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July 31^{st}, 2023

ASYMPTOTIC SCALING IN YANG-MILLS AT LARGE-N_c

The lattice scale and the $\Lambda_{\overline{\mathrm{MS}}}$-parameter at large-$N_c$ from twisted volume reduction

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IFT UAM-CSIC
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THE LATTICE RUNNING COUPLING

The $\Lambda_{\overline{\mathrm{MS}}}$-parameter

RG equations for the 't Hooft coupling $\lambda = g^2 N_c$:

$$\frac{\mathrm{d}\lambda_s}{\mathrm{d}\log(\mu^2)} = \beta_s(\lambda_s) \simeq -b_0\lambda_s^2 - b_1\lambda_s^3 - b_2^{(s)}\lambda_s^4 - \cdots$$

and, upon integration

$$\frac{\Lambda_s}{\mu} = (b_0\lambda_s)^{-\frac{b_1}{2b_0^2}} e^{-\frac{1}{2b_0\lambda_s}} e^{-\int^{\lambda}\mathrm{d}x\left[\frac{1}{2\beta_s(x)}+\frac{1}{2b_0x^2}-\frac{b_1}{2b_0x}\right]}$$

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- Compute the non-perturbative running of the coupling at low energies μ_{had} and match to PT at $\mu_{\mathrm{PT}} \gg \mu_{\mathrm{had}}$

$$\frac{\Lambda_s}{\mu_{\mathrm{had}}} = \frac{\Lambda_s}{\mu_{\mathrm{pt}}} e^{-\int_{\lambda(\mu_{\mathrm{pt}})}^{\lambda(\mu_{\mathrm{had}})} \frac{\mathrm{d}x}{2\beta_s(x)}}$$

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Weak coupling is hard to simulate on the lattice and for feasible simulations

- $\mathcal{O}(a^2)$ corrections (scaling violations)
- Lattice scheme w/ Wilson action has large higher order terms in the β-function

$$\frac{\Lambda_{\overline{\mathrm{MS}}}}{\Lambda_{\mathrm{lat}}} \sim 38.853\, e^{-\frac{3\pi^2}{11N_c^2}}$$

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or

- Simulate large range of bare couplings $b = 1/\lambda$, use improved lattice couplings to improve convergence with PT

$$\lambda_I = \frac{b}{P(b)}, \qquad \lambda_E = 8\big(1 - P(b)\big), \qquad \lambda_{E'} = -8\log P(b)$$

[Allton et al. JHEP 07 (2008) 21], [Gonzalez-Arroyo, Okawa, Phys. Let. B 718 (2013)]

LARGE-N AND VOLUME REDUCTION

Gauge fields on a twisted lattice

[Gonzalez-Arroyo, Okawa, Phys. Rev. D 27 (1983), Phys. Let. B 120 (1983)]

$$S = bN \sum_{n} \sum_{\mu \neq \nu} \mathrm{tr}\left(\mathbb{1} - U_\mu(n) U_\nu(n+\mu) U_\mu^\dagger(n+\nu) U_\nu^\dagger(n)\right)$$

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Use reduction + **twisted BC**

$$U_\mu(n) \to U_\mu$$
$$U_\mu(n+\nu) \to \Gamma_\nu U_\mu \Gamma_\nu^\dagger$$
$$V_\mu = U_\mu \Gamma_\mu$$

$$\Gamma_\mu \Gamma_\nu = z_{\mu\nu} \Gamma_\nu \Gamma_\mu$$

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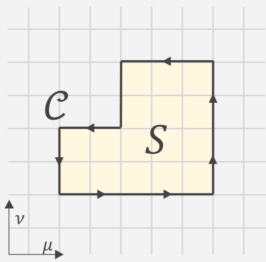
Gauge observables

$$\langle \mathrm{Tr}\, U(\mathcal{C}) \rangle_{\substack{N_c \to \infty \\ V \to \infty}}$$
$$\approx$$
$$z_{\mu\nu}(S) \left\langle \mathrm{Tr}\, V_\mu \ldots V_\nu \ldots V_\mu^\dagger \ldots \right\rangle_{\mathrm{TEK}\,(\mathrm{N_c} \sim \infty)}$$



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Finite-N_c corrections

1^4-lattice with tBC with $N_c \sim 10^2/10^3$ $\approx$ "Effective" periodic lattice with $L = \sqrt{N_c}$

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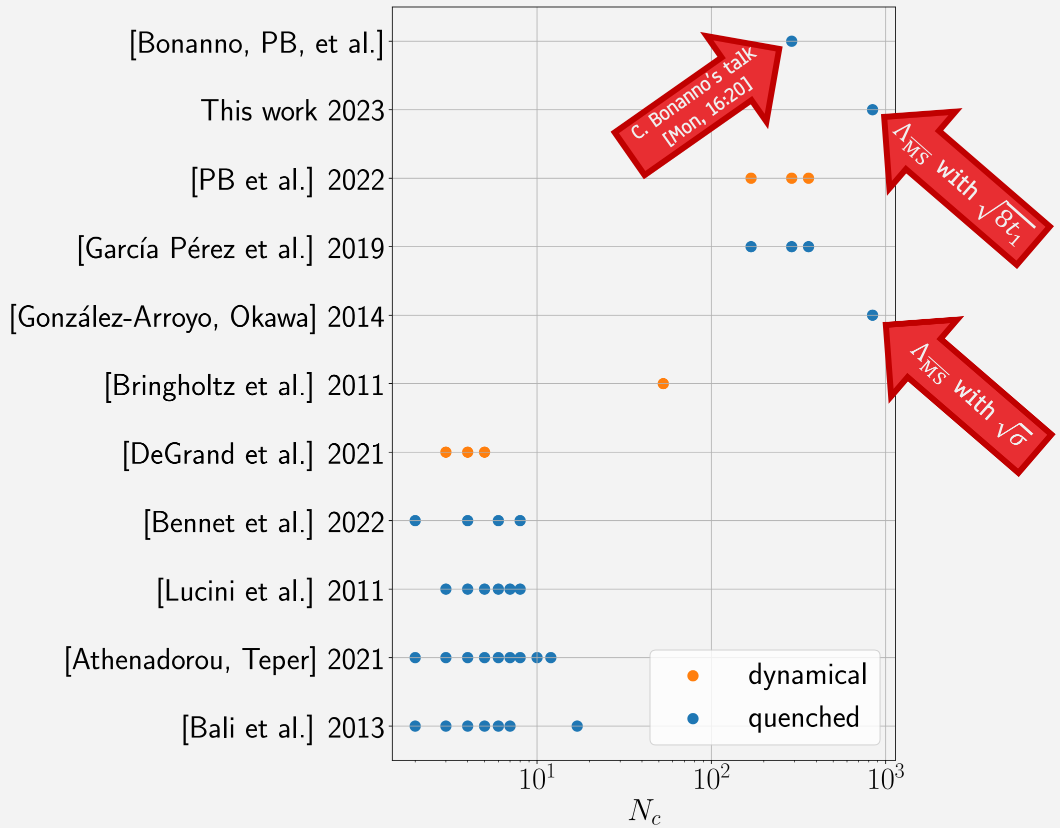
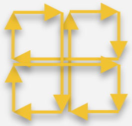


THE LATTICE SCALE

The Wilson flow scale

- Flowed energy density

$$E = -\frac{1}{128}\sum_{\mu\neq\nu} z_{\nu\mu}\,\mathrm{Tr}\,\Big($$

$$-\mathrm{h.c.}\Big)^2$$

- Integrate flow equations to get $E(t)$

- Solve

$$\left.\left\langle \frac{t^2 E\,(t)}{N_c} \right\rangle\right|_{t=t_1} = 0.05$$

