Qubitization Strategies for Bosonic Field Theories

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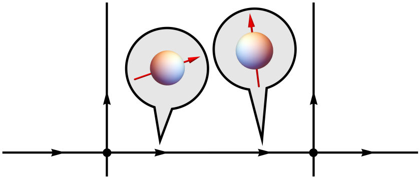


Articles

Basis for this talk

- Andrei Alexandru, Paulo F. Bedaque, Andrea Carosso, Michael J. Cervia, and Andy Sheng
  "Qubitization strategies for bosonic field theories," (2022).
  PRD **107**, 034503 (2023). arXiv:2209.00098

- Edison M. Murairi, Michael J. Cervia, Hersh Kumar, Paulo F. Bedaque, and Andrei Alexandru
  "‘How many quantum gates do gauge theories require?"
  PRD **106**, 094504 (2022). arXiv:2208.11789

- Edison Murairi and Michael J. Cervia
  "Reducing Circuit Depth with Qubitwise Diagonalization."
  arXiv:2306.00170

- Andrei Alexandru, Paulo F. Bedaque, Andrea Carosso, Michael J. Cervia, Edison M. Murairi, and Andy Sheng
  "‘Fuzzy Gauge Theory," *manuscript in preparation* (2023).

"Qubitization"

Bosonic fields on a quantum computer

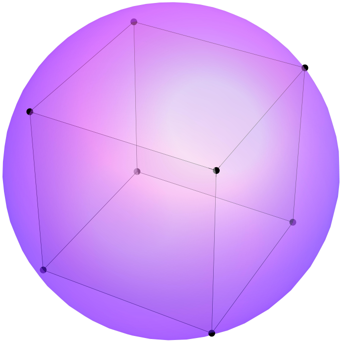
- Lattice: spatial volume $\mathbb{R}^d \to (a\mathbb{Z}_L)^d$ ("domain")
- Bosonic field's Hilbert space $\mathcal{H} \to \mathcal{H}_{\text{reg}}$ ("target")

} *Qubitization*

- Generically, need $\mathcal{H}_{\text{reg}} \to \mathcal{H}$ as well as $L \to \infty$ & $a \to 0$. Not just inconvenient, but...

Each dim of $\mathcal{H}_{\text{reg}}$ may be costly!

More on this later...



$O(3)$ σ-model

A Test for Qubitization of Boson Field Theories

- Continuum $O(3)$ σ-model action

$$S = \frac{1}{2g^2} \int \mathrm{d}x \, \mathrm{d}t \, \partial_\mu \mathbf{n}(t,x) \cdot \partial^\mu \mathbf{n}(t,x),$$

  for coupling g^2 and unit vectors $\mathbf{n} \in \mathcal{S}^2$

- Legendre transform and discretize space: Lattice model Hamiltonian

$$H = \sum_x \left[-\frac{g^2}{2} \nabla^2(x) - \frac{1}{g^2 a^2} \mathbf{n}(x) \cdot \mathbf{n}(x+1) \right]$$

  for gradient $\nabla(x)$ w.r.t. $\mathbf{n}$ at x

- Global Hilbert space for lattice volume N_x has dim $\mathcal{L}^2(\mathcal{S}^2, \mathbb{C})^{\otimes N_x}$ — infinite even for one site!

Harmonic Expansion of the σ-model

Preserving $O(3)$ in a truncated Hilbert space, Take 1

- *Decompose* $\mathcal{H} \equiv \mathcal{L}^2(\mathcal{S}^2, \mathbb{C})$ into:

$$\Psi[\mathbf{n}(\theta,\phi)] = \sum_{\ell=0}^{\ell_{\max}} \sum_{m=-\ell}^{\ell} \psi_{\ell m} Y_\ell^m(\theta,\phi)$$

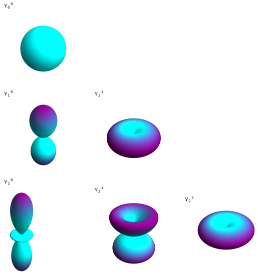
Need to lift truncation $\ell_{\max} \to \infty$.

