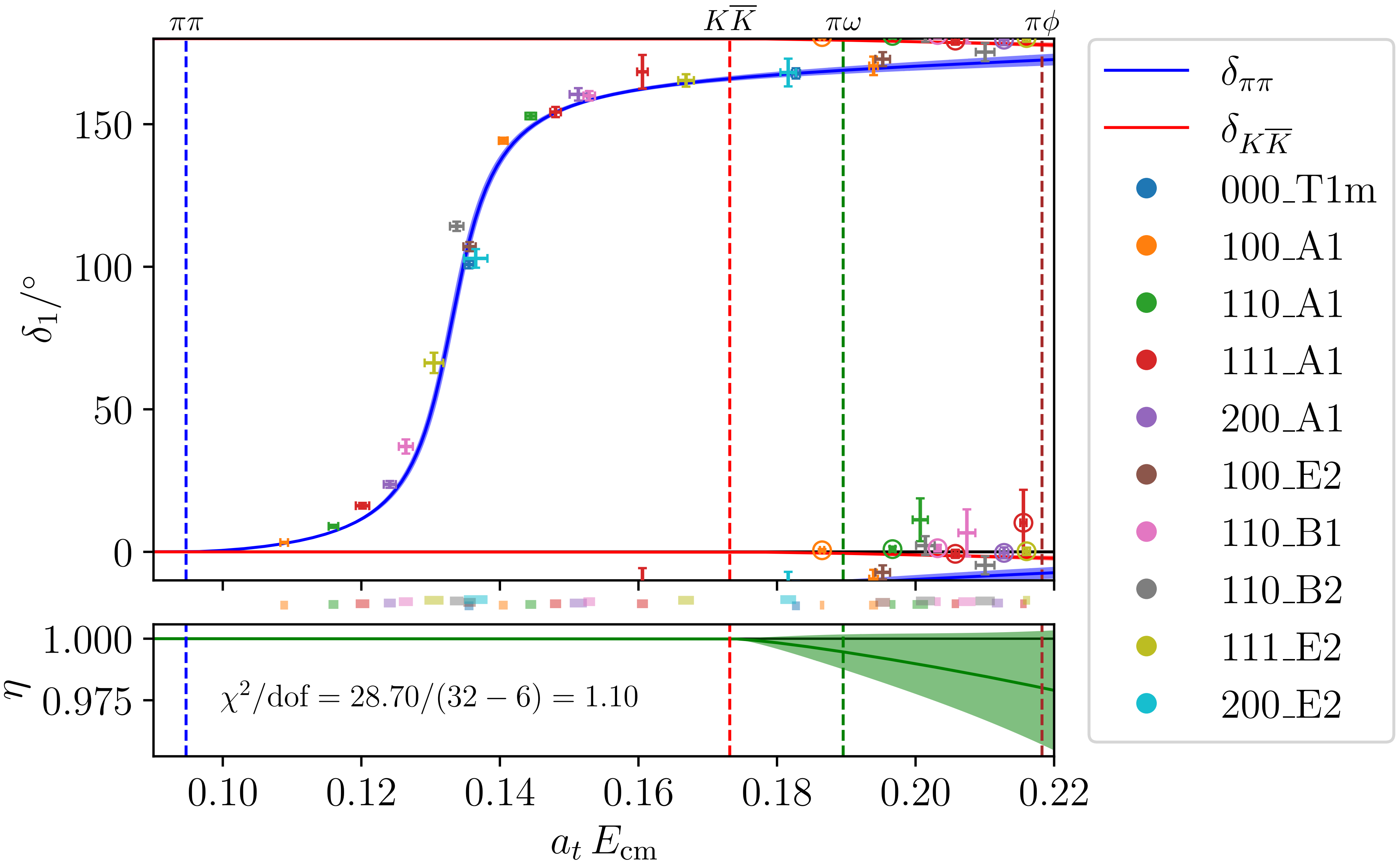
Coupled channel fit

$$\mathcal{M}_{\ell,aa} = \frac{\eta e^{2i\delta_{\ell,a}} - 1}{2i\rho_a}$$

$$\mathcal{M}^{-1}_{\ell,ab} = \frac{1}{(2q^\star_a)^\ell} K^{-1}_{\ell,ab} \frac{1}{(2q^\star_b)^\ell} - i\rho_{\mathrm{CM},ab}$$

$$K_{ab} = \frac{g_a g_b}{-s + m_r^2} + \gamma_{ab}$$

$$\mathcal{M}_{\ell,ab} = \frac{\sqrt{1-\eta^2}\, e^{i(\delta_{\ell,a}+\delta_{\ell,b})}}{2\sqrt{\rho_a \rho_b}}$$

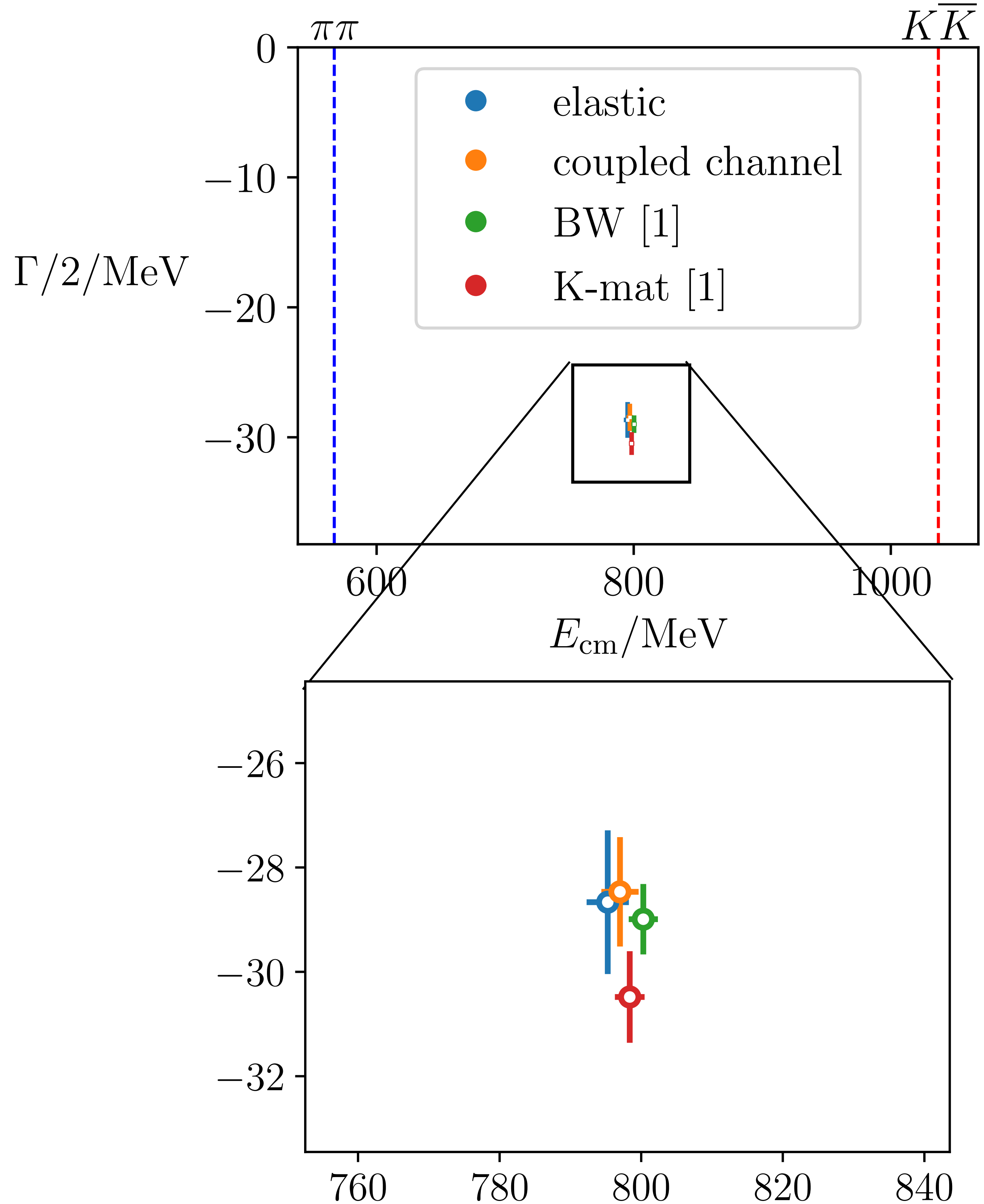


Rho resonance

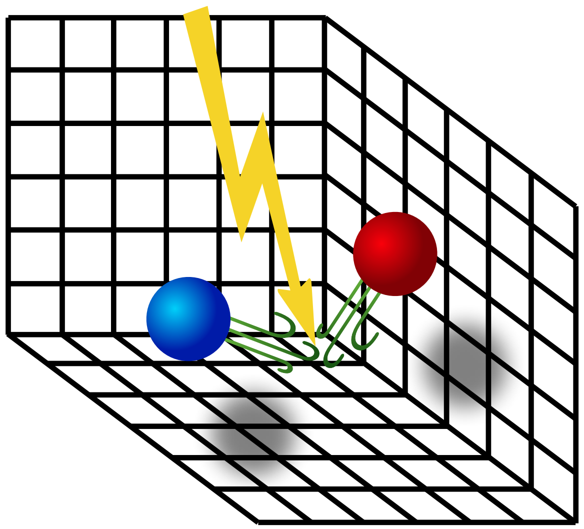
$$\mathcal{M}(s) \sim \frac{g_R^2}{(m_\rho - i\Gamma_\rho/2)^2 - s}$$

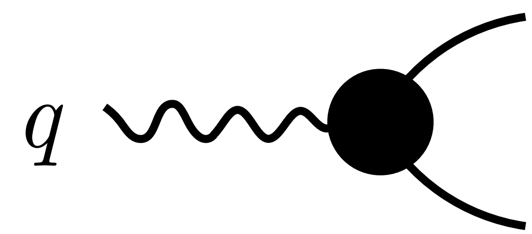
$$g_R = 254 \pm 5 \text{ MeV}$$

$$\text{Re}(m_\rho) = 797 \pm 2.6 \text{ MeV}$$
$$\Gamma_\rho/2 = 28.5 \pm 1.0 \text{ MeV}$$

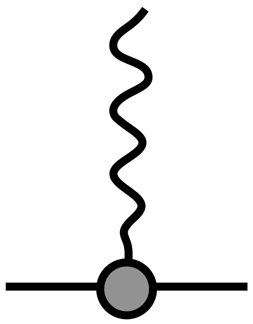


Pair production





$q^2 < 0$



$$\mathcal{H}_L^\mu = K^\mu \mathcal{F}_L$$

$$\mathcal{H}^\mu(q) = K^\mu f(q^2)$$

Finite-volume matrix elements

$$\lim_{t\to\infty} \frac{\left\langle \mathcal{J}(t)\Omega_n^\dagger(0)\right\rangle}{\left\langle \Omega_n(t)\Omega_n^\dagger(0)\right\rangle} = \mathcal{H}_L Z_n^{-1/2}$$

