Qubitization Strategies for Bosonic Field Theories

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Articles
Basis for this talk

Andrei Alexandru, Paulo F. Bedaque, Andrea Carosso, Michael J. Cervia, and Andy Sheng

Edison M. Murairi, Michael J. Cervia, Hersh Kumar, Paulo F. Bedaque, and Andrei Alexandru

Edison Murairi and Michael J. Cervia
“Reducing Circuit Depth with Qubitwise Diagonalization.” arXiv:2306.00170

Andrei Alexandru, Paulo F. Bedaque, Andrea Carosso, Michael J. Cervia, Edison M. Murairi, and Andy Sheng
“Qubitization”
Bosonic fields on a quantum computer

- Lattice: spatial volume $\mathbb{R}^d \rightarrow (a\mathbb{Z}_L)^d$ (“domain”)
- Bosonic field’s Hilbert space $\mathcal{H} \rightarrow \mathcal{H}_{\text{reg}}$ (“target”)
- Generically, need $\mathcal{H}_{\text{reg}} \rightarrow \mathcal{H}$ as well as $L \rightarrow \infty$ & $a \rightarrow 0$. Not just inconvenient, but...

Each dim of $\mathcal{H}_{\text{reg}}$ may be costly!
More on this later...
A Test for Qubitization of Boson Field Theories

- **Continuum** $O(3)$ $\sigma$-model action

\[ S = \frac{1}{2g^2} \int \! dx \, dt \, \partial_\mu n(t, x) \cdot \partial^\mu n(t, x), \]

for coupling $g^2$ and unit vectors $n \in S^2$

- Legendre transform and discretize space: Lattice model Hamiltonian

\[ H = \sum_x \left[ -\frac{g^2}{2} \nabla^2(x) - \frac{1}{g^2a^2} n(x) \cdot n(x + 1) \right] \]

for gradient $\nabla(x)$ w.r.t. $n$ at $x$

- Global Hilbert space for lattice volume $N_x$ has dim $\mathcal{L}^2(S^2, \mathbb{C}) \otimes N_x$ — infinite even even for one site!
Harmonic Expansion of the \( \sigma \)-model

Preserving \( O(3) \) in a truncated Hilbert space, Take 1

- Decompose \( \mathcal{H} \equiv \mathcal{L}^2(S^2, \mathbb{C}) \) into:
  \[
  \Psi[n(\theta, \phi)] = \sum_{\ell=0}^{\ell_{\text{max}}} \sum_{m=-\ell}^{\ell} \psi_{\ell m} Y_{\ell}^{m}(\theta, \phi)
  \]

  \textit{Need to lift truncation} \( \ell_{\text{max}} \rightarrow \infty \).

- Truncate: \( \ell_{\text{max}} = 1, \{Y_0^0, Y_1^{-1}, Y_1^0, Y_1^{+1}\} \)

- Our \textit{reduced} Hamiltonian:
  \[
  H \leftarrow \eta \sum_x \left[ g^2 K(x) \pm \frac{1}{g^2} \sum_{k=1}^{3} y_k(x) y_k(x+1) \right]
  \]

  where \( K = \text{diag}(0, 1, 1, 1) \) and \( y_k \leftrightarrow n_k \).
A Fuzzy Sphere $\sigma$-model
Preserving $O(3)$ in a truncated Hilbert space, Take 2

- **Promote** coordinates $n_k$ to spin-1/2 operators $J_k$:
  \[
  \sum_k n_k J_k^* n_k = 1 \quad \longleftrightarrow \quad \sum_k J_k J_k = 1
  \]

  **SPHERE**

  **FUZZY SPHERE**

- *New* local Hilbert space: complex $2 \times 2$ matrices
- Spherical symmetry with *only four* points: $\mathbb{1}$ and Paulis $\sigma_1, \sigma_2, \sigma_3$
- *Distinct* truncated Hamiltonian:
  \[
  H = \eta \sum_x \left[ g^2 K(x) \pm \frac{3}{4g^2} \sum_{k=1}^3 J_k(x) J_k(x + 1) \right]
  \]
  where $K = [J_k, [J_k, \cdot]]$ and $J_k$ replaces $n_k$
Problems for both models:

- \( \dim \mathcal{H}_1 = 4 \), lattice volume \( N_x \rightarrow \dim \mathcal{H} = 4^{N_x} \)
- Monte Carlo sign problem

**Solution: Matrix Product State (MPS) for global wave function \( \Psi \):**

\[
|\Psi\rangle = \sum_{a_1,\ldots,a_N=1}^{4} A(1)^{a_1} \cdots A(N)^{a_N} |a_1,\ldots,a_N\rangle
\]

- Well-established variational algo (DMRG) for lowest-lying states!
- *N.B.* Open boundary condition...
Correlation Length(s) in Hamiltonian Lattice Theories
Renormalization of the speed of light

<table>
<thead>
<tr>
<th>Mass “Gap”</th>
<th>vs.</th>
<th>Energy Gap</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inverse correlation length;</td>
<td></td>
<td>Difference in lowest energies:</td>
</tr>
<tr>
<td>$C(x, y) = \langle \Psi_0</td>
<td>O(x)O(y)</td>
<td>\Psi_0 \rangle$</td>
</tr>
</tbody>
</table>

Relativistic theory: $\eta(g^2)\Delta(g^2) = m(g^2)$, so tune the scale factor $\eta$;

$$\eta(g^2) = \frac{am(g^2)}{a\Delta(g^2)}$$
Particle mass in the continuum limit
A First Test

Inverse spatial corr lengths have expected form: \( am(g^2) = \frac{A}{g^2} e^{-B/g^2} \)

Continuum limit expected as \( g^2 \to 0 \): \( am(g^2) \to 0 \)

mass/less

- Early failure for \( \ell \) truncation...
- Fuzzy model passes so far...

Each 2 qubits/site!

AA, PFB, AC, MJC, AS (2022)
Assess “step scaling” of model across wide energy range:
\[ \frac{E(L_x)}{E(2L_x)} \text{ vs. } \frac{1}{L_x E(L_x)} \]

(e.g., Monte Carlo simulations as the basis for comparison)
Step-scaling Curves
Comparison of models

Fuzzy sphere truncation

Spherical harmonics truncation

(Blue): Continuum-limit behavior derived from MC simulations

Smallest volumes: \( L_x/a \sim 4 \),

Fuzzy sphere calculations approach continuum-limit behavior much further into UV regime

AA, PFB, AC, MJC, AS (2022)
Another picture of the fuzzy $\sigma$-model

Heisenberg comb

Local change of basis $\rightarrow$ Heisenberg comb:

$$H = \eta \sum_x g^2 \vec{S}(x, 0) \cdot \vec{S}(x, 1) + \frac{1}{g^2} \vec{S}(x, 0) \cdot \vec{S}(x + 1, 0)$$

Recently investigated by Bhattacharya et al. (2021)
Stay tuned for more on this realization... up next!
Resource Estimates for Truncated Models

CNOT costs to simulate one time step

- Foundation for resource estimates of quantum simulations
- *Automatic* procedure — stay tuned for code *you* can use!

EMM, **MJC**, HK, PFB, AA (2022)

EMM & **MJC** (2023)
Lessons Learned

Fuzzy 2-sphere leads to continuum $O(3)$ $\sigma$-model description

- $\ell_{\text{max}}$ mass $> 0$, less descriptive of UV
- Cautionary tale about the importance of small qubitization schemes!
- THANK YOU -
Another difference between models

Fuzzy vs $\ell$ truncation: ferromagnetic vs antiferromagnetic

Beside algebraic closure, another distinction for the $\ell$ truncation:

- Mapping between (anti)ferromagnetic models: global operator

$$O = \bigotimes_{n=1}^{N_x/2} (U_{2n-1} \otimes 1_{2n}),$$

where locally $U K U^\dagger = K$, yet $U y_k U^\dagger = -y_k$

- $U$ does exist for harmonic expansion
  e.g., $\ell_{\text{max}} = 1$: $U = \text{diag}(1, -1, -1, -1)$

- (Anti)ferromagnetic phases of $\ell$ truncation have an equivalence
  $\implies$ if one phase fails to describe theory, so does the other
Basis of Comparison