- Truncate: $\ell_{\max} = 1$, $\{Y_0^0, Y_1^{-1}, Y_1^0, Y_1^{+1}\}$
- Our *reduced* Hamiltonian:

$$H \leftarrow \eta \sum_x \left[g^2 K(x) \pm \frac{1}{g^2} \sum_{k=1}^{3} y_k(x) y_k(x+1) \right]$$

where $K = \mathrm{diag}(0,1,1,1)$ and $y_k \leftrightarrow n_k$.

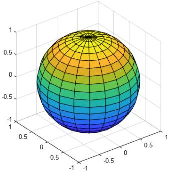
Spherical harmonics



A Fuzzy Sphere σ-model

Preserving $O(3)$ in a truncated Hilbert space, Take 2

- *Promote* coordinates n_k to spin-$1/2$ *operators* J_k:



coordinate n_k		$J_k \in \mathrm{SU}(2)$
$\sum_k n_k n_k = 1$	$\longleftrightarrow$	$\sum_k J_k J_k = \mathbb{1}$
SPHERE		*FUZZY SPHERE*

- *New* local Hilbert space: complex 2×2 matrices
- Spherical symmetry with *only four* points: $\mathbb{1}$ and Paulis σ_1, σ_2, σ_3
- *Distinct* truncated Hamiltonian:

$$H = \eta \sum_x \left[g^2 K(x) \pm \frac{3}{4g^2} \sum_{k=1}^{3} J_k(x) J_k(x+1) \right]$$

where $K = [J_k, [J_k, \cdot]]$ and J_k replaces n_k

MPS Ansatz: A Practical Note

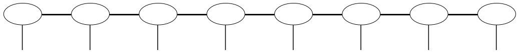
Working around the exponential Hilbert Space

Problems for both models:

- $\dim \mathcal{H}_1 = 4$, lattice volume $N_x \longrightarrow \dim \mathcal{H} = 4^{N_x}$
- Monte Carlo sign problem

Solution: Matrix Product State (MPS) for global wave function Ψ:

$$|\Psi\rangle = \sum_{a_1,\ldots,a_N=1}^{4} A(1)^{a_1} \cdots A(N)^{a_N} \, |a_1,\ldots,a_N\rangle$$

$=$ 

- Well-established variational algo (DMRG) for lowest-lying states!
- *N.B.* Open boundary condition...

Mass "Gap" vs. Energy Gap

Inverse correlation length;

$$C(x,y) = \langle \Psi_0 | O(x) O(y) | \Psi_0 \rangle$$

Difference in lowest energies:

$$a\Delta \equiv \lim_{L\to\infty} [\hat{E}_1(L) - \hat{E}_0(L)]$$

Relativistic theory: $\eta(g^2)\Delta(g^2) = m(g^2)$, so tune the scale factor η;

$$\eta(g^2) = \frac{am(g^2)}{a\Delta(g^2)}$$

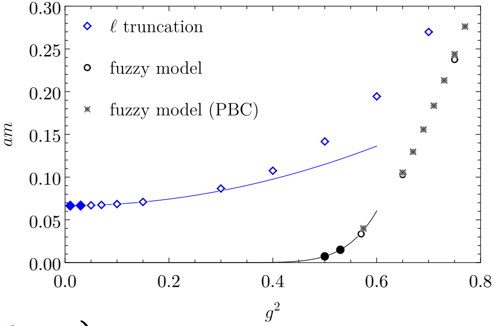
Particle mass in the continuum limit

A First Test

Inverse spatial corr lengths have expected form: $am(g^2) = \frac{A}{g^2} e^{-B/g^2}$

Continuum limit expected as $g^2 \to 0$: $am(g^2) \to 0$

mass*less*



- Early failure for ℓ truncation...
- Fuzzy model passes so far...