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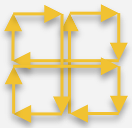
$$E = -\frac{1}{128}\sum_{\mu\neq\nu} z_{\nu\mu}\,\mathrm{Tr}\,\big(\ $$  $$\ -\mathrm{h.c.}\big)^2$$

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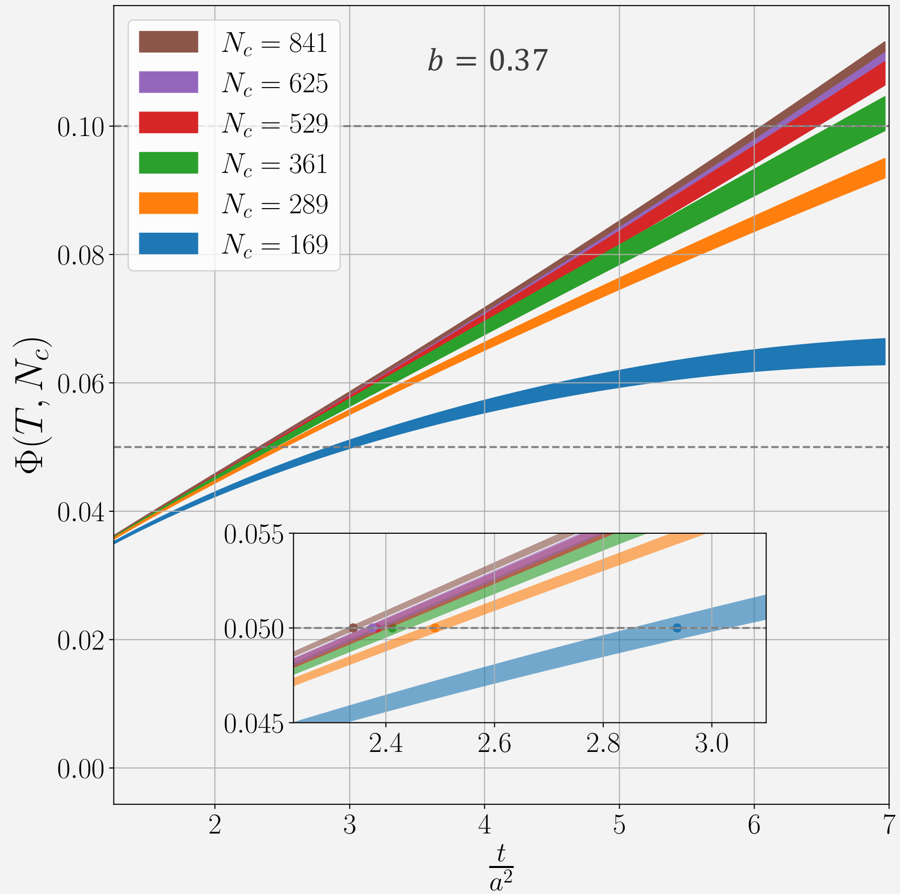

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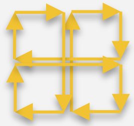
$$E = -\frac{1}{128}\sum_{\mu\neq\nu} z_{\nu\mu}\mathrm{Tr}\,($$  $$-\mathrm{h.c.})^2$$

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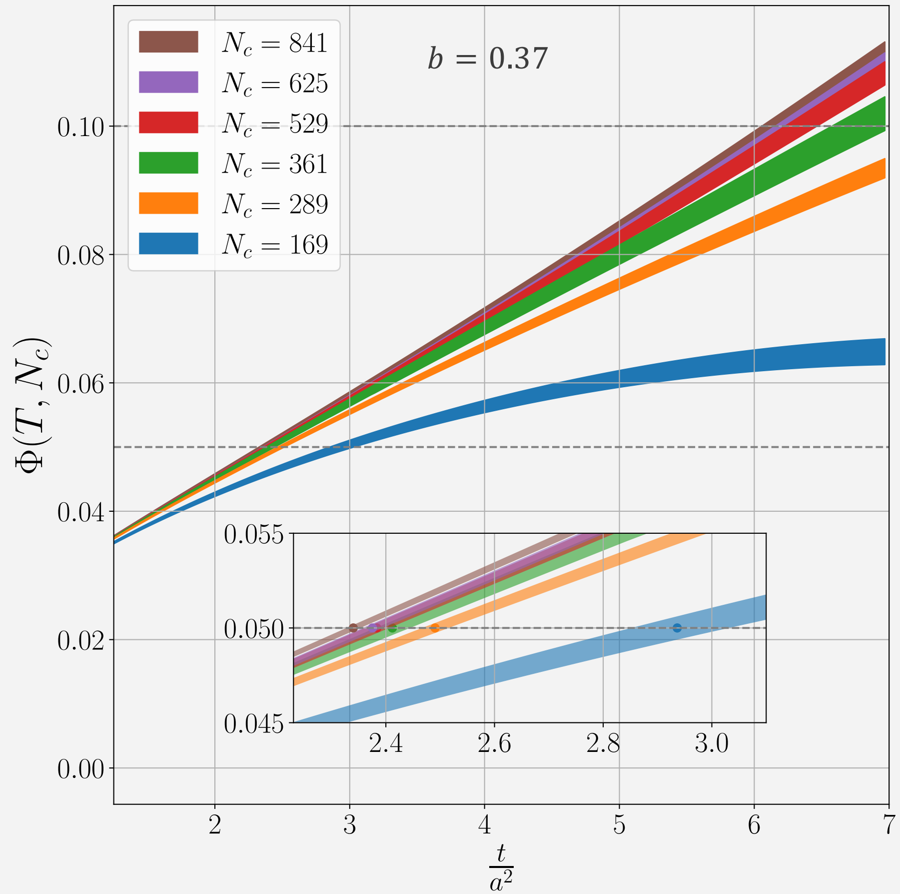
- Treat finite "volume" effects



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- (twisted) flowed running coupling

$$\lambda_{\rm gf}\left(\mu = \frac{1}{\sqrt{8t}}\right) \equiv \frac{128\pi^2 N_c^2}{N_c^2 - 1}\left\langle \frac{t^2 E(t)}{N_c} \right\rangle$$

- On a $V = 1^4$ torus with twisted BC at finite N_c
  [García Pérez, Ibañez, *JHEP* 03 (2019) 200]

$$\hat{\lambda}(\mu) \equiv \mathcal{N}^{-1}\left(\frac{\sqrt{8t}}{\sqrt{N_c}}\right)\left\langle \frac{t^2 E(t)}{N_c} \right\rangle \xrightarrow{N_c \to \infty} \lambda_{\rm gf}(\mu)$$

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  [PB, García Pérez, González-Arroyo, Ishikawa, Okawa JHEP 07 (2022) 074]