$$f = \frac{\mathcal{M}}{q^{\star 2}} \frac{1}{\tilde{r}_n} \mathcal{F}_L$$

Lellouch-Lüscher factor

$q^\star$

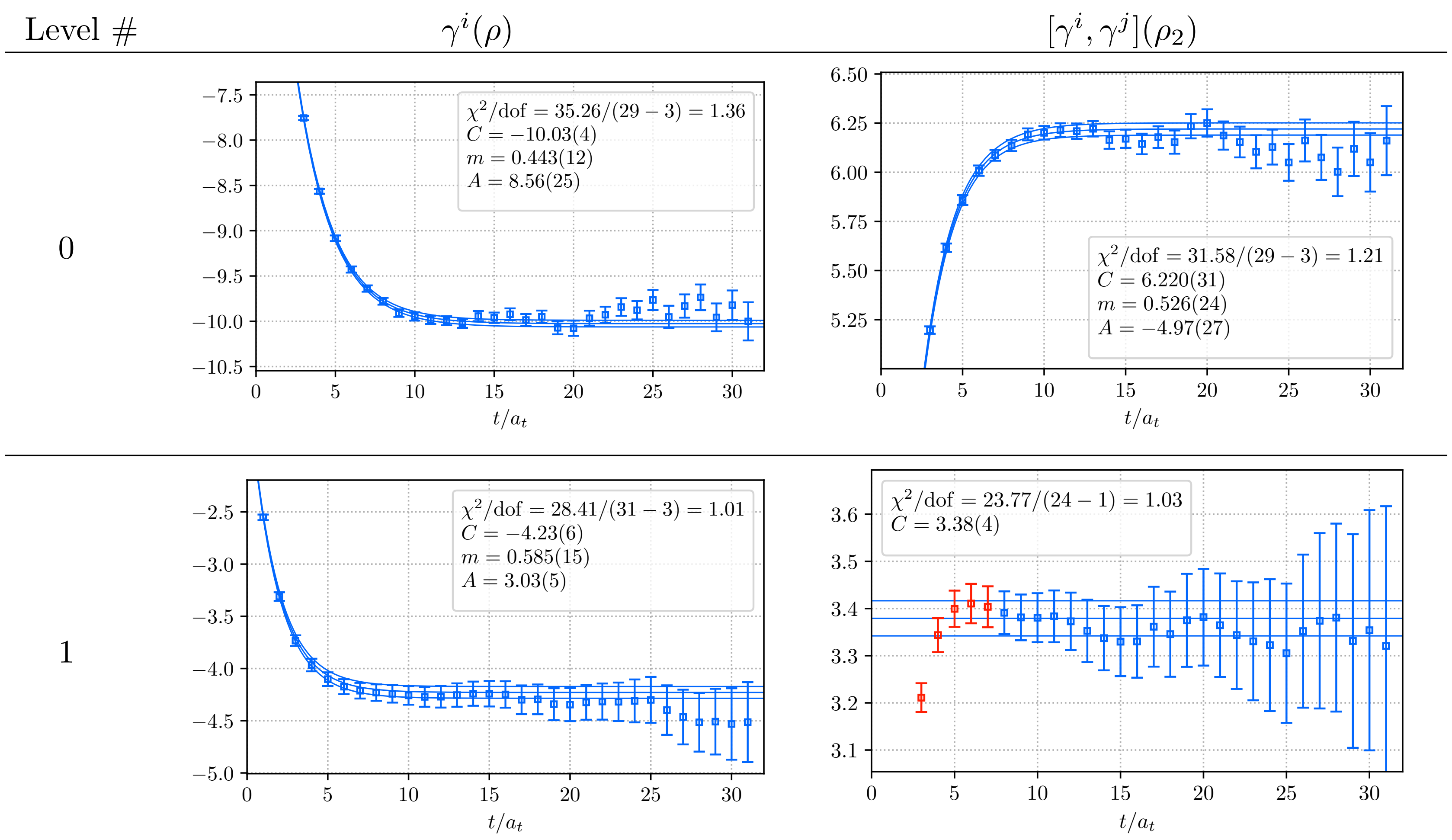
Relative momentum in cm frame

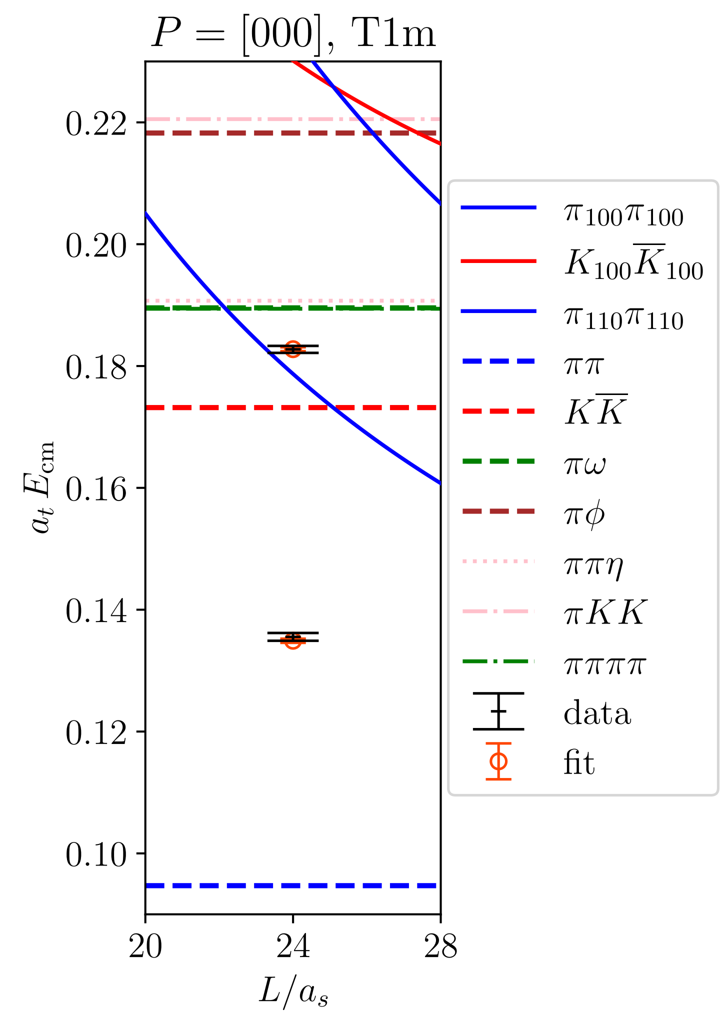
Matrix elements

$$\frac{\langle \mathcal{J}(t)\Omega_n^\dagger(0)\rangle}{\left\langle \Omega_n(t)\Omega_n^\dagger(0)\right\rangle} = \langle 0|\mathcal{J}(0)|n\rangle\, Z_n^{-1/2} + \mathcal{O}(e^{-(E_N-E_n)t})$$

Optimized operators

Fit function: $C + Ae^{-mt}$





anisotropy

$$\langle 0|\mathcal{J}^\rho_{\text{impro}}|\mathfrak{n}\rangle = \langle 0|\rho|\mathfrak{n}\rangle - \frac{1}{4}(1-\xi)a_t E_n\,\langle 0|\rho_2|\mathfrak{n}\rangle$$

Matrix elements

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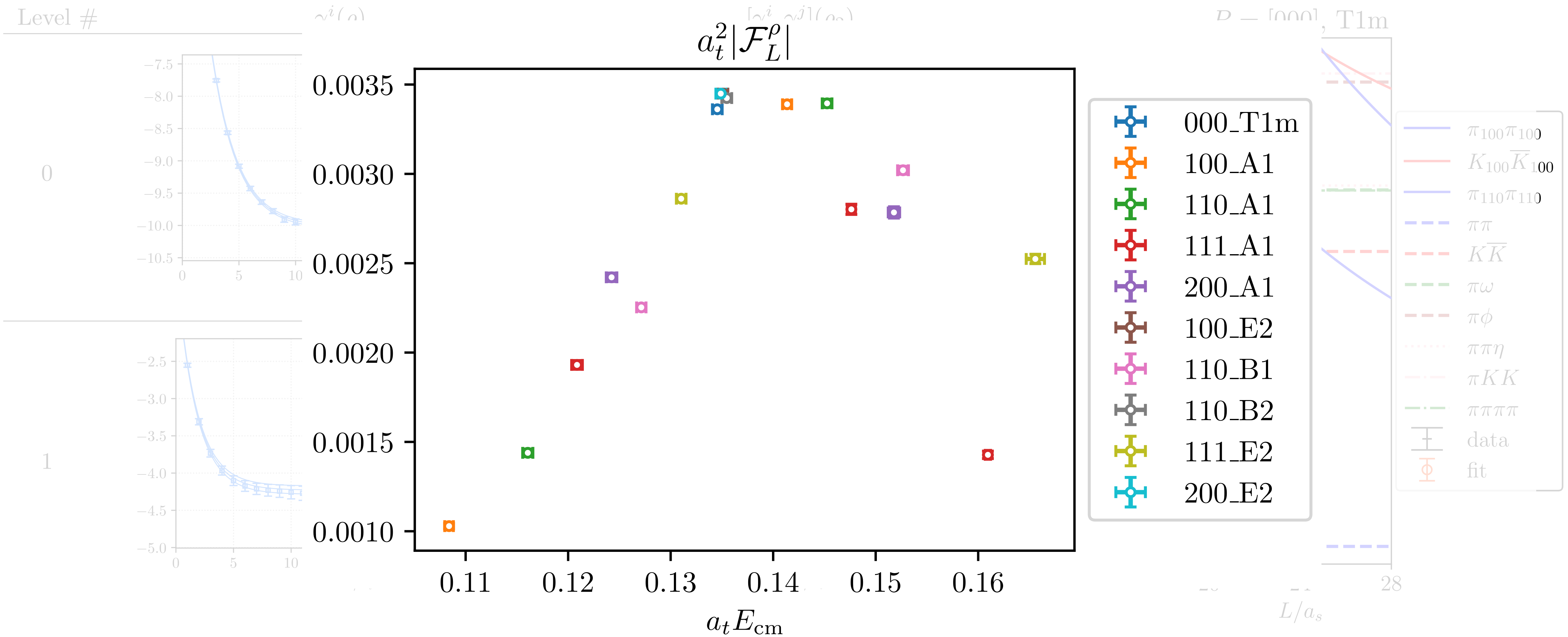

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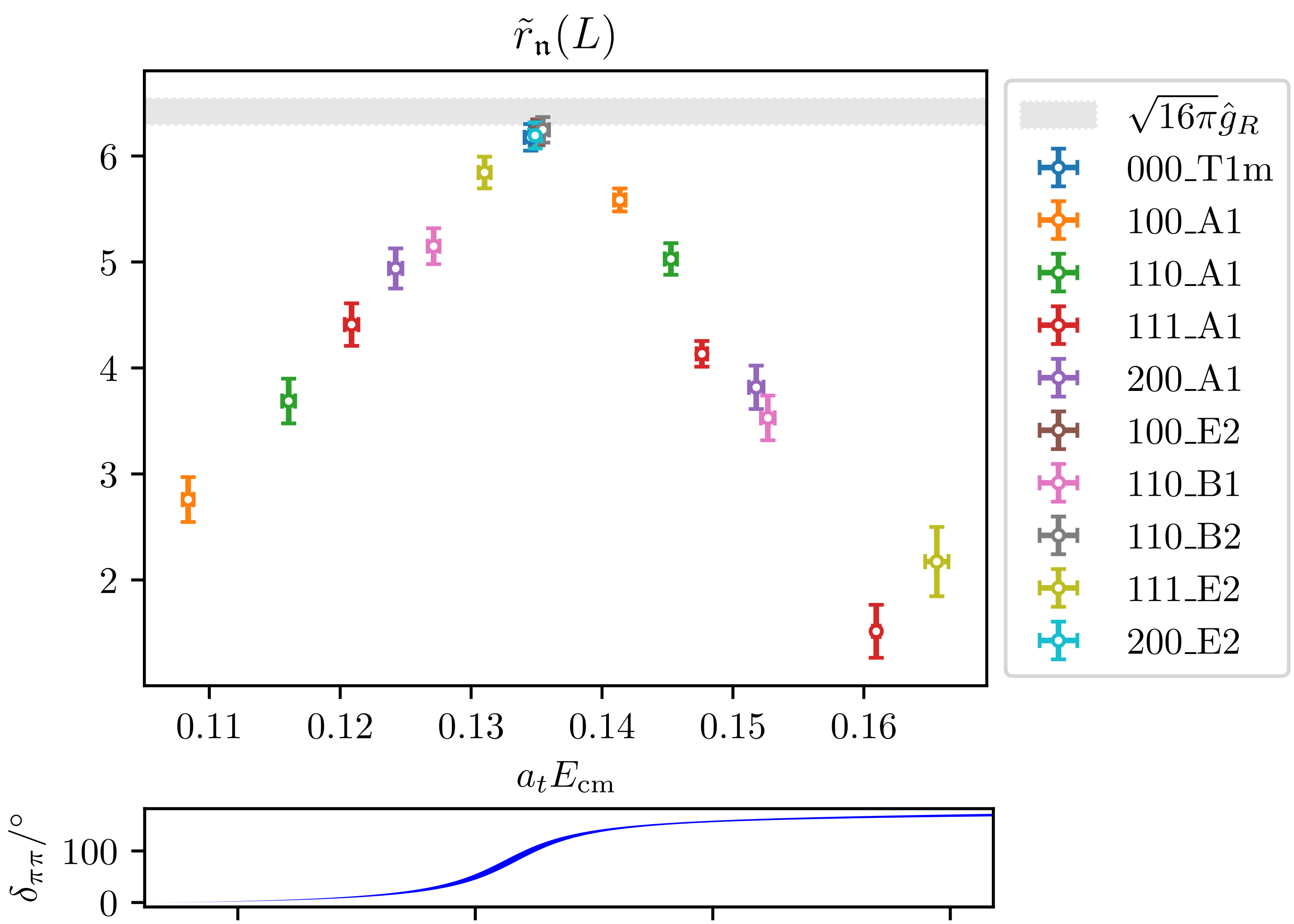
Finite volume correction

- Lellouch-Lüscher factor

$$\mu_0'^{\star} = \frac{\partial}{\partial E^\star}\left(\mathcal{M}^{-1} + F\right)\Big|_{E^\star = E_n^\star}$$

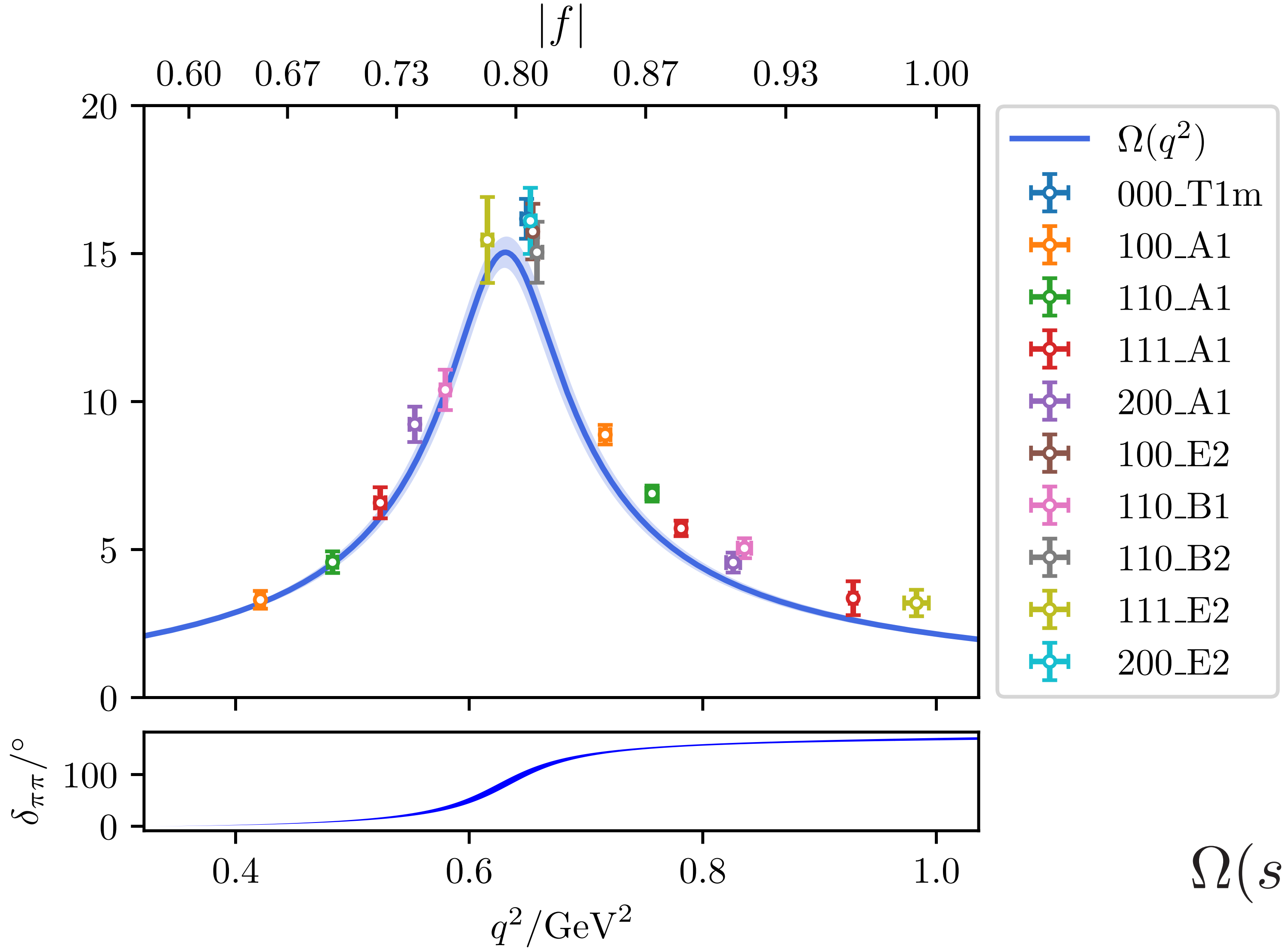
$$\tilde{r}_{\mathfrak{n}} = \frac{1}{q^\star}\sqrt{\frac{-2E_n^\star}{\mu_0'^{\star}}} \sim \sqrt{16\pi g_R/q^\star} \equiv \sqrt{16\pi}\hat{g}_R$$