} *Each 2 qubits/site!*

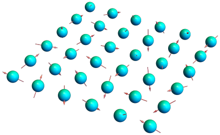
AA, PFB, AC, **MJC**, AS (2022)

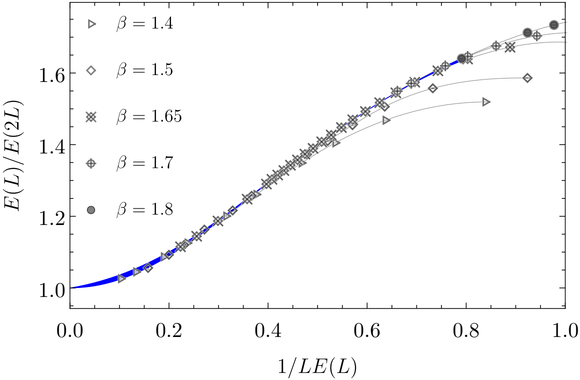
Step-scaling Curves

A Comprehensive Test

Assess "step scaling" of model across *wide energy range*:
$E(L_x)/E(2L_x)$ vs. $1/L_xE(L_x)$

(e.g., Monte Carlo simulations as the basis for comparison)



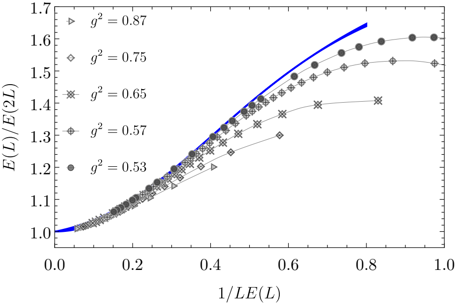


←infrared (IR) ultraviolet (UV)→

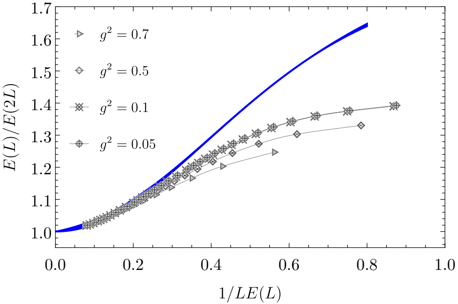
Step-scaling Curves

Comparison of models

Fuzzy sphere truncation



Spherical harmonics truncation

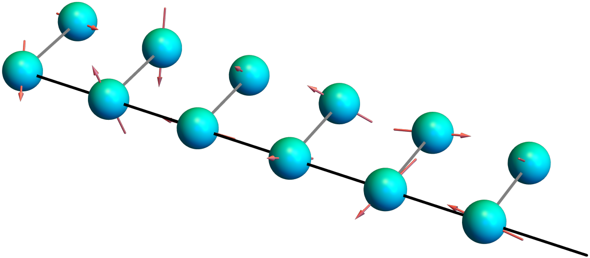


- (Blue): Continuum-limit behavior derived from MC simulations
- Smallest volumes: $L_x/a \sim 4$,
- Fuzzy sphere calculations approach continuum-limit behavior much further into UV regime

AA, PFB, AC, **MJC**, AS (2022)

Local chage of basis $\rightarrow$ Heisenberg comb:

$$H = \eta \sum_x g^2 \vec{S}(x,0) \cdot \vec{S}(x,1) + \frac{1}{g^2} \vec{S}(x,0) \cdot \vec{S}(x+1,0)$$

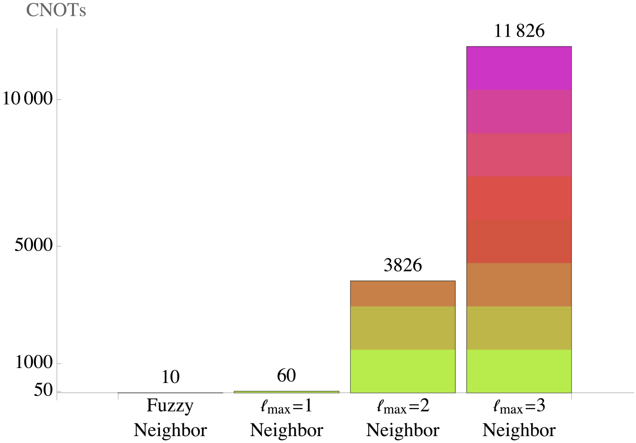


Recently investigated by Bhattacharya *et al* (2021)
Stay tuned for more on this realization... up next!