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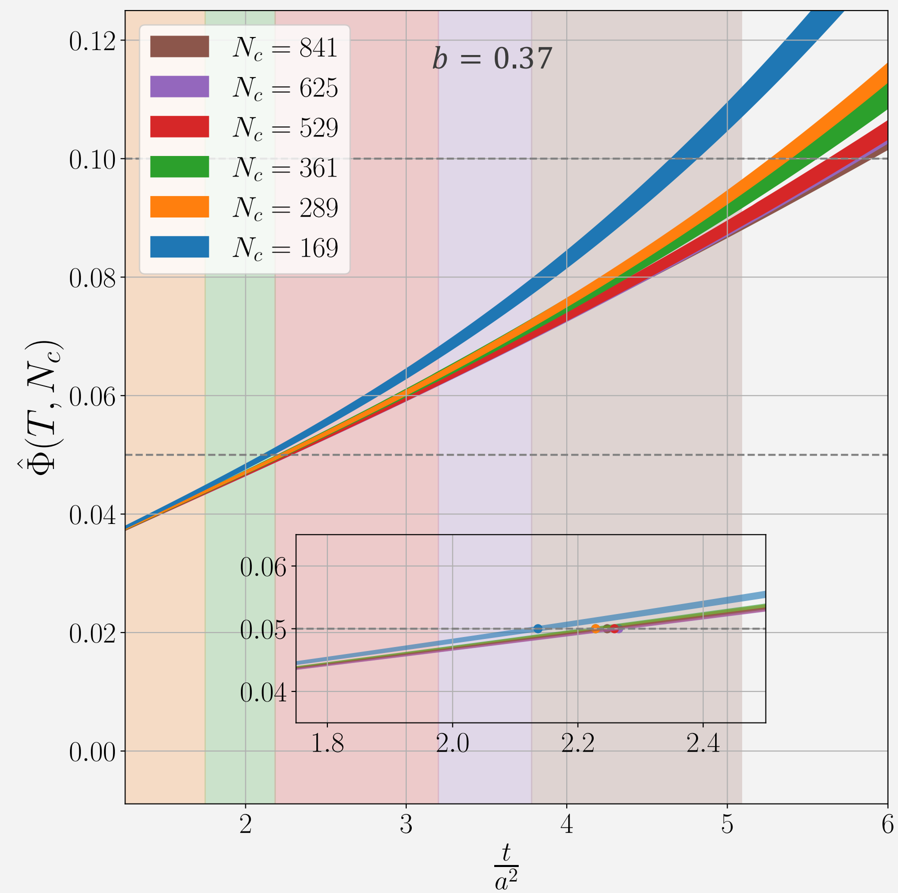

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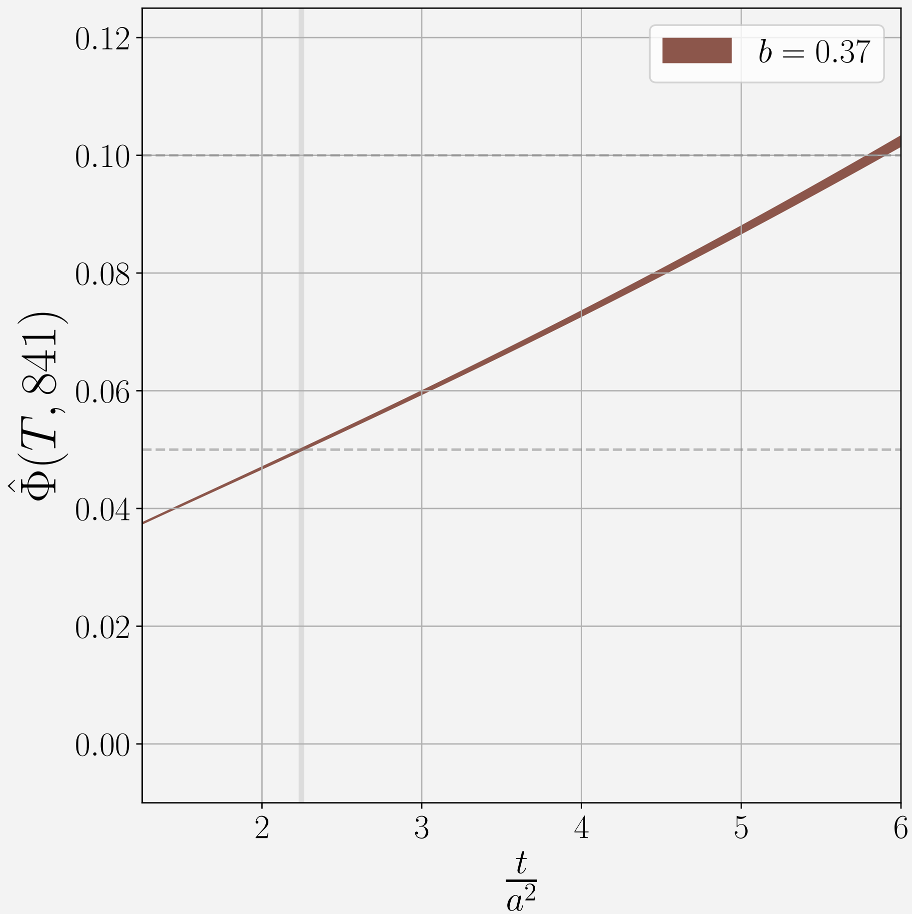

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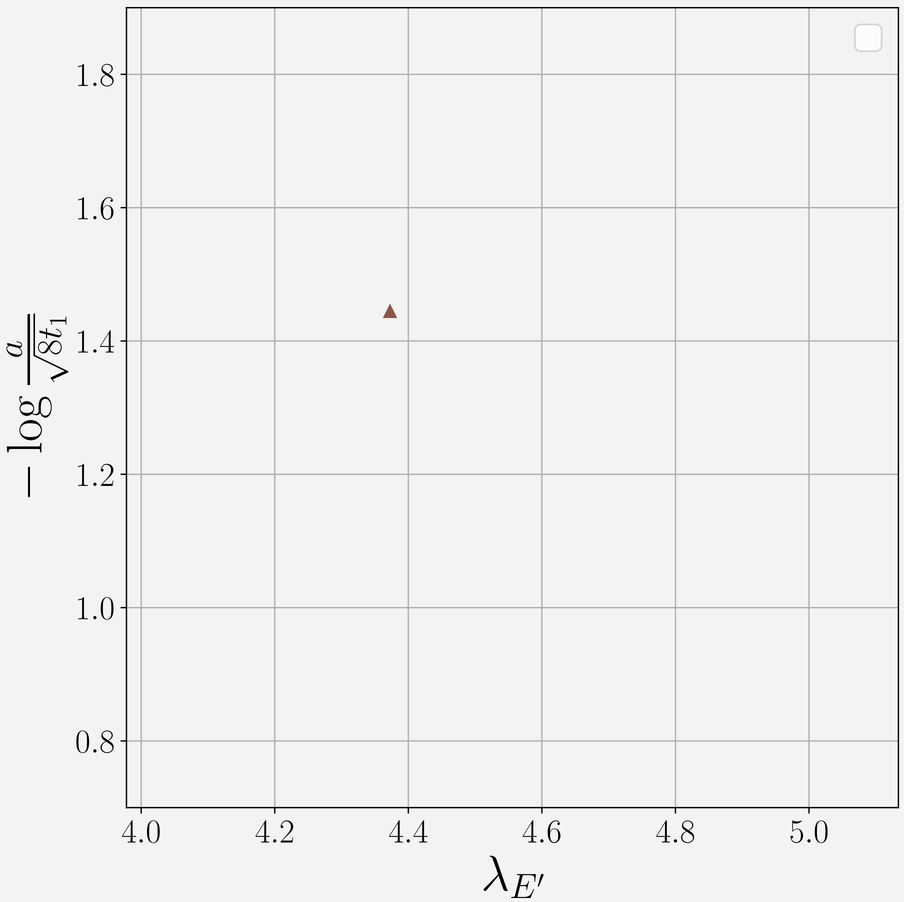