In the narrow width limit



Timelike form factor

$$f = \frac{\mathcal{M}}{q^{\star 2}} \frac{1}{\tilde{r}_n} \mathcal{F}_L$$



Watson's theorem:

$$f = \Omega \times \mathcal{F}$$

$$\Omega(s) = \exp\left(\frac{s}{\pi} \int_{s_\mathrm{thr}}^{\infty} \mathrm{d}s' \frac{\delta(s')}{s'(s'-s)} \right)$$

High energy ps behavior inspired by [2]

Timelike form factor

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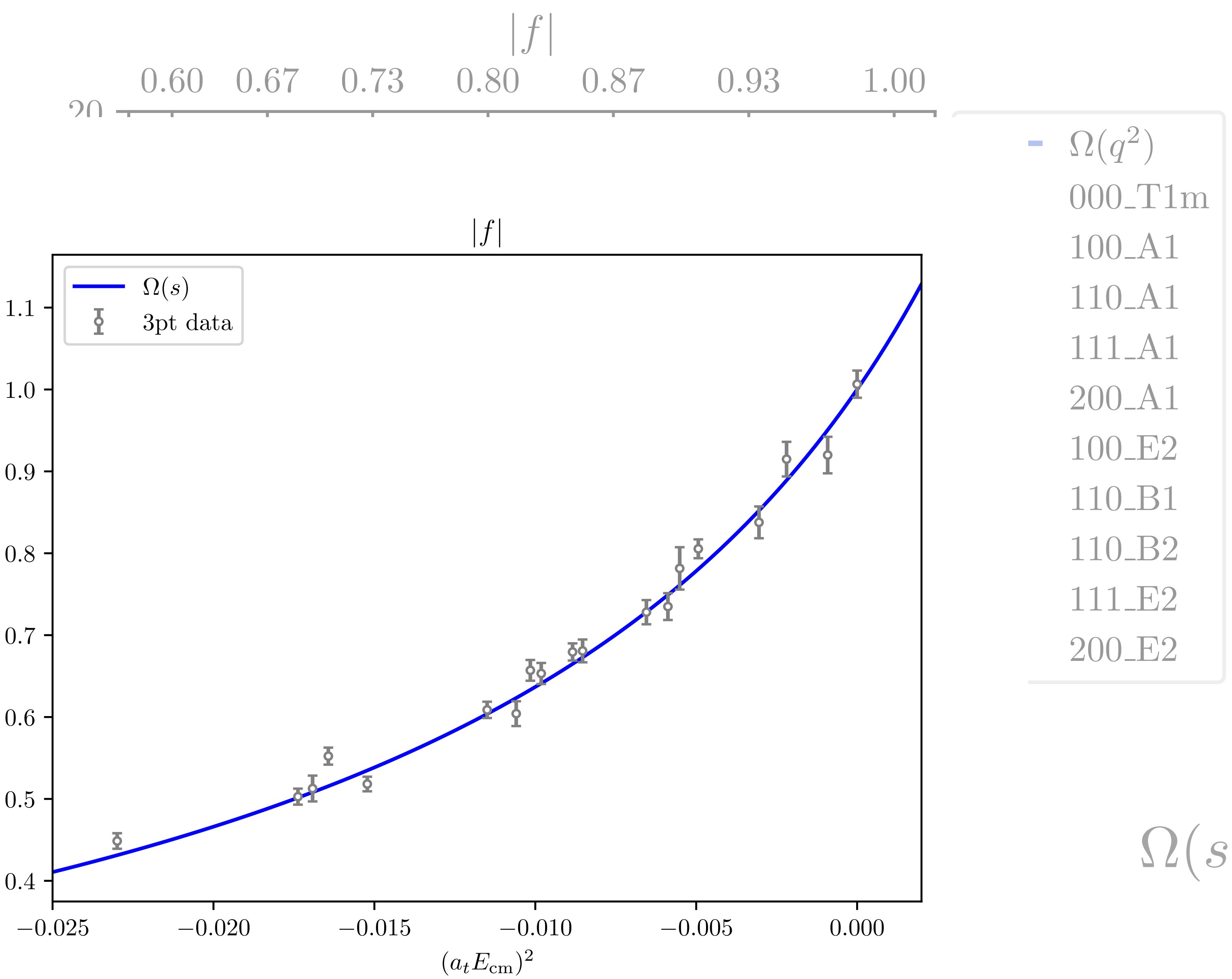

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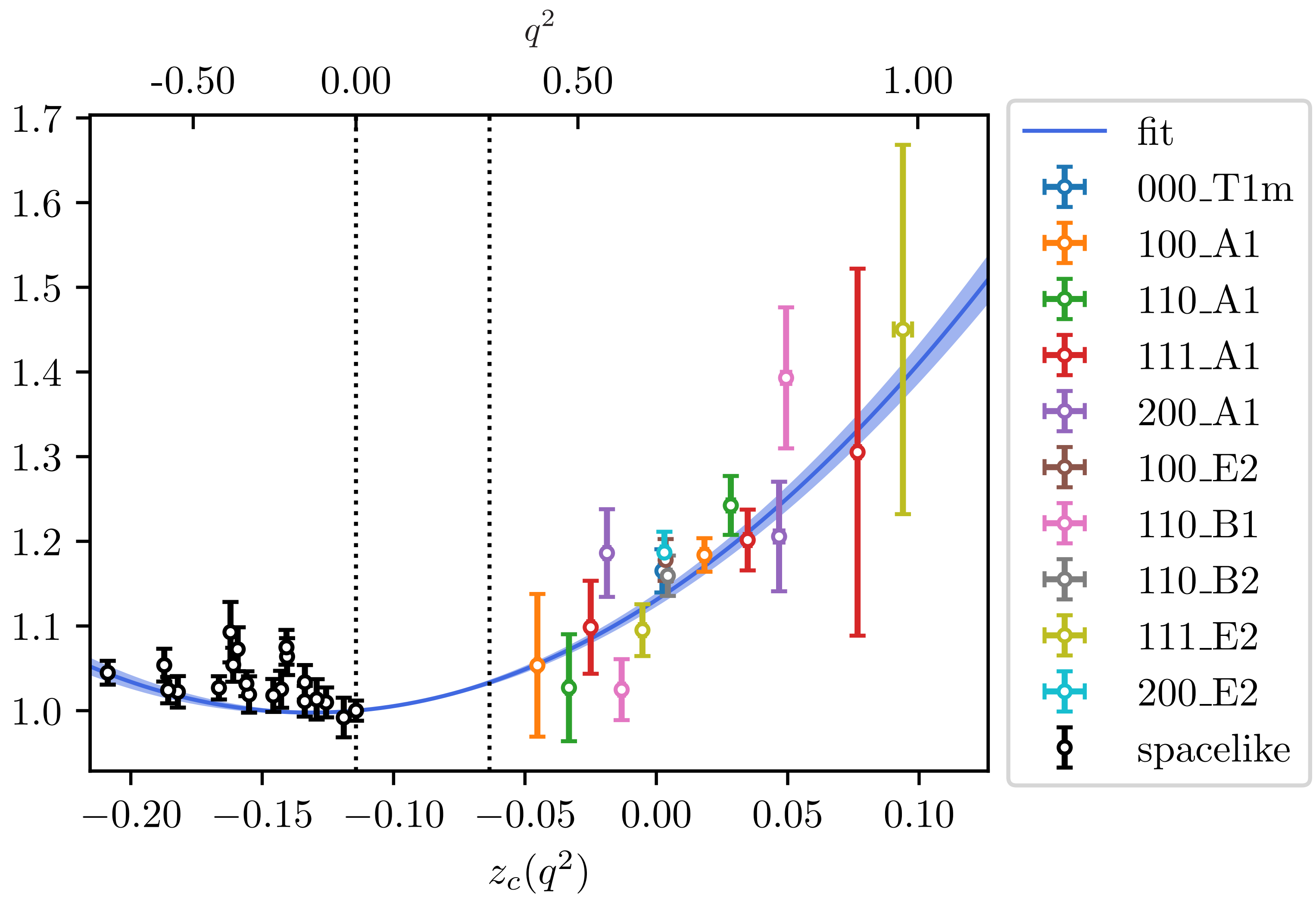
High energy ps behavior inspired by [2]

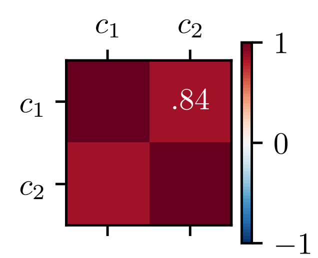
Smooth function fit $\mathcal{F} = f/\Omega$



$$\chi^2/n_{\text{dof}} = 87.7/(37-2) = 2.51$$

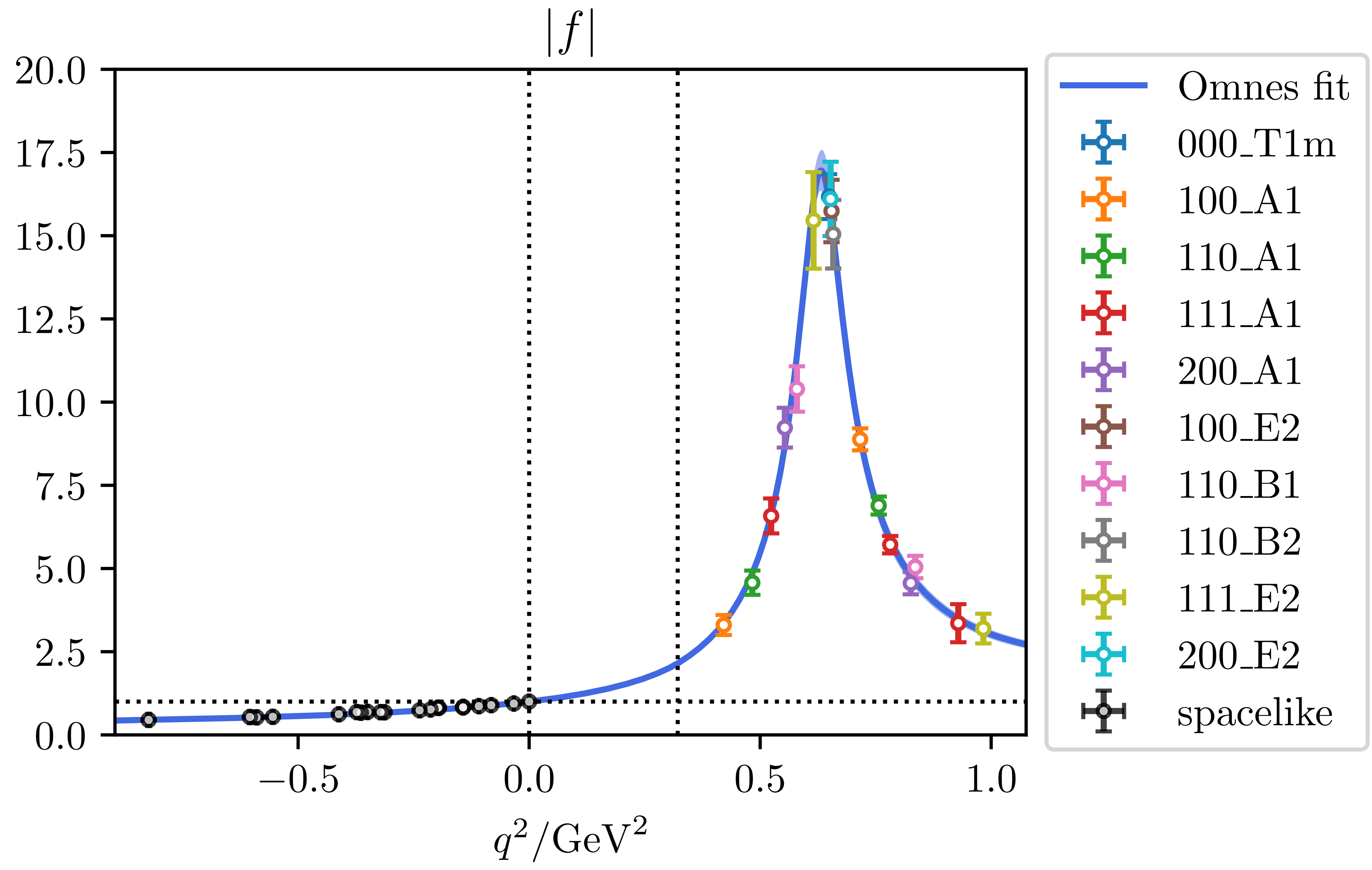
$$\mathcal{F}(q^2) = Q + \sum_{n=1}^{N} c_n\left(z_c(q^2) - z_c(0)\right)^n$$