Monte Carlo methods

- Lattice $O(3) \sigma$-model action
  \[ S = -\beta \sum_{t,x} [n(t, x) \cdot n(t + 1, x) + n(t, x) \cdot n(t, x + 1)] \]

- Boundaries: periodic time, “open” space ⇐⇒ open for MPS
- Monte Carlo simulations w/ no sign problem
- Measure time-slice correlators
  \[ C(t) = \frac{1}{N^2} \sum_{x,y} \langle n(t, x) \cdot n(0, y) \rangle \]
  Fit $\exp[-t\Delta(N_x)]$ for energy gaps.
Measurements via MPS
Extrapolation w.r.t. bond dimension

MPS ansatz becomes exact for a bond dimension cutoff

\[ D \rightarrow D_0 \equiv p \lfloor L/2 \rfloor, \]

for an open chain of \( L \) sites each with local Hilbert space dimension \( p \).

Fit approximate quantity \( \Delta \) as a function of bond dimension \( D \ll D_0 \)

\[ \Delta(D) = \Delta + \frac{A}{D^B} \]

(e.g., fuzzy, \( g^2 = 0.53, \ L/a = 800 \) \( \rightarrow \)

\[ \Longrightarrow \text{Uncertainty: } \epsilon_\Delta = \frac{\Delta(D_{\text{max}}) - \Delta}{2} \]
Euclidean Action $\sigma$-model

Monte Carlo methods

- Lattice $O(3)$ $\sigma$-model action

$$ S = -\beta \sum_{t,x} \left[ n(t, x) \cdot n(t+1, x) + n(t, x) \cdot n(t, x+1) \right] $$

for $\beta > 0$ and $n \in S^2$ on a $N_t \times N_x$ lattice, periodic in time yet “open” in space.

- Monte Carlo simulations via Wolff cluster algorithm

- Measure time-slice correlators

$$ C(t) = \frac{1}{N_x^2} \sum_{x,y} \langle n(t, x) \cdot n(0, y) \rangle $$

fit to $\exp[-t\Delta(L_x)]$ for energy gaps.

- Suppress finite-temperature $T = 1/L_t$ effects $O[\exp(-m/T)]$ with large $m(\beta)/T \gtrsim 8$

- Suppress spatial boundary effects with $\delta E(L_x)/T \gtrsim 8$ for free scalar energy gap $\delta E \equiv \omega_3 - \omega_1 \approx 8\pi^2/mL_x^2$
Fit energy gap $\Delta$ as a function of volume $L_x$:

$$a\Delta(L_x) = a\Delta + \frac{A}{(L_x/a)^B}$$

(e.g., fuzzy, $g^2 = 0.53$) →

- Corrections to fit function are $O(e^{-mL_x})$, so take $mL_x \gtrsim 5$.
- Reasonable description down to $mL_x \sim 2$
Spatial Correlation Lengths
Mass Determination

- Given the ground state $|\Psi_0\rangle$ of the lattice (approximated as a MPS), the spatial correlation function:

$$C(x, y) = \langle \Psi_0 | O(x) O(y) | \Psi_0 \rangle \zeta^{x-y},$$

where $\zeta = -1$ for antiferromagnetic models, and $O = y_3, j_3$.

- In 1+1 dimensions, fit to expected form $K_0 \equiv \text{BesselK}[0, \cdot]$:

$$C(r) \equiv \sum_{x \in X(r)} C(x, x + r) = A K_0(r/\xi)$$

for inverse corr length $1/\xi = am$, where $r = |x - y|$.

- Avoid edge effects:
  - Select only $x \in X(r)$ such that $[x, x + r]$ is centered around $L/2$
  - Take fit window to $r \in [x_0, x_0 + w]$ with sufficiently large $x_0$
Mass Fitting Procedure

Illustrating the failure of an exponential correlator fit

Fitted mass $m$ and c.f., effective mass (i.e., log-derivative) $m_{\text{eff}}$

e.g.,

$g^2 = 0.75,$
$L_x/a = 60,$
$D = 800$

Bottom: Fit window of $[x_0, x_0 + 10]$ for $C(r) \propto K_0(r/\xi)$

AA, PFJ, AC, MJC, AS (2022)