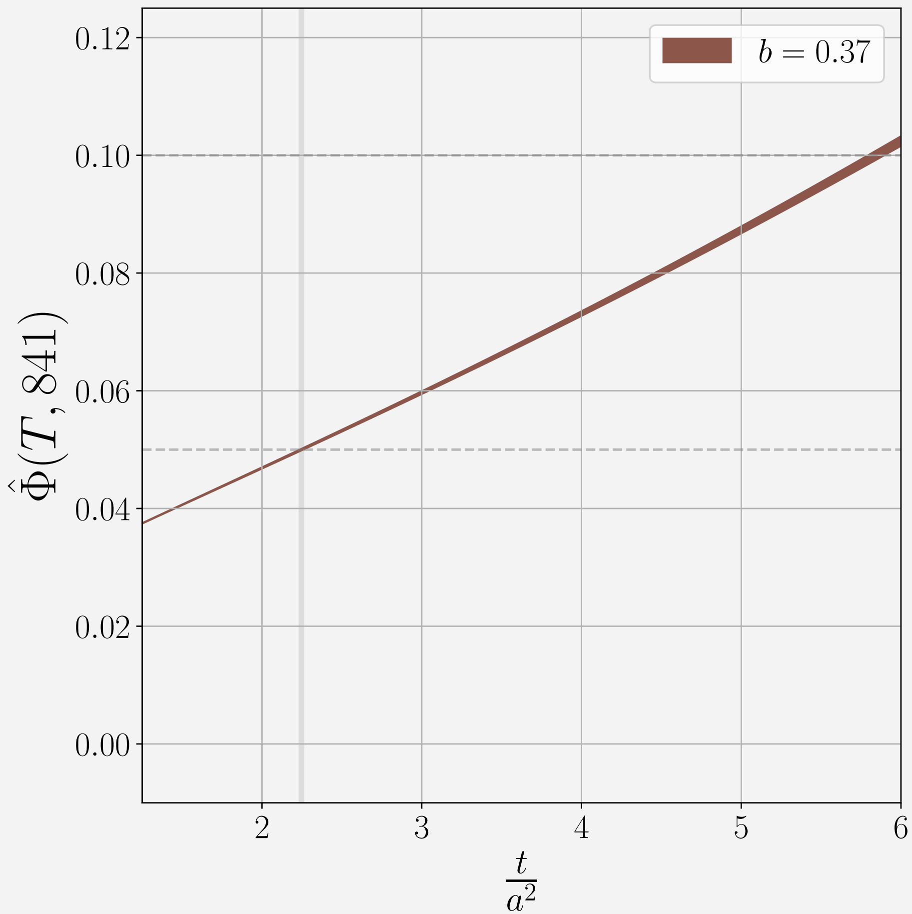

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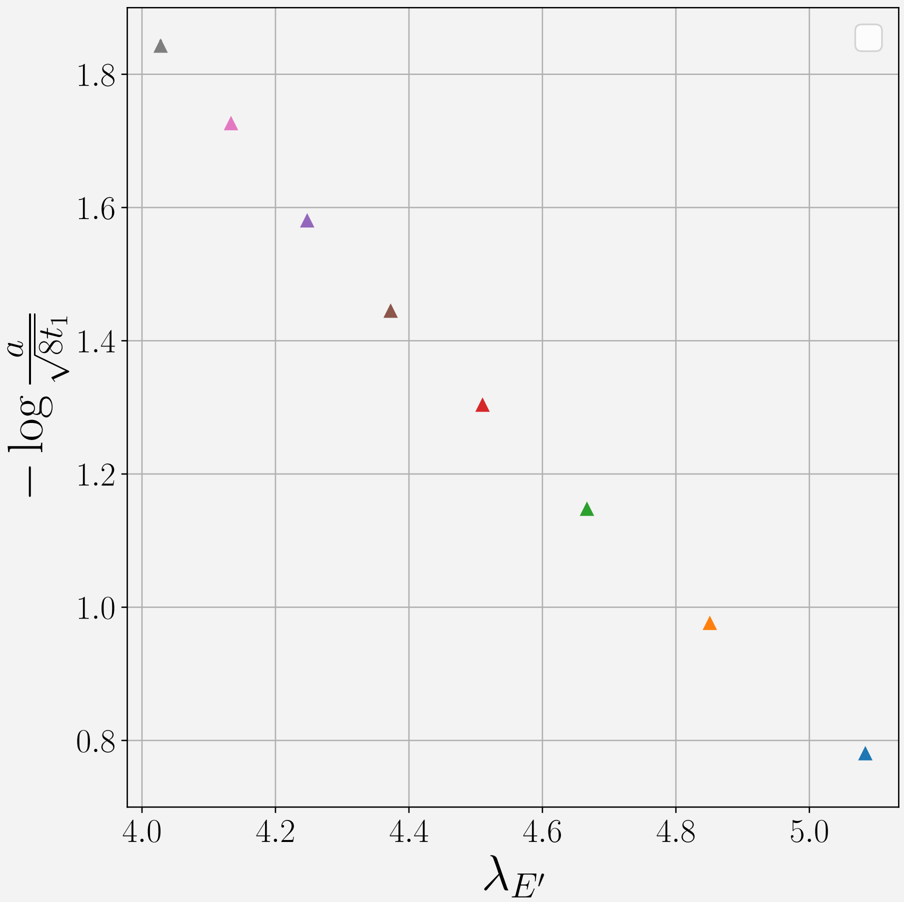
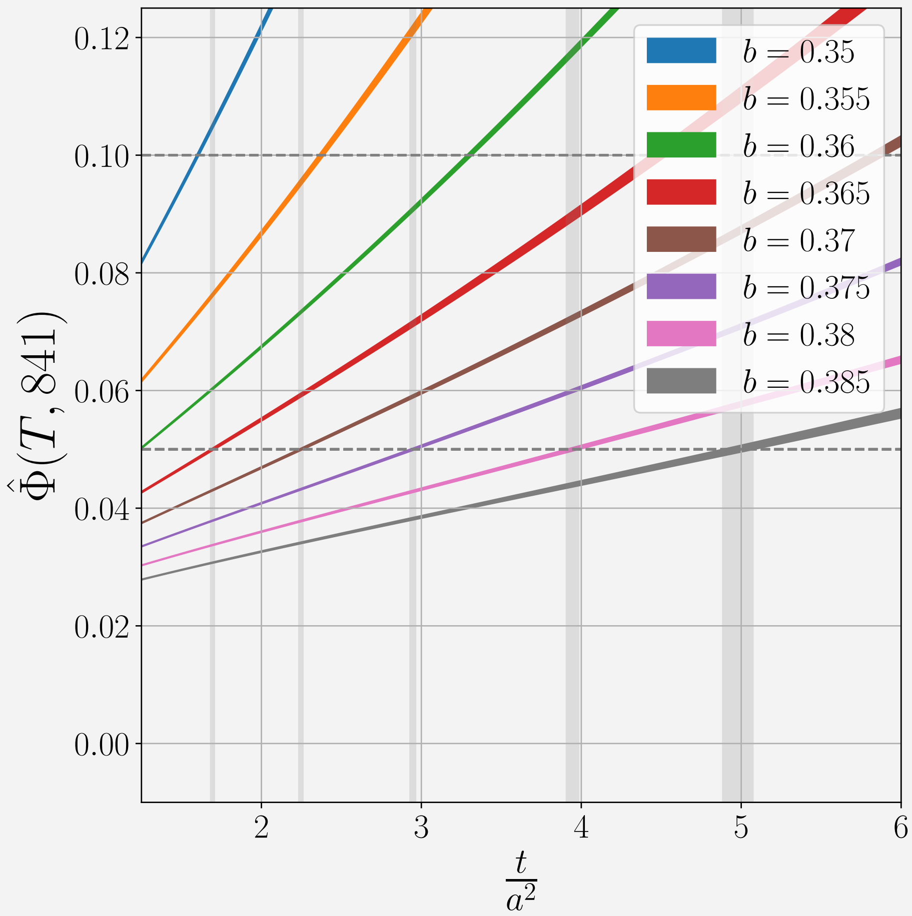
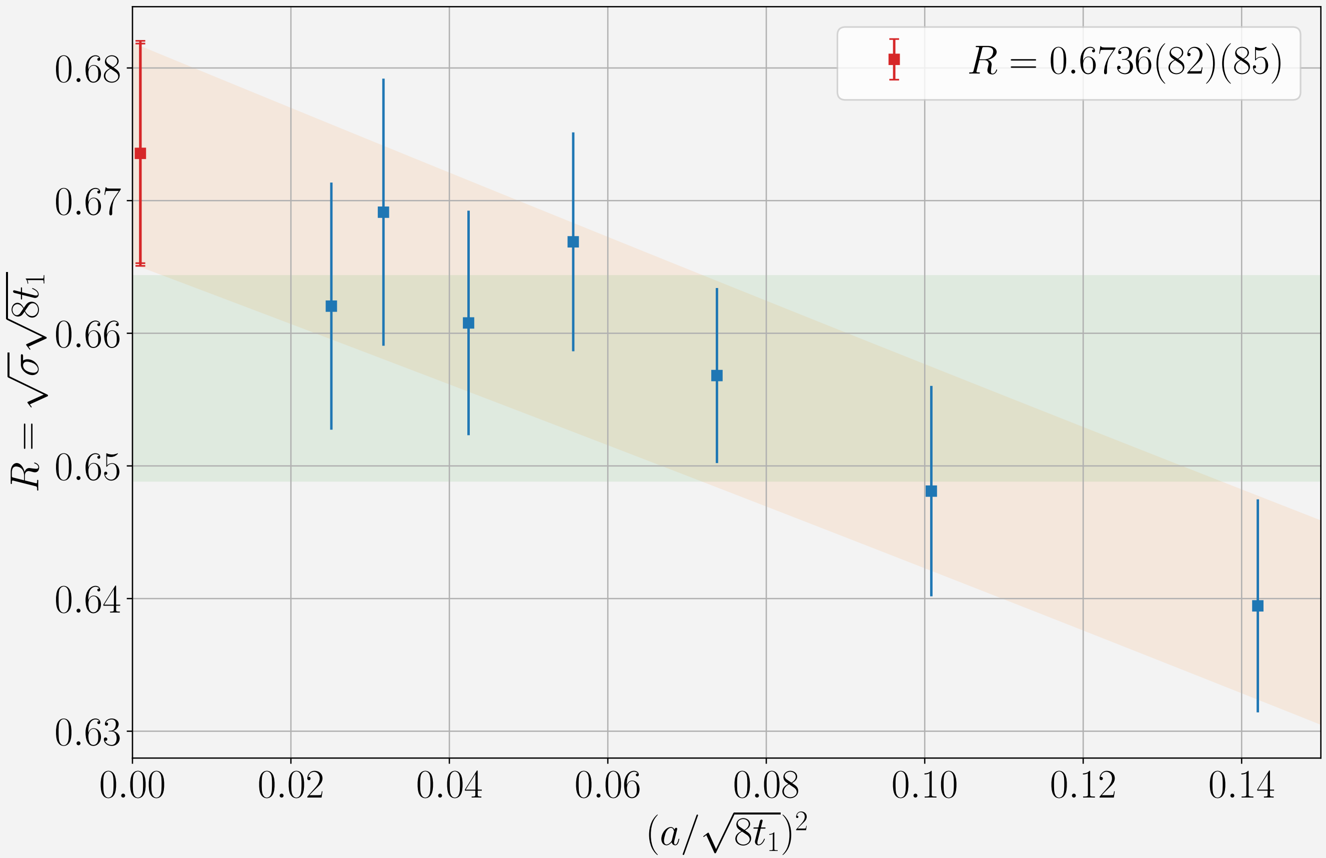