Q [fixed] $= 1(0)$
$c_1 = 2.03(11)$
$c_2 = 7.7(5)$



$$z_c(q^2) = \frac{\sqrt{s_c - s_{in}} - \sqrt{s_c - q^2}}{\sqrt{s_c - s_{in}} + \sqrt{s_c - q^2}}$$

Form factor fit: spacelike and timelike region



$$\sqrt{\langle r_\pi^2 \rangle} = 0.616(7)\ \mathrm{fm}$$

Form factor fit: spacelike and timelike region

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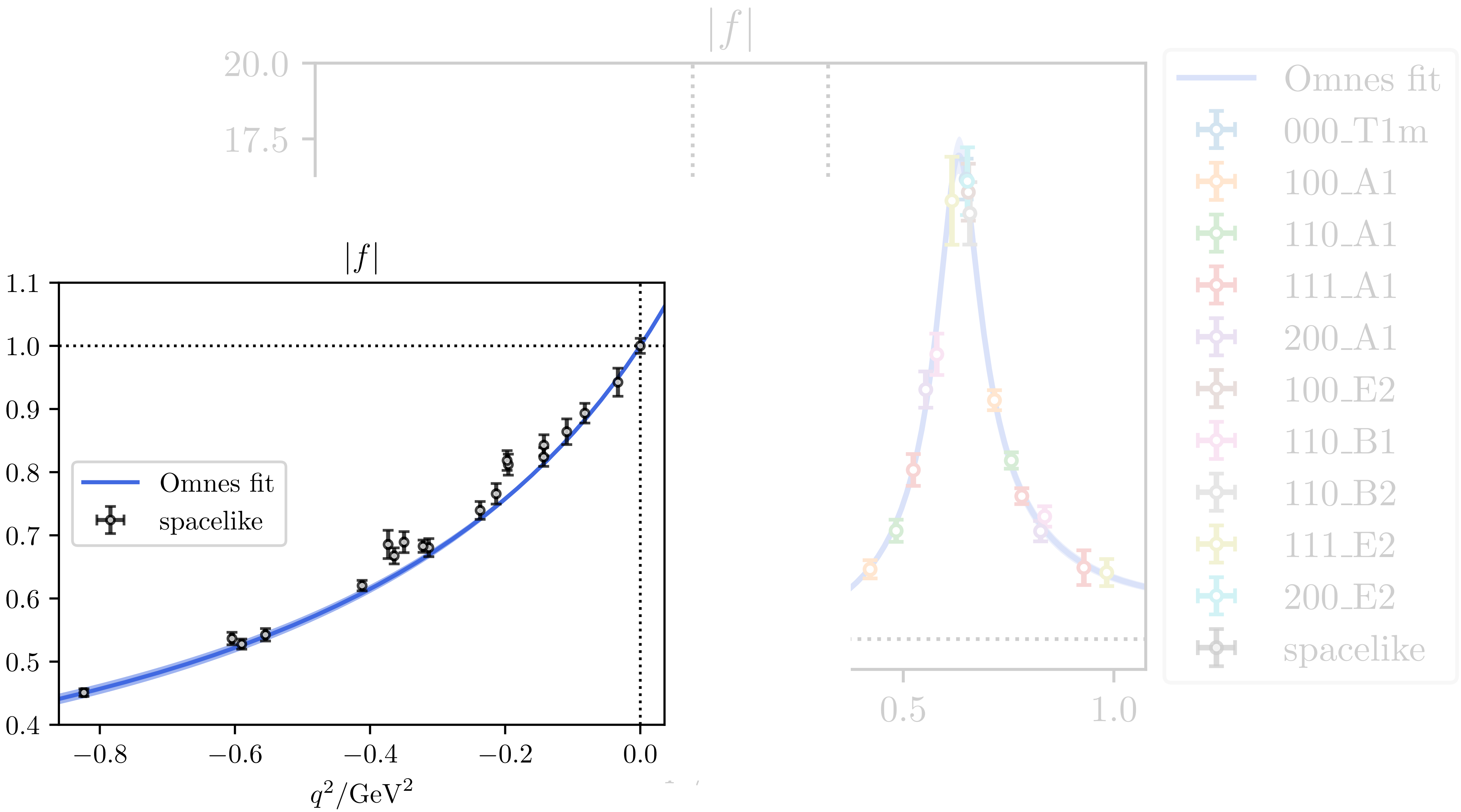
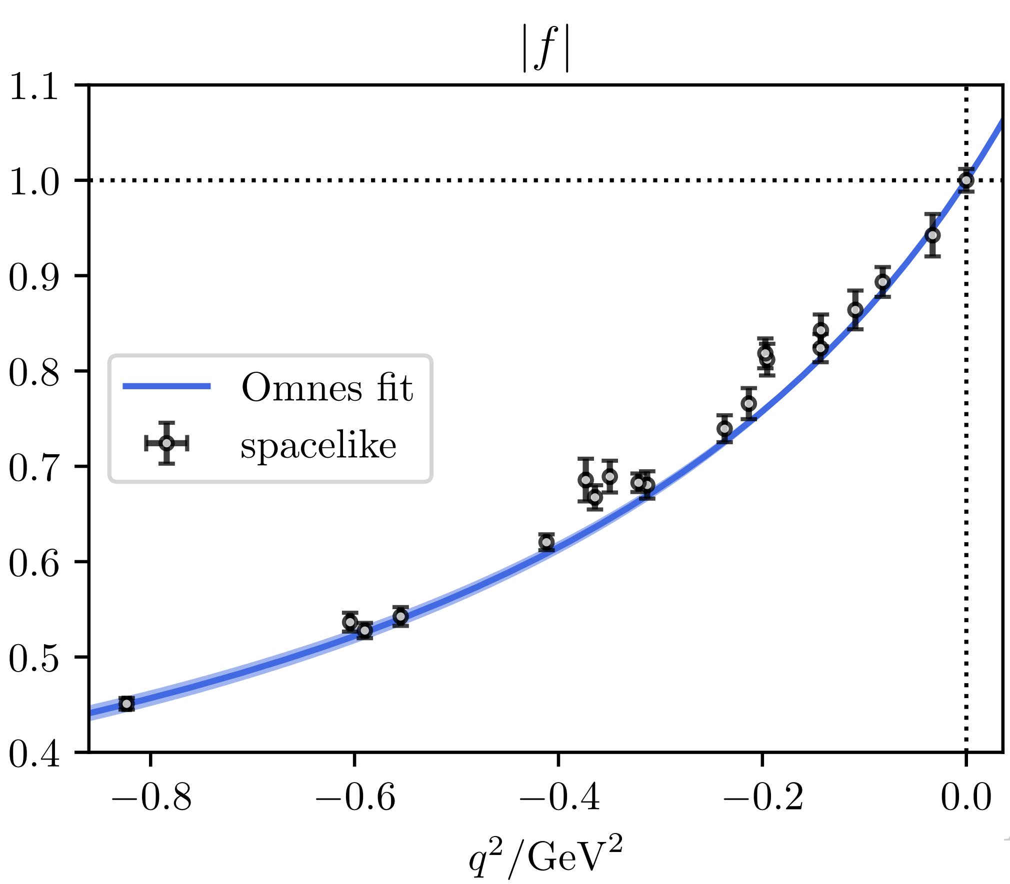
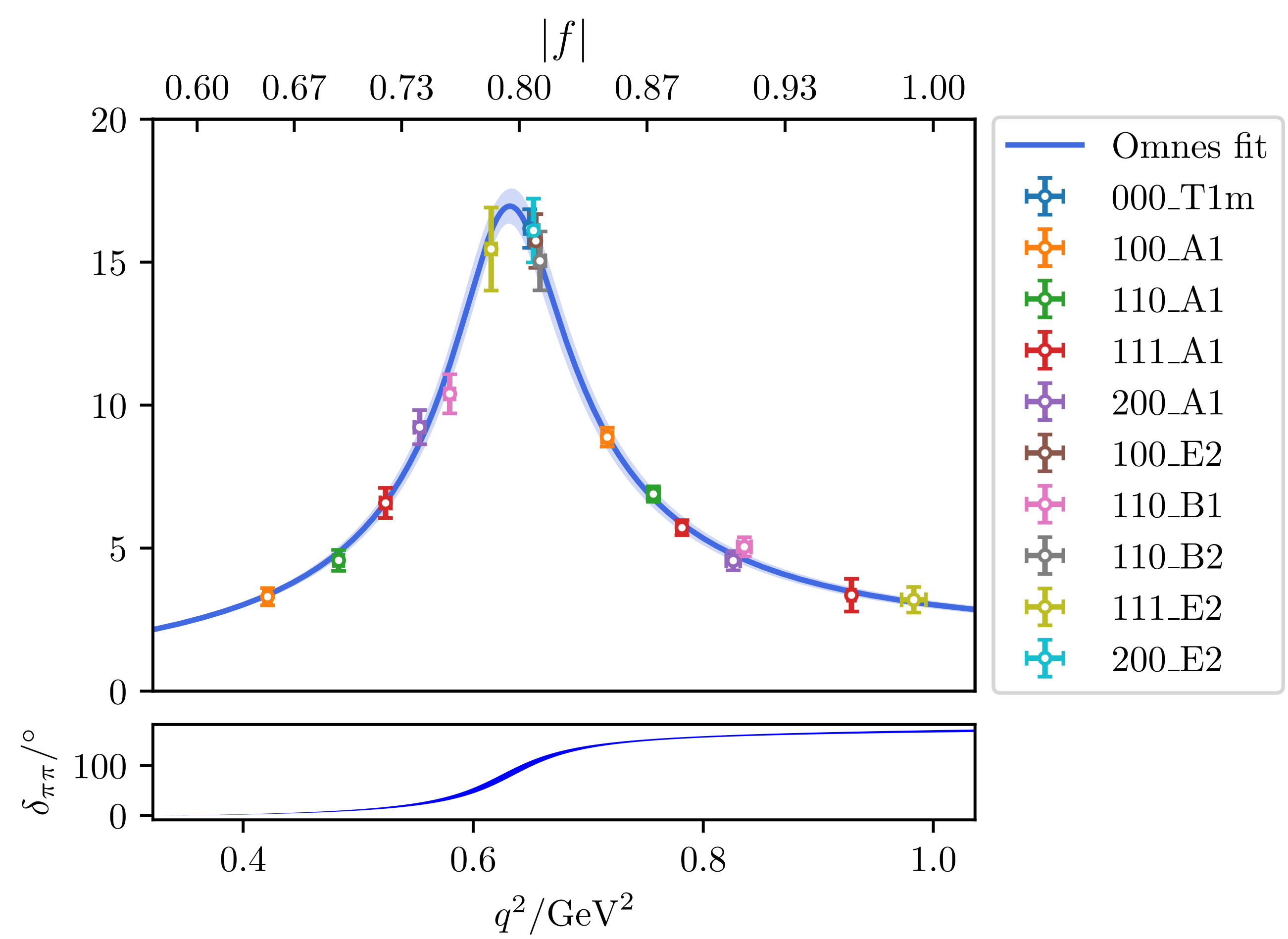