Resource Estimates for Truncated Models

CNOT costs to simulate one time step

- Foundation for resource estimates of quantum simulations
- *Automatic* procedure — stay tuned for code *you* can use!



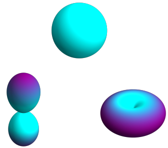
EMM, **MJC**, HK, PFB, AA (2022)

EMM & **MJC** (2023)

Lessons Learned

A Summary

- Fuzzy 2-sphere leads to continuum $O(3)$ σ-model description



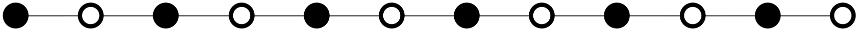
- ℓ_{max} mass > 0, less descriptive of UV
- Cautionary tale about the importance of *small* qubitization schemes!

- THANK YOU -

Another difference between models

Fuzzy vs ℓ truncation: ferromagnetic vs antiferromagnetic

Beside algebraic closure, another distinction for the ℓ truncation:



- Mapping between (anti)ferromagnetic models: global operator

$$O = \bigotimes_{n=1}^{N_x/2} (U_{2n-1} \otimes \mathbb{1}_{2n}),$$

  where locally $U\,K\,U^\dagger = K$, yet $U\,y_k\,U^\dagger = -y_k$

- U *does* exist for harmonic expansion
  e.g., $\ell_{\max} = 1$: $U = \mathrm{diag}(1, -1, -1, -1)$
- (Anti)ferromagnetic phases of ℓ truncation have an equivalence
  $\implies$ if one phase fails to describe theory, so does the other

Basis of Comparison

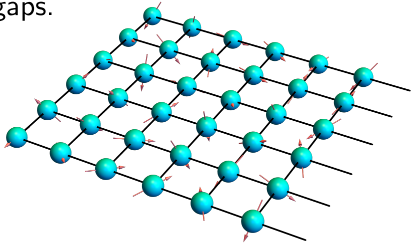
Monte Carlo methods

- Lattice $O(3)$ σ-model action

$$S = -\beta \sum_{t,x} \left[\mathbf{n}(t,x) \cdot \mathbf{n}(t+1,x) + \mathbf{n}(t,x) \cdot \mathbf{n}(t,x+1) \right]$$

- Boundaries: periodic time, "open" space $\iff$ open for MPS
- Monte Carlo simulations w/ no sign problem
- Measure time-slice correlators $C(t) = \frac{1}{N_x^2} \sum_{x,y} \langle \mathbf{n}(t,x) \cdot \mathbf{n}(0,y) \rangle$

  Fit $\exp[-t\Delta(N_x)]$ for energy gaps.



Measurements via MPS

Extrapolation w.r.t. bond dimension

MPS ansatz becomes exact for a bond dimension cutoff

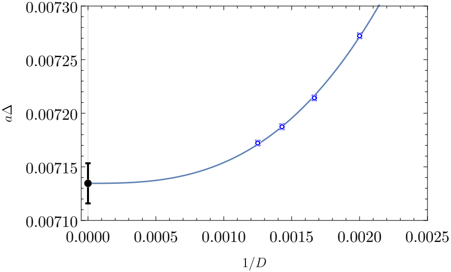
$$D \to D_0 \equiv p^{\lfloor L/2 \rfloor},$$

for an open chain of L sites each with local Hilbert space dimension p.

Fit approximate quantity Δ as a function of bond dimension $D \ll D_0$

$$\Delta(D) = \Delta + \frac{A}{D^B}$$

(e.g., fuzzy, $g^2 = 0.53$, $L/a = 800$) $\to$



$$\implies \text{Uncertainty: } \epsilon_\Delta = \frac{\Delta(D_{\max}) - \Delta}{2}$$