SCALING



$R = 0.6736(82)(85)$
0.68
0.67
0.66
0.65
0.64
0.63
$R = \sqrt{\sigma}\sqrt{8t_1}$
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
$(a/\sqrt{8t_1})^2$

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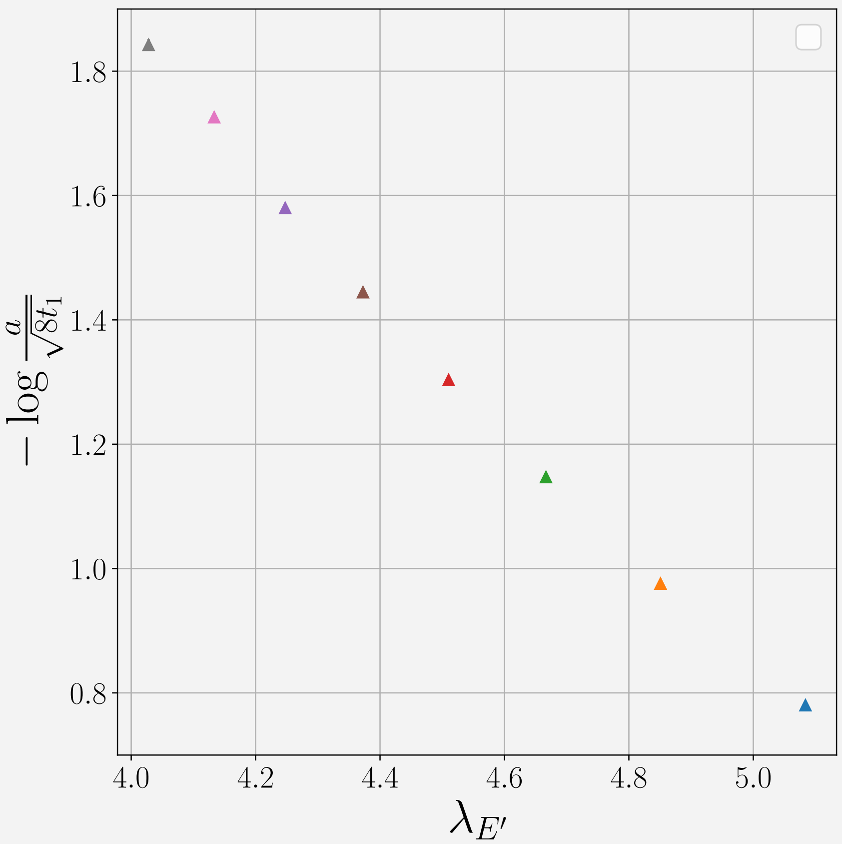


ASYMPTOTIC SCALING

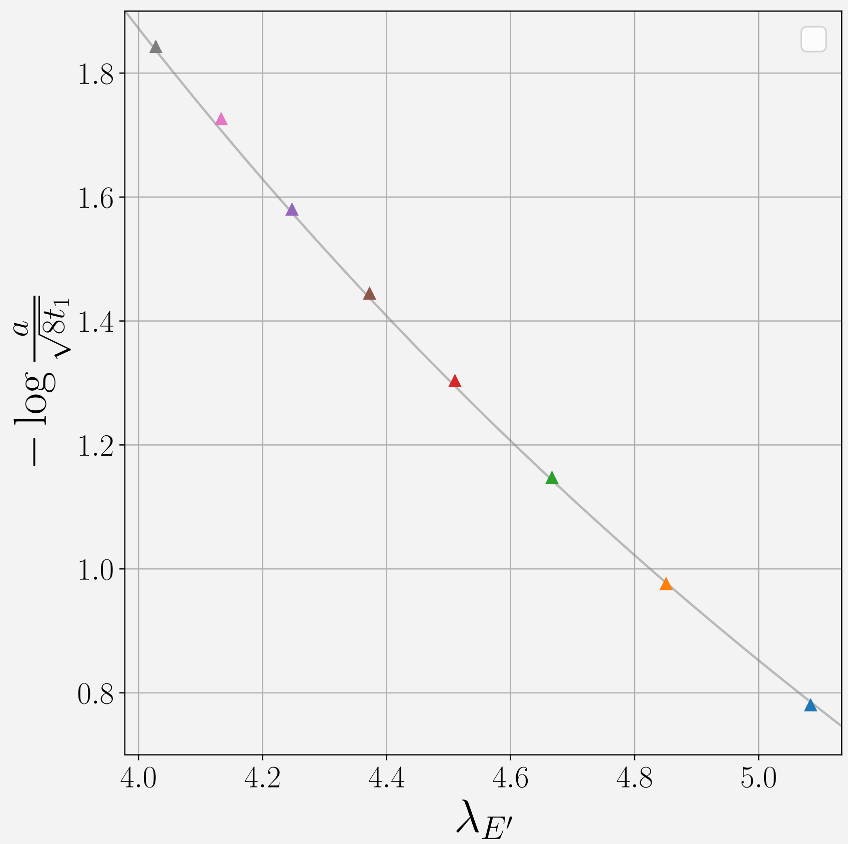
Fit PT to data

- Integrate the perturbative β-function at $\mathcal{O}(\lambda^4)$

$$-\log\frac{a}{\sqrt{8t_1}} = \log\Lambda_s\sqrt{8t_1} + \frac{1}{2b_0\lambda_s} + \frac{b_1}{2b_0^2}\log(b_0\lambda_s) + \frac{c_1^{(s)}}{2b_0}\lambda_s$$