$$\sqrt{\langle r_\pi^2 \rangle} = 0.616(7)\ \mathrm{fm}$$

Form factor fit: spacelike and timelike region



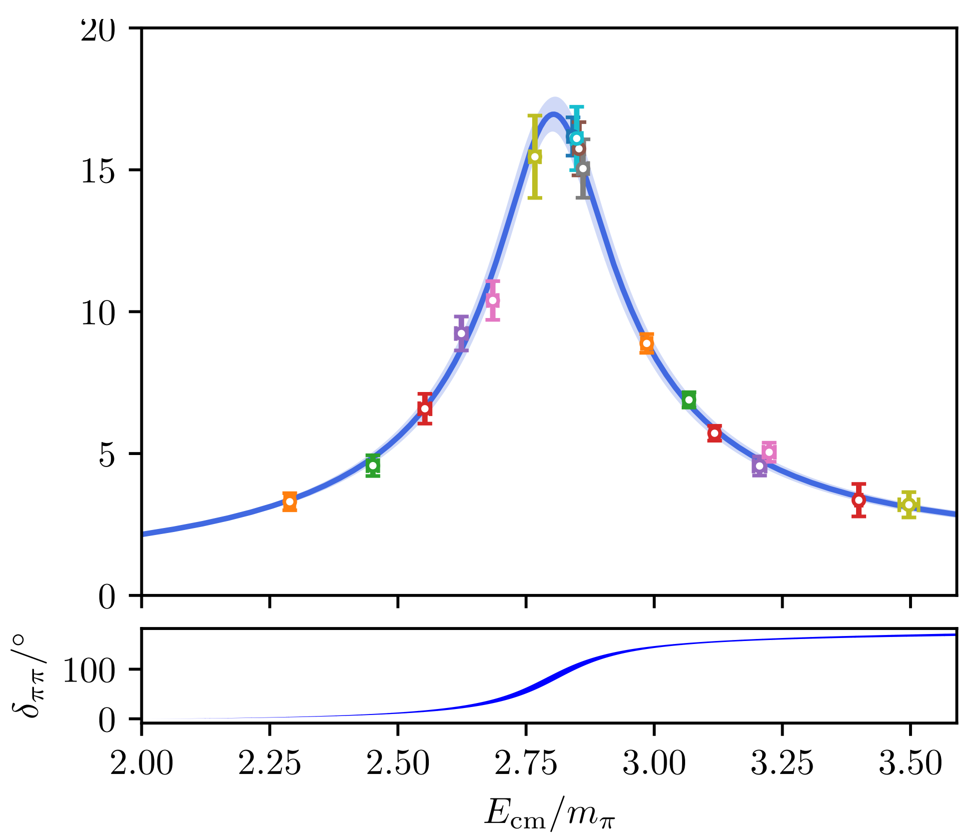


$$\sqrt{\langle r_\pi^2 \rangle} = 0.616(7)\ \mathrm{fm}$$

Consistency with existing results

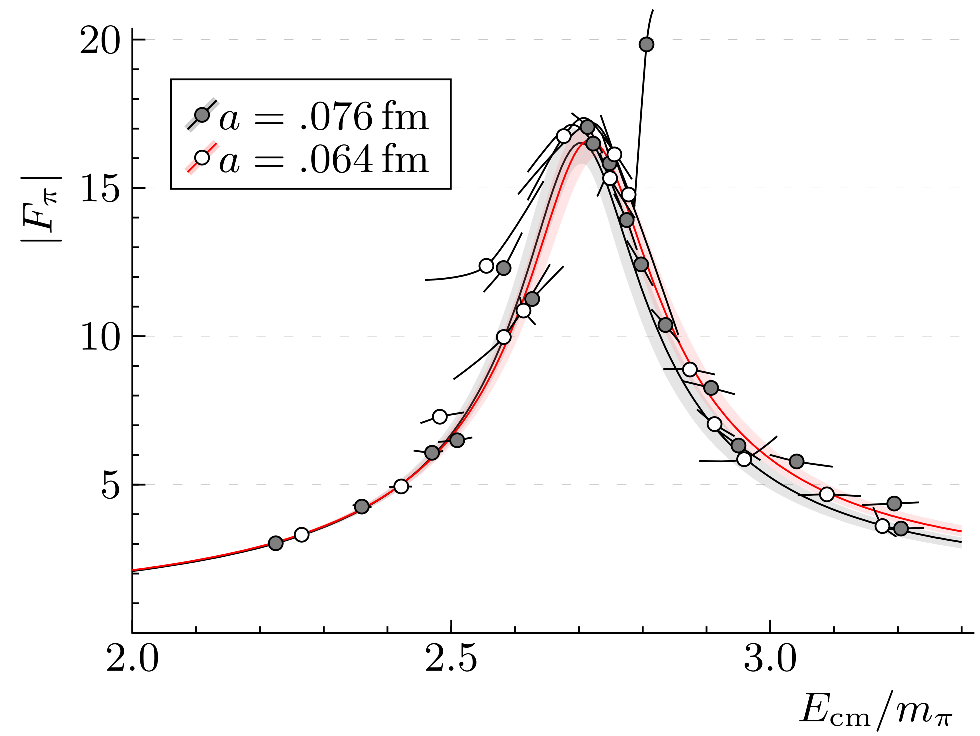
$$m_\pi \approx 280 \text{ MeV}$$

$m_\pi = 284\,\text{MeV},\ m_K = 519\,\text{MeV}$



This work

$m_\pi = 280\,\text{MeV},\ m_K = 460\,\text{MeV}$

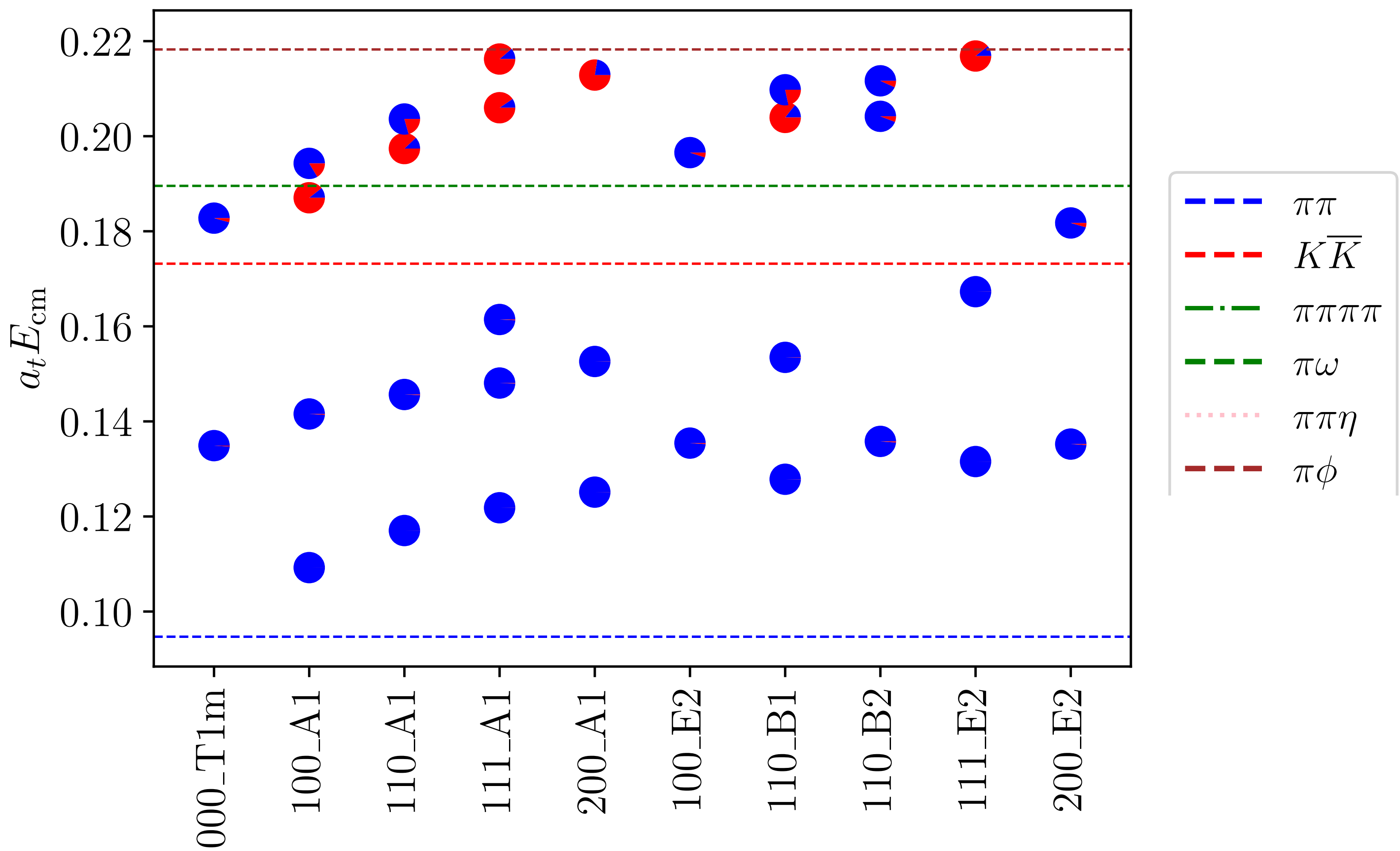


[arXiv:1808.05007] Andersen, et al.

Coupled channel Finite Volume correction

$$\lambda_0'^{\star} w_0 w_0^{\intercal} = \frac{\partial}{\partial E^{\star}} \left(\mathcal{M} + F^{-1} \right)$$

$$\sum_a w_{0,a}^2 = 1$$



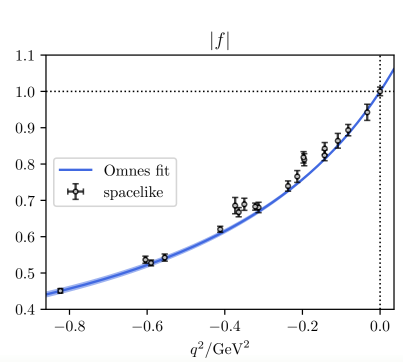
$$\mathcal{F}_L \propto \sum_a w_{0,a} f_a$$

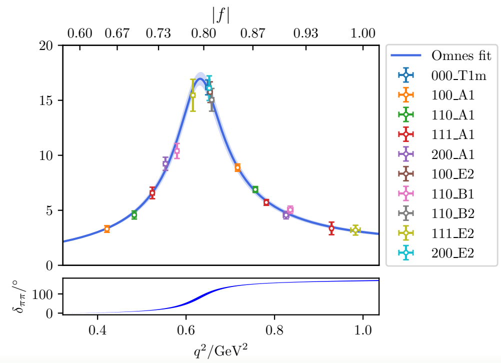
Summary and outlook

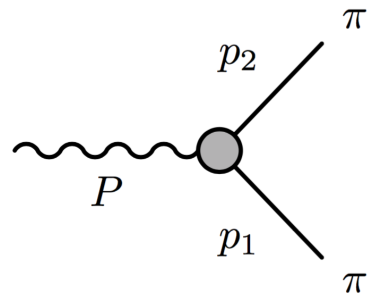
- Isovector p-wave coupled channel $\pi\pi/KK$
  - Finite-volume spectrum
  - Scattering amplitude

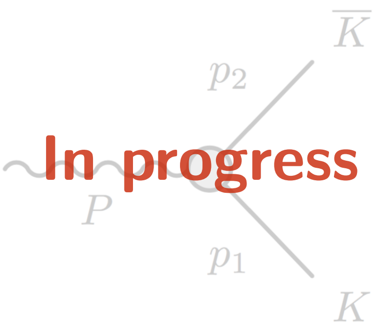
- Pair production amplitude
  - Zero-to-two finite volume matrix elements
  - Lellouch-Lüscher factor
  - Form factor fit across spacelike and timelike region

- Future work to extend the analysis to the coupled channel region.



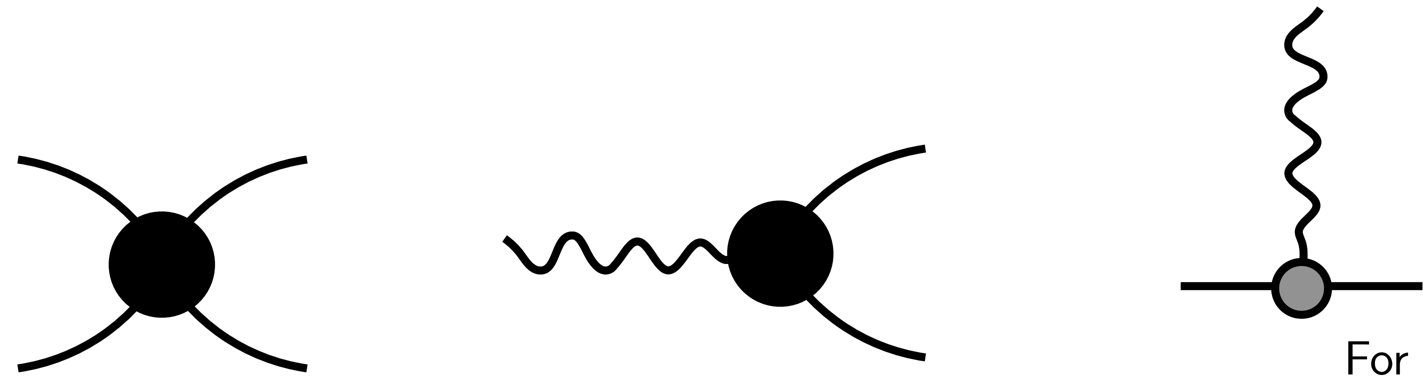






Back up slides

MC statistics



For renormalization of the current

MC configurations	400	348	348
# time sources	4	1	1
Correlation timeslice extent	40	32	32