- Lattice $O(3)$ σ-model action

$$S = -\beta \sum_{t,x} \left[\mathbf{n}(t,x) \cdot \mathbf{n}(t+1,x) + \mathbf{n}(t,x) \cdot \mathbf{n}(t,x+1) \right]$$

  for $\beta > 0$ and $\mathbf{n} \in \mathcal{S}^2$ on a $N_t \times N_x$ lattice,
  periodic in time yet "open" in space.
- Monte Carlo simulations via Wolff cluster algorithm
- Measure time-slice correlators

$$C(t) = \frac{1}{N_x^2} \sum_{x,y} \langle \mathbf{n}(t,x) \cdot \mathbf{n}(0,y) \rangle \,,$$

  fit to $\exp[-t\Delta(L_x)]$ for energy gaps.
- Suppress finite-temperature $T = 1/L_t$ effects $\mathcal{O}[\exp(-m/T)]$ with large $m(\beta)/T \gtrsim 8$
- Suppress spatial boundary effects with $\delta E(L_x)/T \gtrsim 8$ for free scalar energy gap $\delta E \equiv \omega_3 - \omega_1 \approx 8\pi^2/mL_x^2$

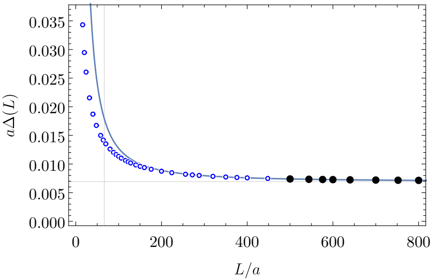
Energy Gaps in the Infinite-Volume Limit

[Given: For each fixed L_x, we have extrapolated w.r.t. D already.]

Fit energy gap Δ as a function of volume L_x:

$$a\Delta(L_x) = a\Delta + \frac{A}{(L_x/a)^B}$$

(e.g., fuzzy, $g^2 = 0.53$) $\rightarrow$



- Corrections to fit function are $\mathcal{O}(e^{-mL_x})$, so take $mL_x \gtrsim 5$.
- Reasonable description down to $mL_x \sim 2$

Spatial Correlation Lengths

Mass Determination

- Given the ground state $|\Psi_0\rangle$ of the lattice (approximated as a MPS), the spatial correlation function:

$$C(x,y) = \langle \Psi_0 | O(x) O(y) | \Psi_0 \rangle \, \zeta^{x-y},$$

  where $\zeta = -1$ for antiferromagnetic models, and $O = y_3, j_3$.
- In 1+1 dimensions, fit to expected form $K_0 \equiv \texttt{BesselK}[0, \cdot]$:

$$C(r) \equiv \sum_{x \in \mathcal{X}(r)} C(x, x+r) = A \, K_0(r/\xi)$$

  for inverse corr length $1/\xi = am$, where $r = |x - y|$.
- Avoid edge effects:
  - Select only $x \in \mathcal{X}(r)$ such that $[x, x+r]$ is centered around $L/2$
  - Take fit window to $r \in [x_0, x_0 + w]$ with sufficiently large x_0

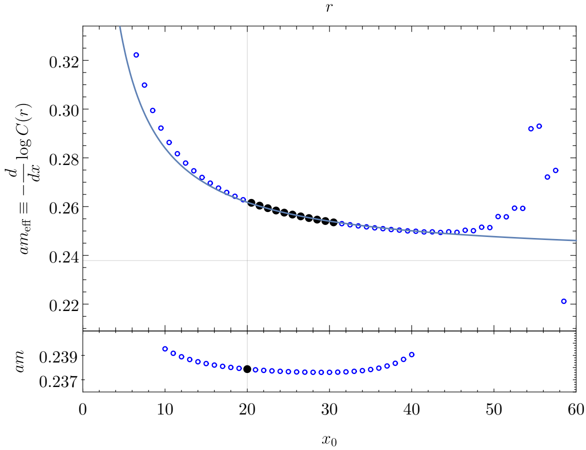
Mass Fitting Procedure

Illustrating the failure of an exponential correlator fit

Fitted mass m and c.f., effective mass (i.e., log-derivative) m_{eff}

e.g.,

$$g^2 = 0.75,$$
$$L_x/a = 60,$$
$$D = 800$$



Bottom: Fit window of $[x_0, x_0 + 10]$ for $C(r) \propto K_0(r/\xi)$

AA, PFJ, AC, **MJC**, AS (2022)