- Try to fit against one improved coupling

$$\lambda_I = \frac{\lambda}{P(\lambda)}, \qquad \lambda_E = 8\big(1 - P(\lambda)\big), \qquad \lambda_{E'} = -8\log P(\lambda)$$



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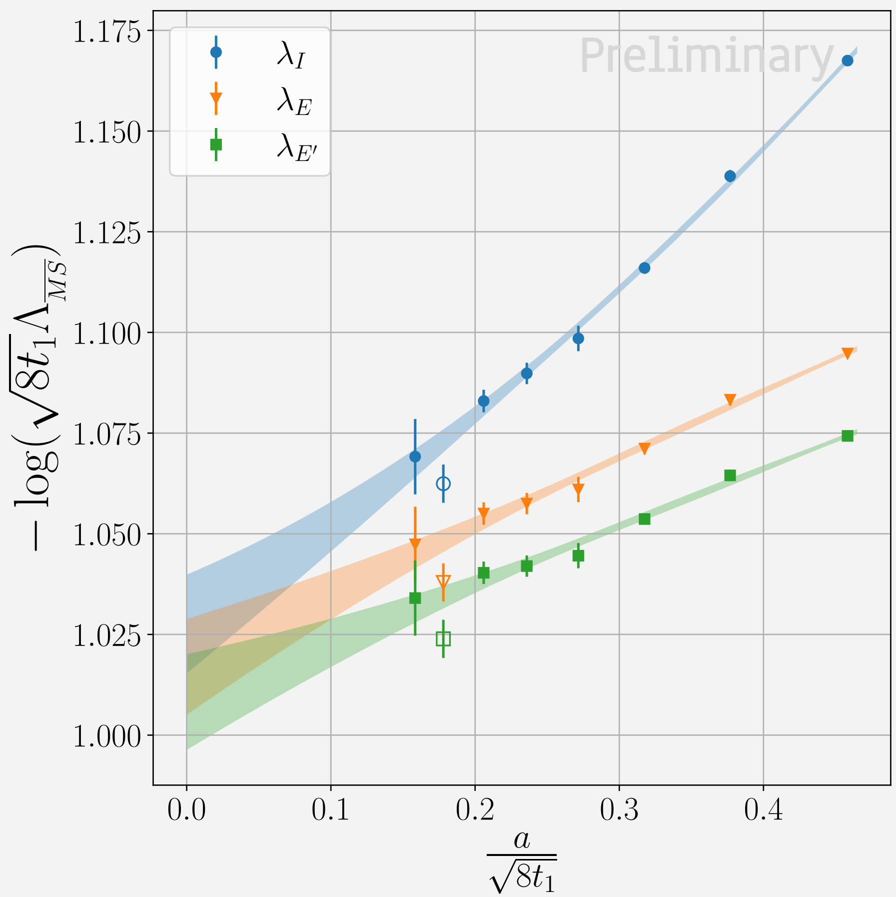
- Subtract $\log f(\lambda_s)$ to data, convert to $\overline{\text{MS}}$, take the limit

$$\Lambda_{\overline{\text{MS}}}\sqrt{8t_1} = \lim_{\lambda_s\to 0}\frac{\Lambda_{\overline{\text{MS}}}}{\Lambda_s} f(\lambda_s)\frac{\sqrt{8t_1}}{a}$$



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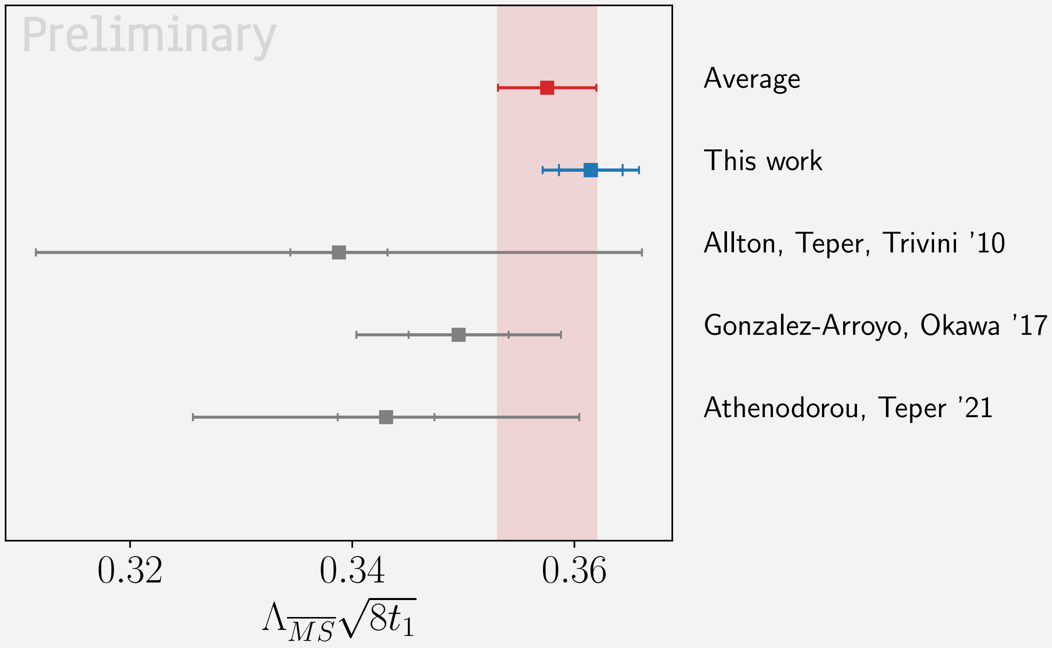
- Convert to common units to compare



THANK YOU

OVERVIEW

1
THE Λ-PARAMETER

2
LARGE-N_c LIMIT AND REDUCED MODELS

3
WILSON FLOW SCALE

4
RESULTS

BACKUP MATERIAL

THE LATTICE SCALE

The Wilson flow scale

- Build up a new flow observable which defines a coupling free from finite volume ($\mathcal{N}$) effect at leading order