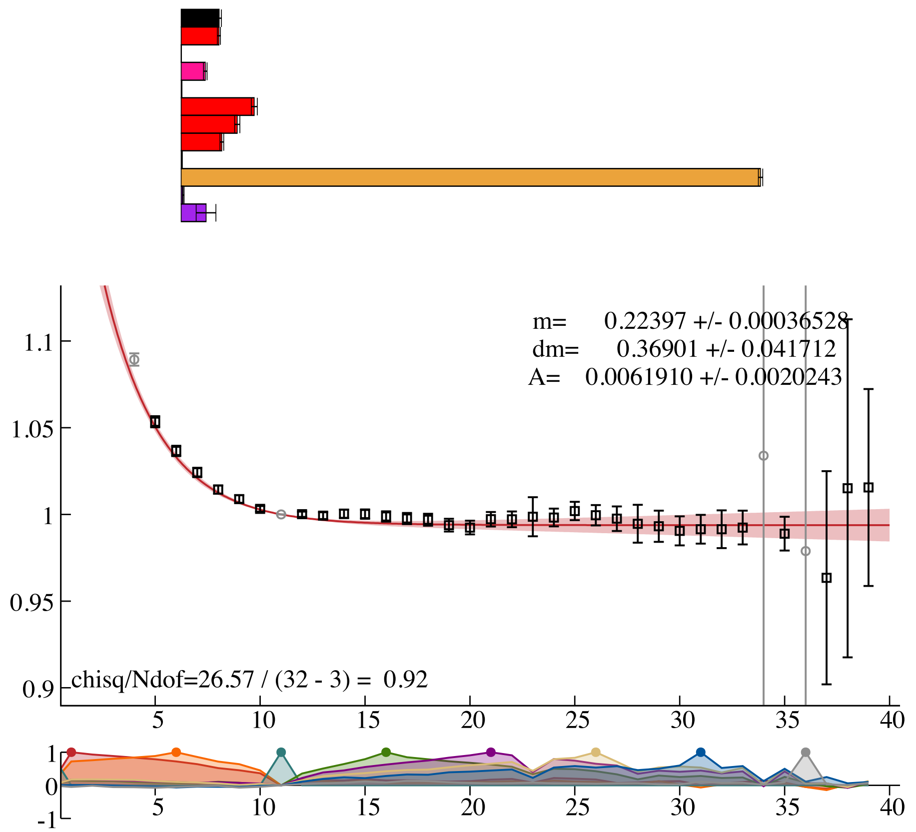
Operator basis

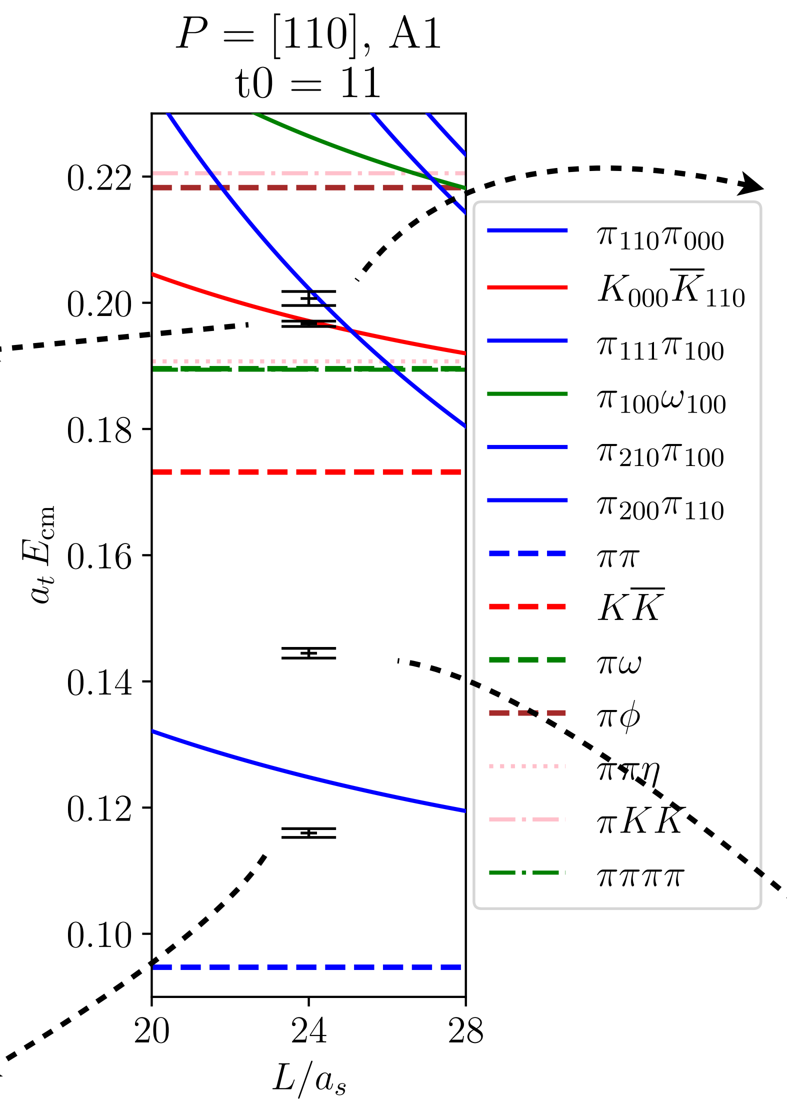
	$a_t E_{\text{th}}$	E_{th}/MeV
$\pi\pi$	0.0947	567
$K\overline{K}$	0.1732	1037
$\pi\pi\pi\pi$	0.1894	1134
$\pi\omega$	0.1896	1135
$\pi\pi\eta$	0.1908	1142
$\pi\phi$	0.2183	1307
$\pi K\overline{K}$	0.2206	1320
$\pi\eta\omega$	0.2856	1710

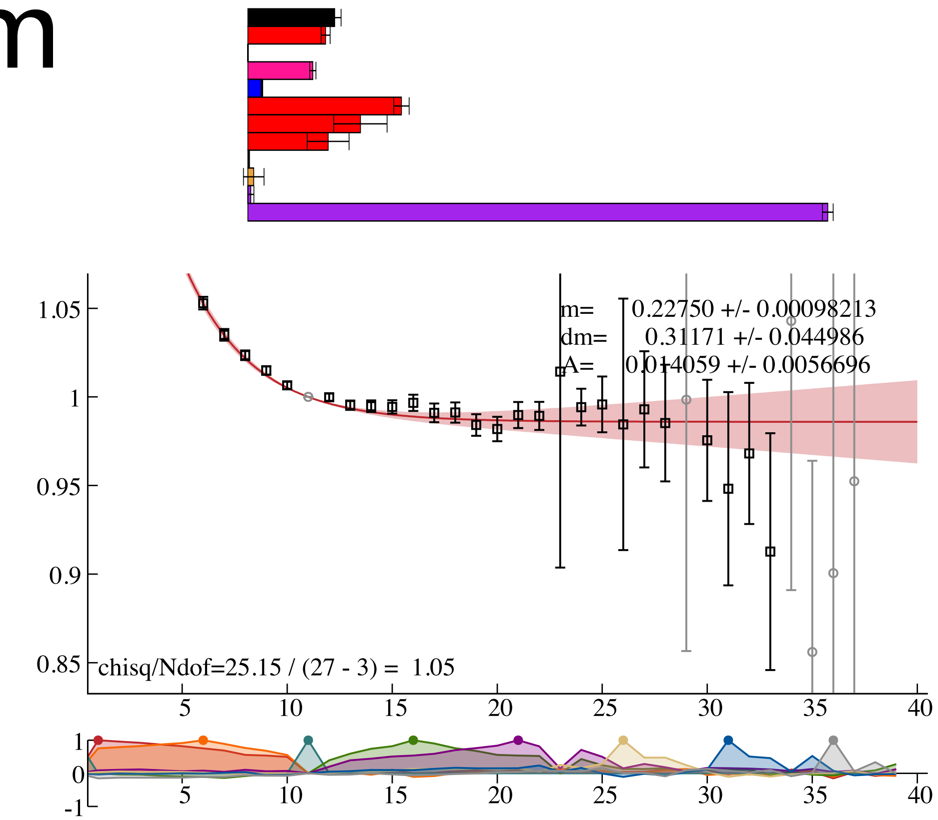
[000] T_1^-	[100] A_1	[110] A_1	[111] A_1	[200] A_1
$11 \times \bar{\psi}\mathbf{\Gamma}\psi$	$8 \times \bar{\psi}\mathbf{\Gamma}\psi$	$9 \times \bar{\psi}\mathbf{\Gamma}\psi$	$10 \times \bar{\psi}\mathbf{\Gamma}\psi$	$11 \times \bar{\psi}\mathbf{\Gamma}\psi$
$\pi_{[100]}\pi_{[100]}$	$\pi_{[100]}\pi_{[000]}$	$\pi_{[110]}\pi_{[000]}$	$\pi_{[111]}\pi_{[000]}$	$\pi_{[200]}\pi_{[000]}$
	$K_{[100]}\overline{K}_{[000]}$	$K_{[110]}\overline{K}_{[000]}$	$\pi_{[110]}\pi_{[100]}$	$K_{[200]}\overline{K}_{[000]}$
	$\pi_{[110]}\pi_{[100]}$	$\pi_{[111]}\pi_{[100]}$	$K_{[111]}\overline{K}_{[000]}$	
			$K_{[110]}\overline{K}_{[100]}$	

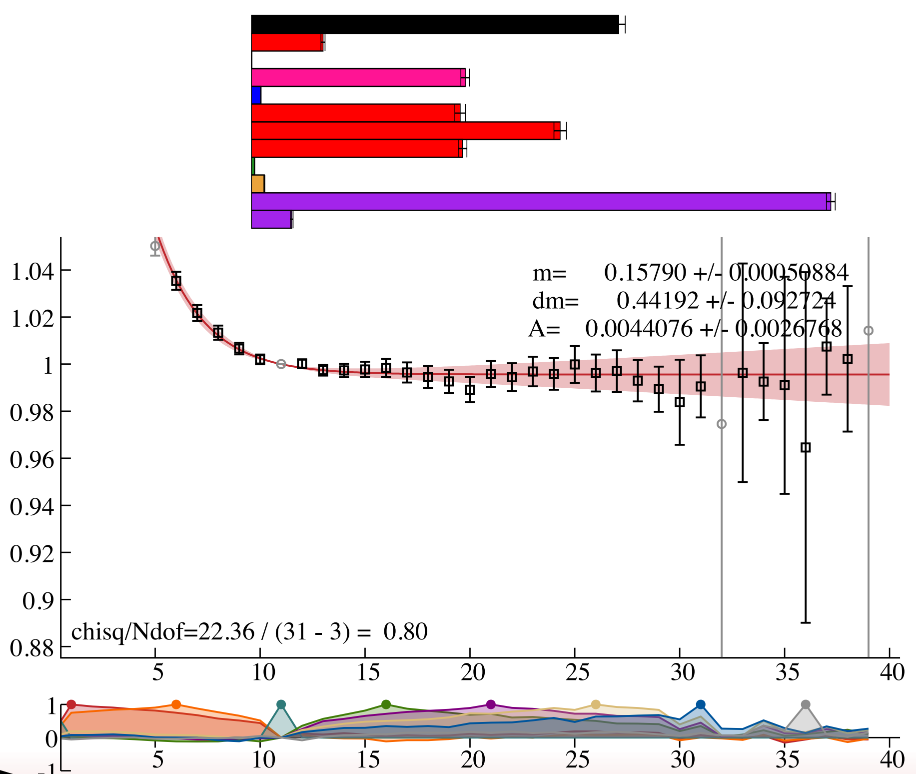
[100] E_2	[110] B_1	[110] B_2	[111] E_2	[200] E_2
$17 \times \bar{\psi}\mathbf{\Gamma}\psi$	$12 \times \bar{\psi}\mathbf{\Gamma}\psi$	$15 \times \bar{\psi}\mathbf{\Gamma}\psi$	$12 \times \bar{\psi}\mathbf{\Gamma}\psi$	$15 \times \bar{\psi}\mathbf{\Gamma}\psi$
$\pi_{[110]}\pi_{[100]}$	$\pi_{[100]}\pi_{[100]}$	$\omega_{[110]}\pi_{[000]}$	$\pi_{[110]}\pi_{[100]}$	$\pi_{[110]}\pi_{[110]}$
	$\omega_{[110]}\pi_{[000]}$	$\pi_{[111]}\pi_{[100]}$	$K_{[110]}\overline{K}_{[100]}$	
	$K_{[100]}\overline{K}_{[100]}$	$\pi_{[110]}\pi_{[110]}$		
	$\pi_{[110]}\pi_{[110]}$	$\phi_{[110]}\pi_{[000]}$		
	$\phi_{[110]}\pi_{[000]}$			
	$\omega_{[100]}\pi_{[100]}$			