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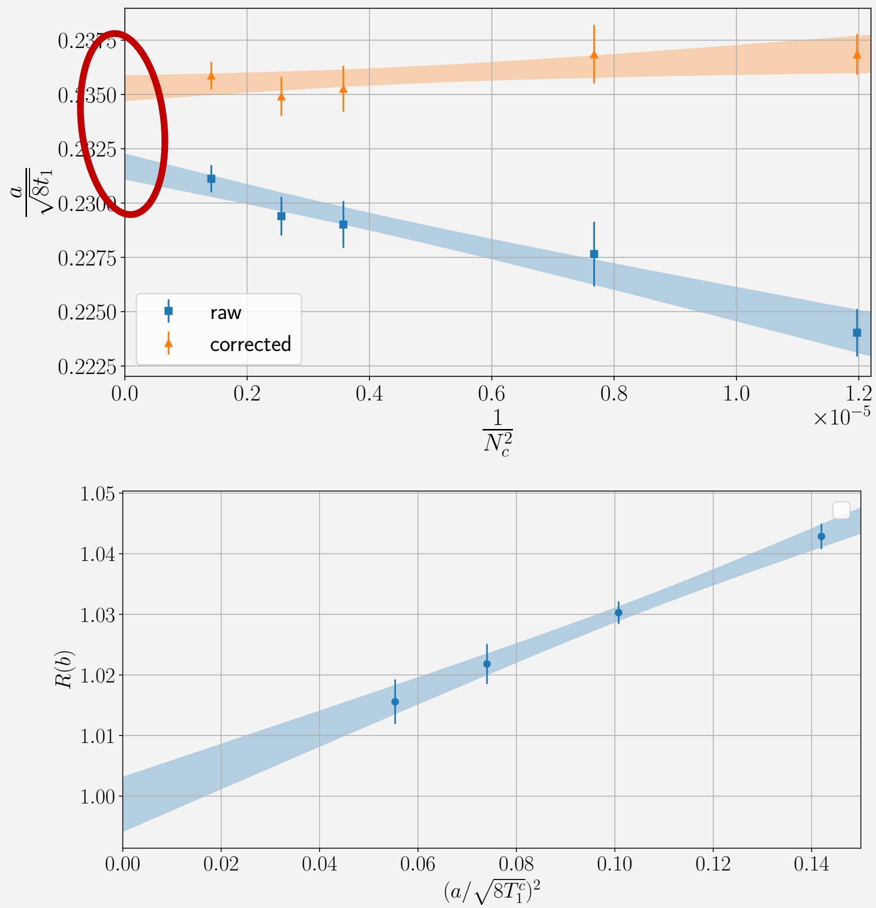
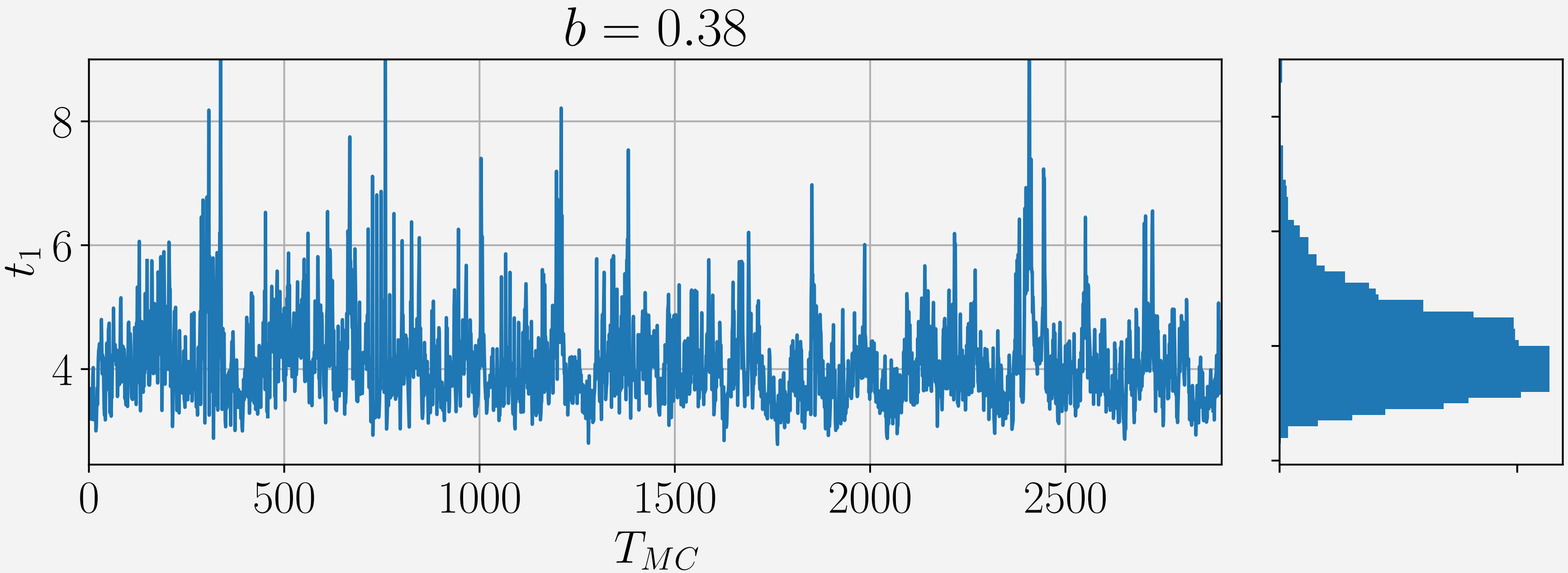
- Controls both finite-volume / lattice artefacts
- Remnant finite-volume effect are $< 3‰$ in the scaling window



$b = 0.38$



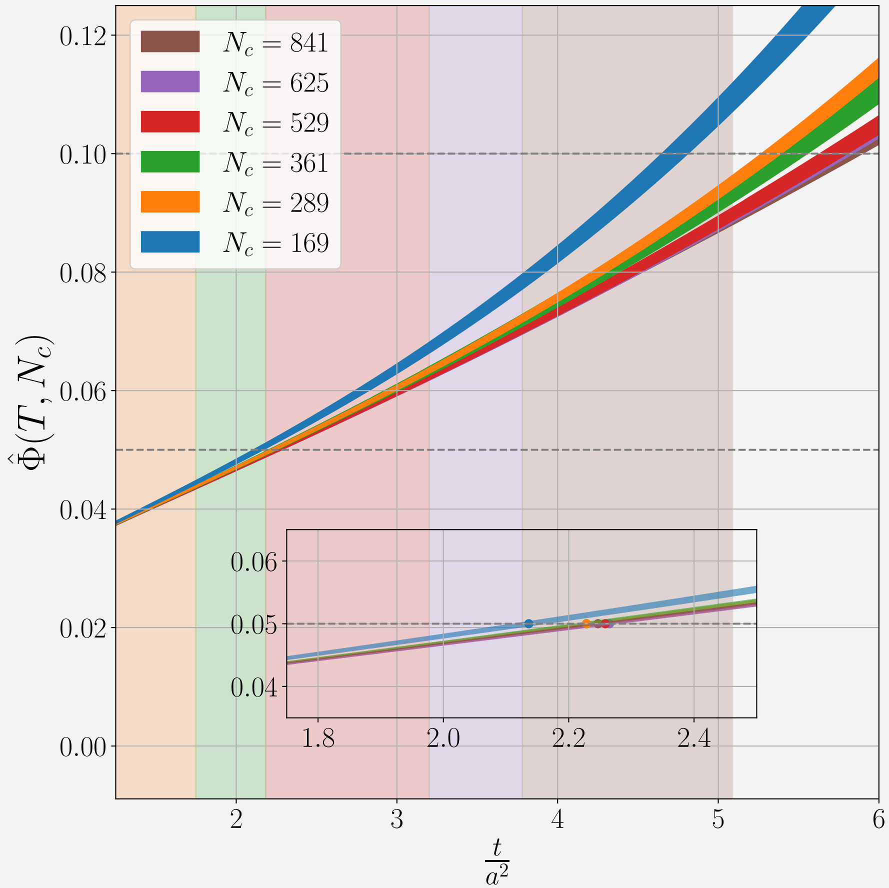
THE LATTICE SCALE

The Wilson flow scale

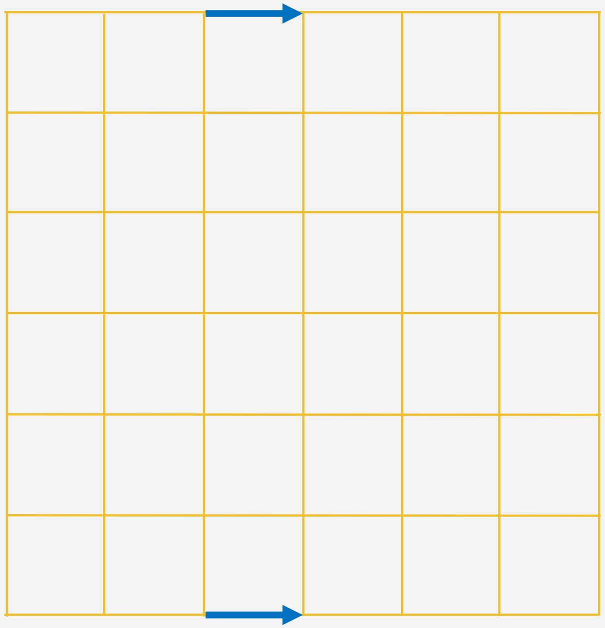
- Build up a new flow observable which defines a coupling free from finite volume ($\mathcal{N}$) effect at leading order

$$\hat{\Phi}(t) \equiv \frac{3/128\pi^2}{\mathcal{N}_{\text{lat}}\left(\sqrt{8t}/\sqrt{N_c}\right)} \left\langle \frac{t^2 E(t)}{N_c} \right\rangle \qquad T \in \left[1.25, \gamma^2 \frac{N_c}{8}\right]$$