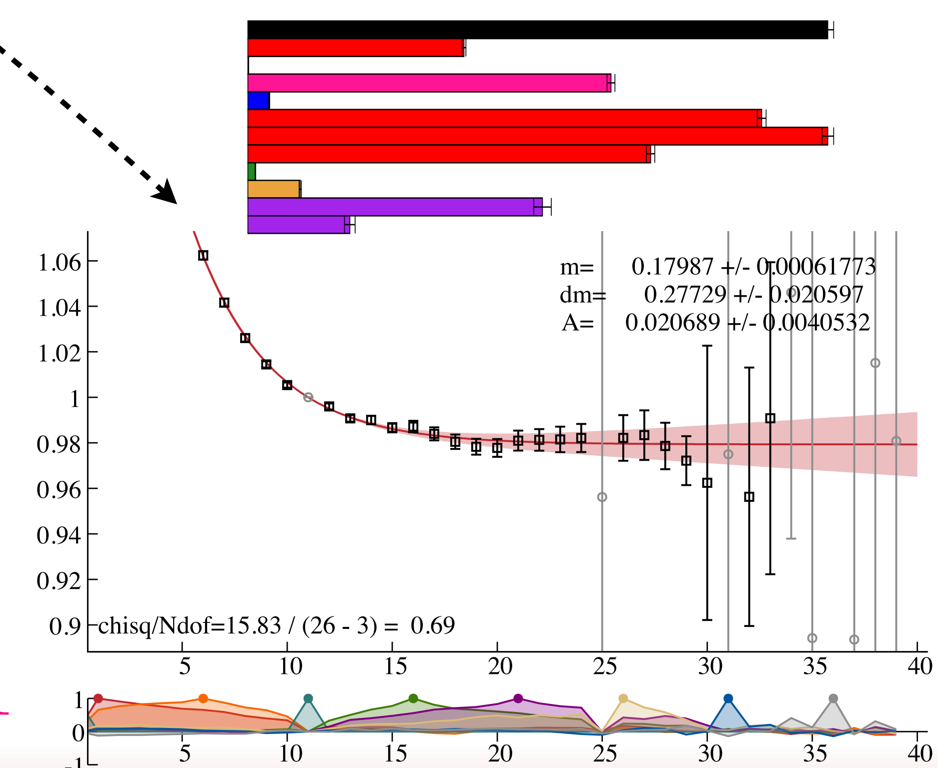
Typical spectrum











Multimeson $\pi\pi$ $K\overline{K}$

qbarq ρ ρ_2 ρ_3 b_0

hybrids

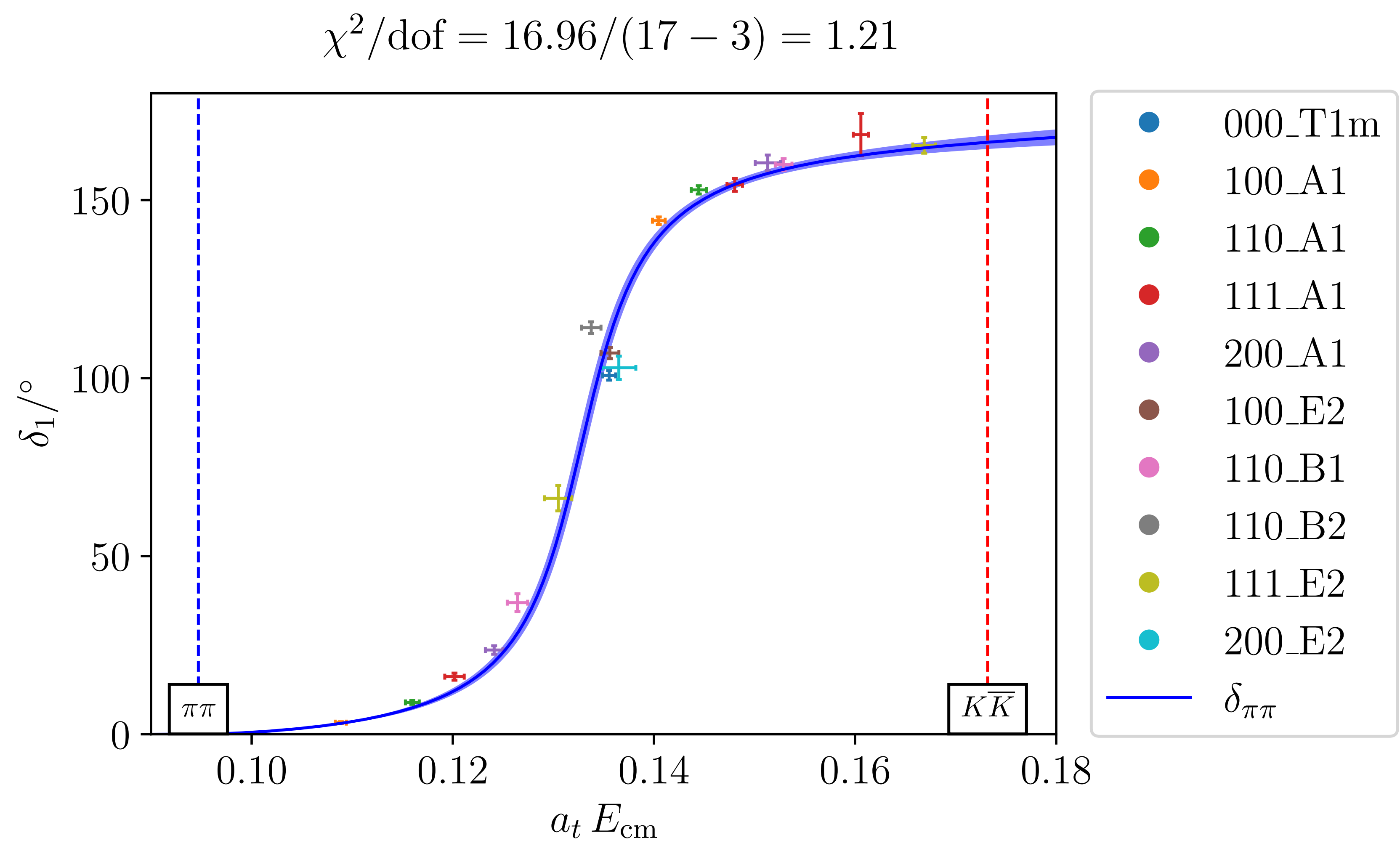
$$(\gamma^0\gamma^5 \times D^{[2]}_{J=1})^{J=1}$$

Elastic scattering

$$\mathcal{M}_\ell = \frac{E_{\text{cm}}}{2q^\star} e^{i\delta_\ell} \sin\delta_\ell$$

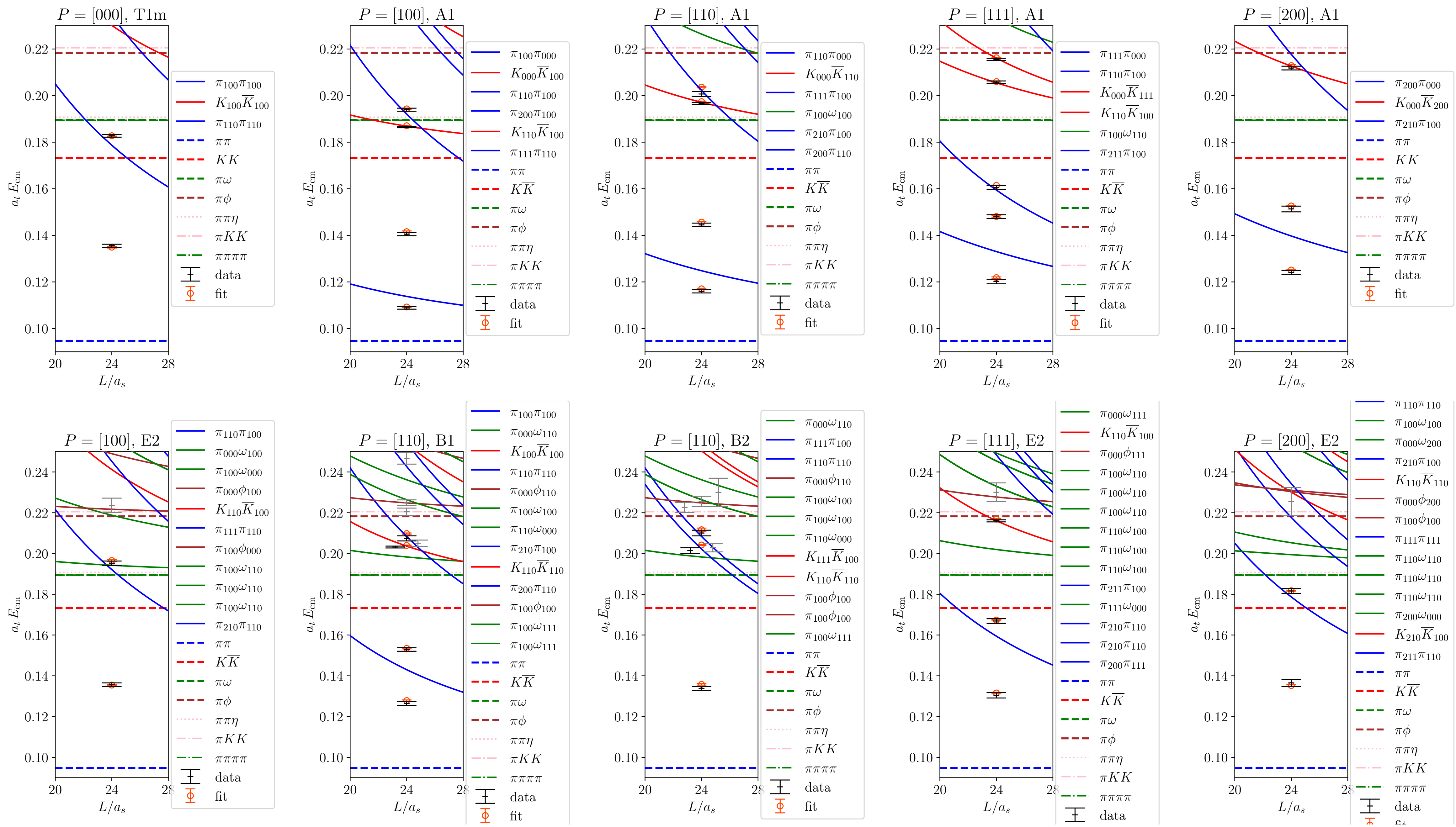
$$\mathcal{M}_\ell^{-1} = \frac{1}{(2q^\star)^\ell} K_\ell^{-1} \frac{1}{(2q^\star)^\ell} - i\rho_{\text{CM}}$$

$$K_1 = \frac{g^2}{-s + m^2} + \gamma$$



Coupled channel fit: $\pi\pi$ $K\overline{K}$

$$\mathcal{M}^{-1}_{\ell,ab} = \frac{1}{(2q^\star_a)^\ell} K^{-1}_{\ell,ab} \frac{1}{(2q^\star_b)^\ell} - i\rho_{\mathrm{CM},ab} \qquad K_{ab} = \frac{g_a g_b}{-s + m_r^2} + \gamma_{ab} \qquad \chi^2/\mathrm{dof} = 28.70/(32-6) = 1.10$$

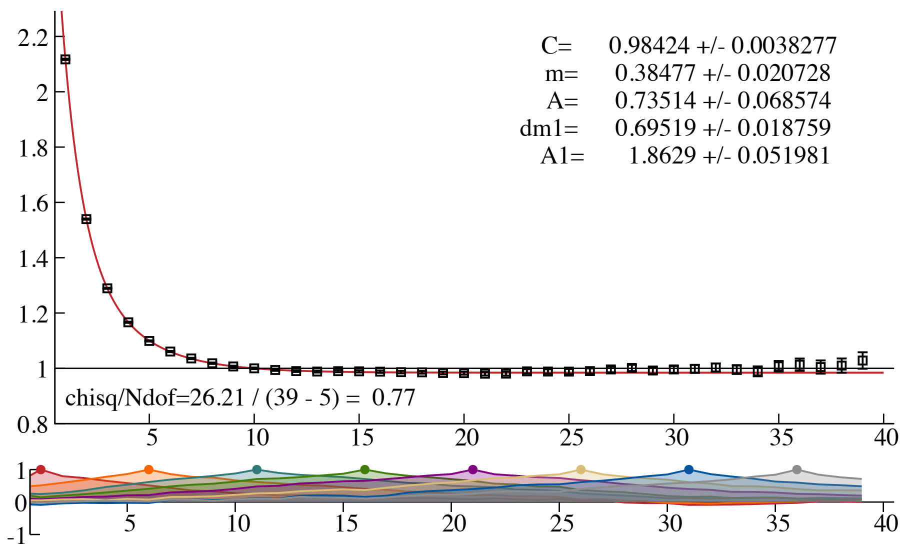


Optimized operators

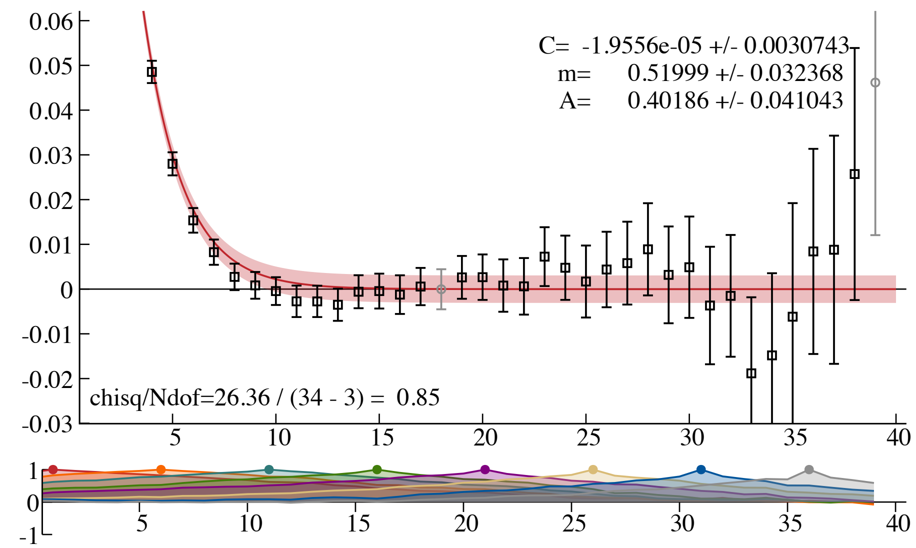
[000] T1m irrep

$$e^{\sqrt{E_n E_m} t} \langle 0 | \Omega_n(t) \Omega_m^\dagger(0) | 0 \rangle = \delta_{n,m} + \mathcal{O}(e^{-(E_N - \sqrt{E_n E_m})t})$$

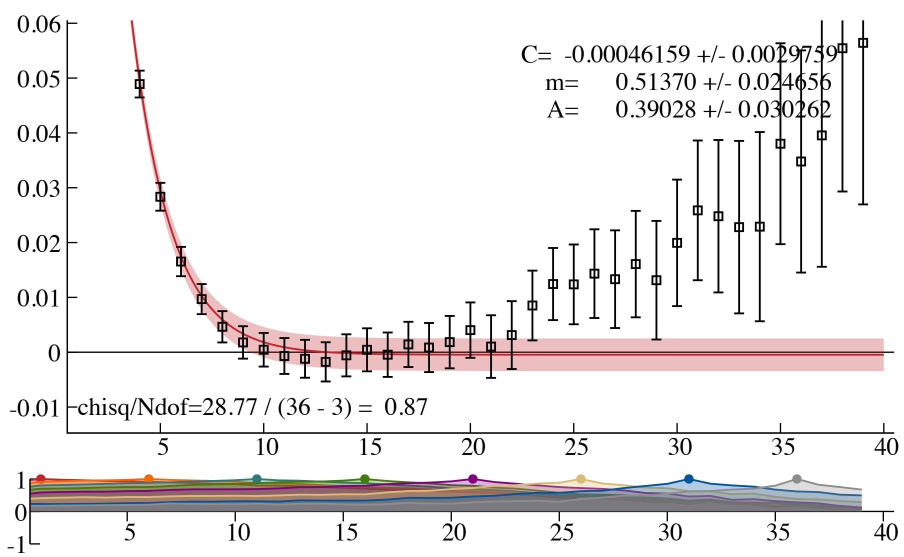
C00



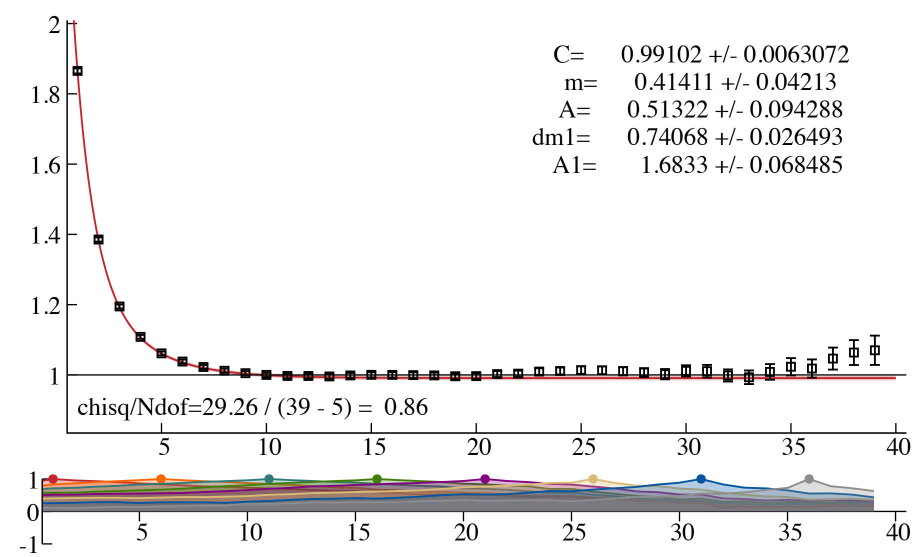
C01



C10



C11

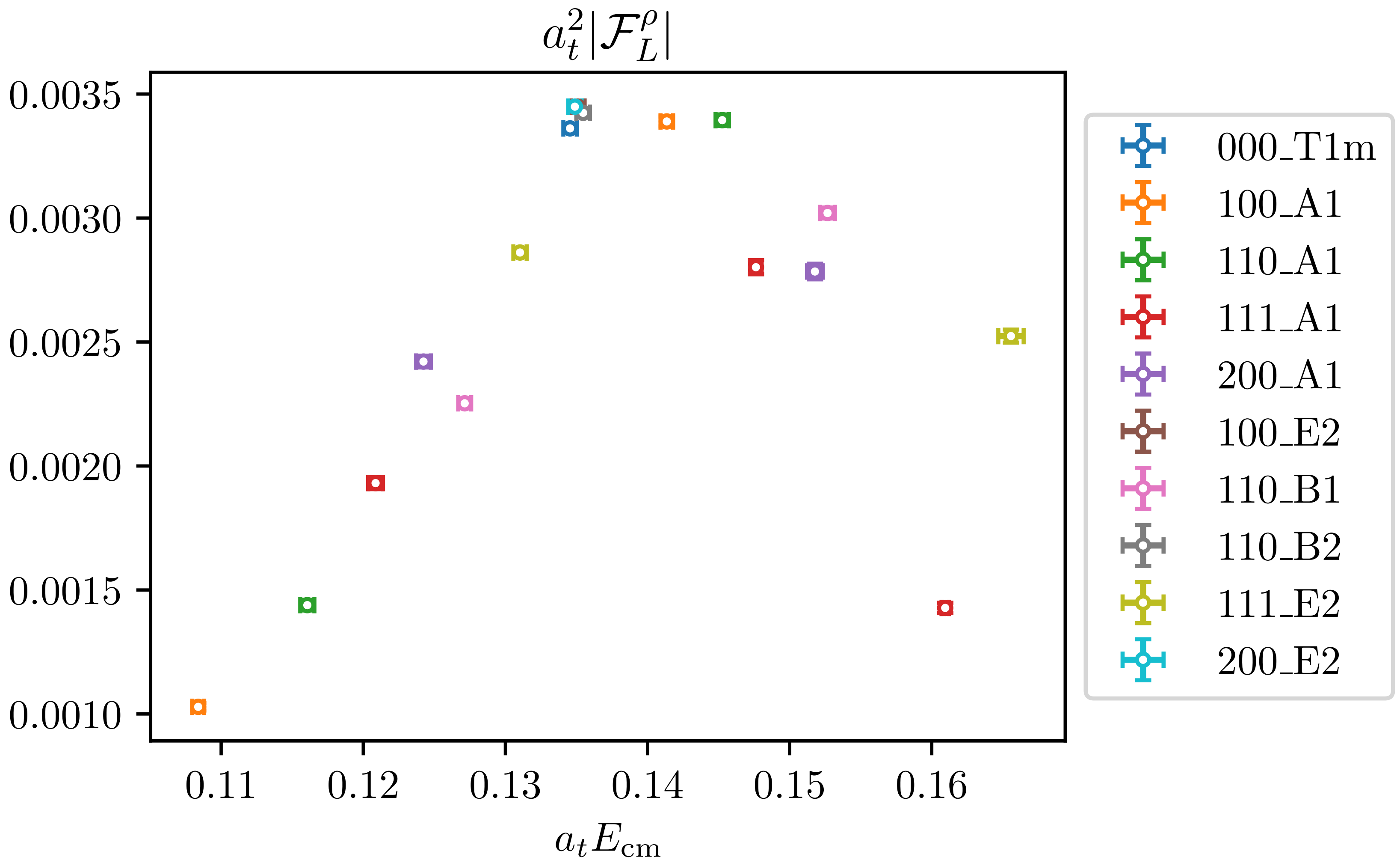


Invariant matrix element

$$\mathcal{H}_L^\mu = K^\mu \mathcal{F}_L$$

$$\langle 0|\mathcal{J}^\mu(0)|n, P, 1\lambda\rangle = \frac{2}{\sqrt{3}}\epsilon^\mu(P,\lambda)\mathcal{F}_L(P^2)$$

$$\mathcal{F}_L(P^2) = \sqrt{\frac{2E_n}{L^3}}\frac{Z_n^{1/2} Z_V C_n^{\mathbf{d},\Lambda,\rho}}{K_n(\vec{d},\Lambda)}$$



Typical three point fit

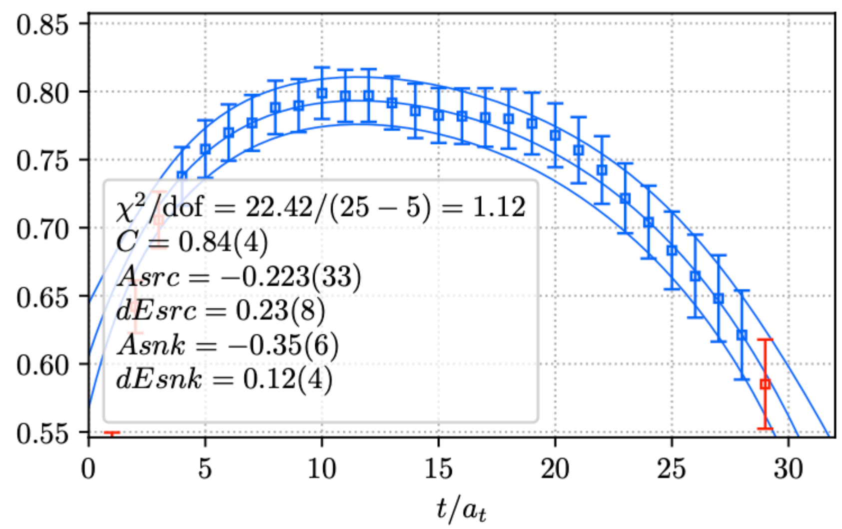
$$\left\langle \pi_{[1-10]} \middle| \mathcal{J}^{\rho}_{\text{impro}} \middle| \pi_{[200]} \right\rangle$$

$$\frac{\left\langle \pi(\tau, \vec{p_f}) \mathcal{J}(t) \pi^\dagger(0, \vec{p_i}) \right\rangle_{\text{rel}}}{\langle \pi(\tau - t, \vec{p_f}) \pi^\dagger(0, \vec{p_f}) \rangle \langle \pi(t, \vec{p_i}) \pi^\dagger(0, \vec{p_i}) \rangle} = Z_f^{-1/2} Z_i^{-1/2} \langle \pi(\vec{p_f}) | \mathcal{J}(0) | \pi(\vec{p_i}) \rangle + \dots$$

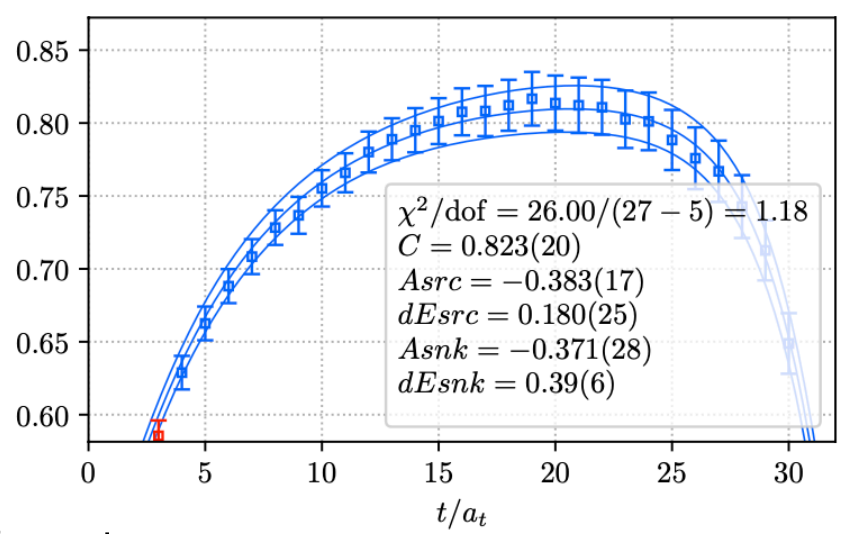
$$C + A_{src} e^{-dE_{src} t} + A_{snk} e^{-dE_{snk}(\tau - t)}$$

Current irrep

A1



B1



B2 kinematic factor is zero.

Kinematic factor

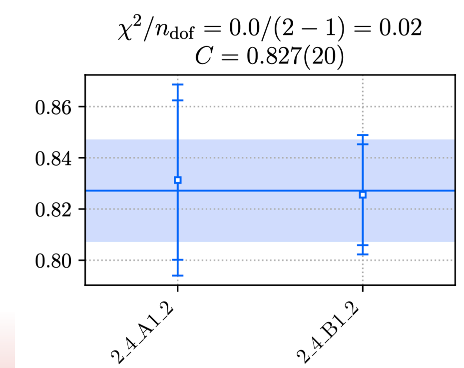
$$\langle \pi(p_1) | \mathcal{J}^i(\vec{q}) | \pi(p_2) \rangle = (p_1 + p_2)^i f(Q^2)\,,$$

$$\mathcal{S}^{\Lambda,\mu}_m \varepsilon^m_i(\vec{q}) \langle \pi(p_1) | \mathcal{J}^i(\vec{q}) | \pi(p) \rangle = \langle \pi(p_1) | \mathcal{J}^{\Lambda,\mu}(\vec{q}) | \pi(p_2) \rangle\,,$$

Improvement

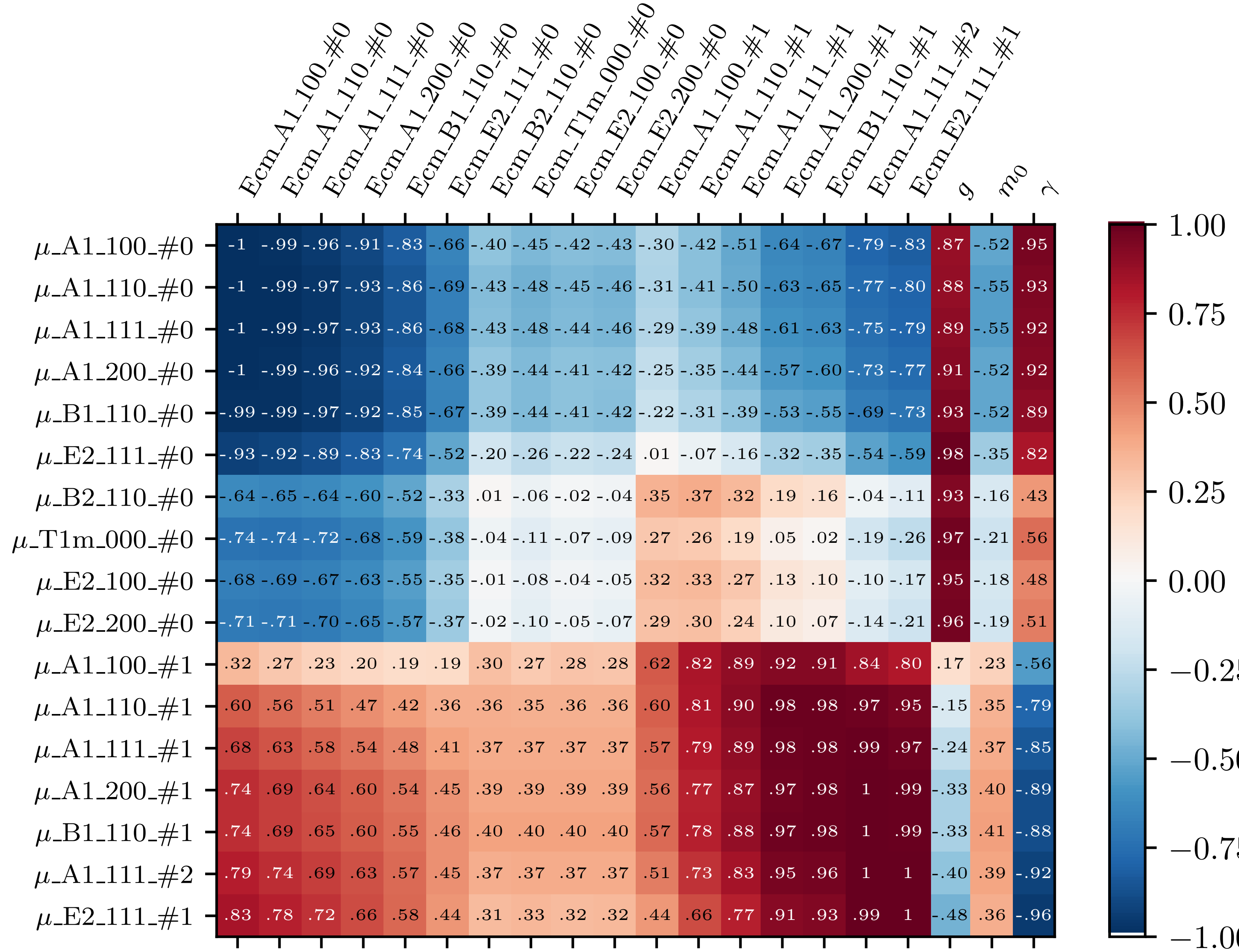
$$\langle \mathfrak{m} | \mathcal{J}^{\rho}_{\text{impro}} | \mathfrak{n} \rangle = \langle \mathfrak{m} | \rho | \mathfrak{n} \rangle + \frac{1}{4}(1 - \xi) a_t (E_m - E_n) \langle \mathfrak{m} | \rho_2 | \mathfrak{n} \rangle$$

Average over irreps



Correlation between LL slopes and energies

- Slopes away from the resonance are highly correlated to the Lüscher energy.
- Meanwhile those close to the resonance have high correlation to g.



Correlation matrix between slopes, Lüscher energies and scattering parameters