- Controls both finite-volume / lattice artefacts
- Remnant finite-volume effect are $< 3‰$ in the scaling window



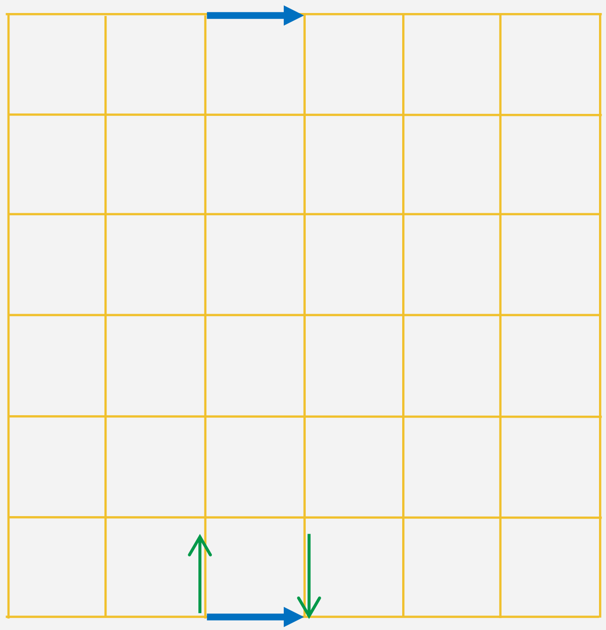
TWISTED BC & VOLUME REDUCTION



PERIODIC boundary conditions

$$U_\mu(n + L\,\hat{\nu}) \;=\; U_\mu(n)$$

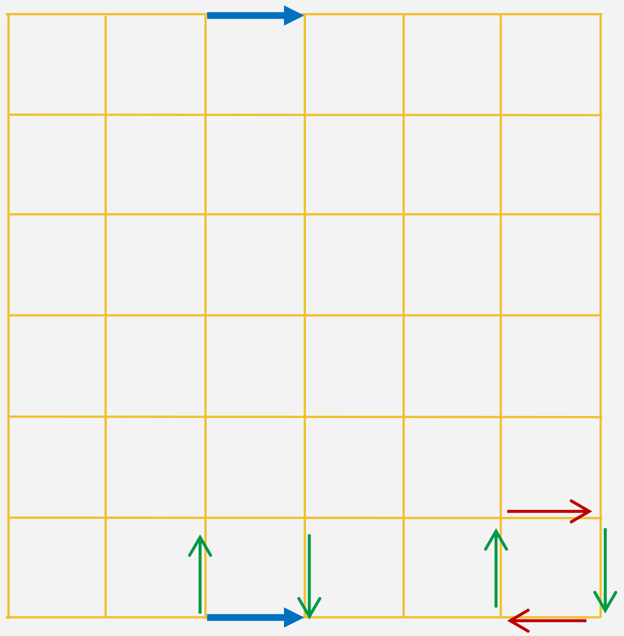
TWISTED BC & VOLUME REDUCTION



TWISTED boundary conditions

$$U_\mu(n + L\,\hat{\nu}) = \Gamma_\nu\, U_\mu(n)\, \Gamma_\nu^\dagger$$

TWISTED BC & VOLUME REDUCTION

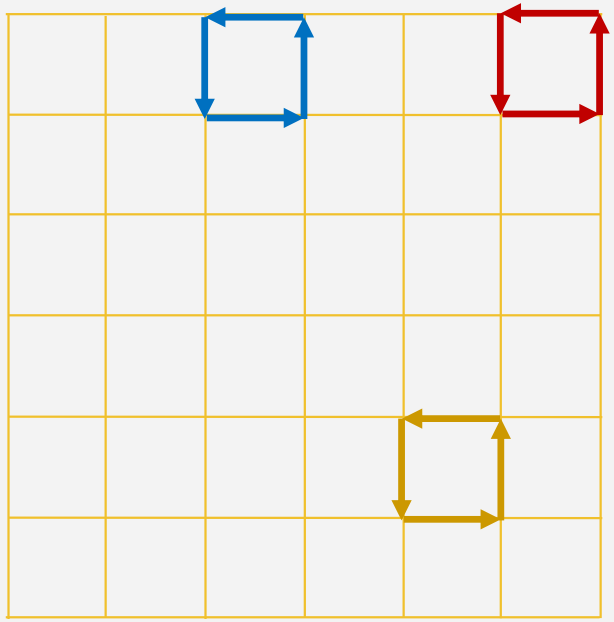


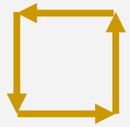
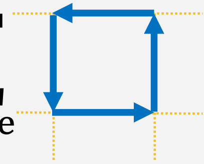
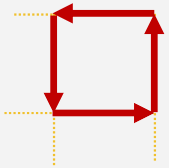
$$\Gamma_\mu \Gamma_\nu = e^{\frac{2\pi i}{N_c}\epsilon_{\mu\nu}} \ \Gamma_\nu \Gamma_\mu$$

TWISTED boundary conditions

$$U_\mu(n + L\,\hat{\nu}) \ = \Gamma_\nu \, U_\mu(n)\, \Gamma_\nu^\dagger$$

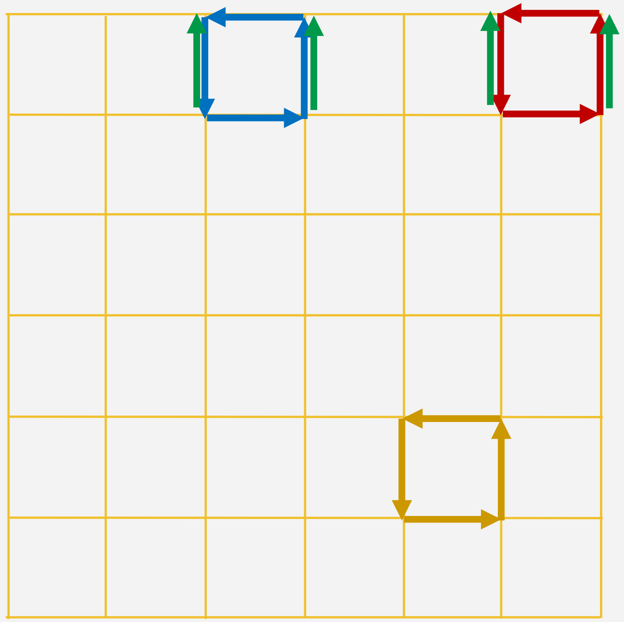
TWISTED BC & VOLUME REDUCTION



$$S_w = -bN_c \mathrm{Re}\,\mathrm{Tr}\Big[\sum_{\text{inner}}$$  $$+\sum_{\text{edge}}$$  $$+\sum_{\text{corner}}$$  $$\Big]$$

TWISTED BC & VOLUME REDUCTION

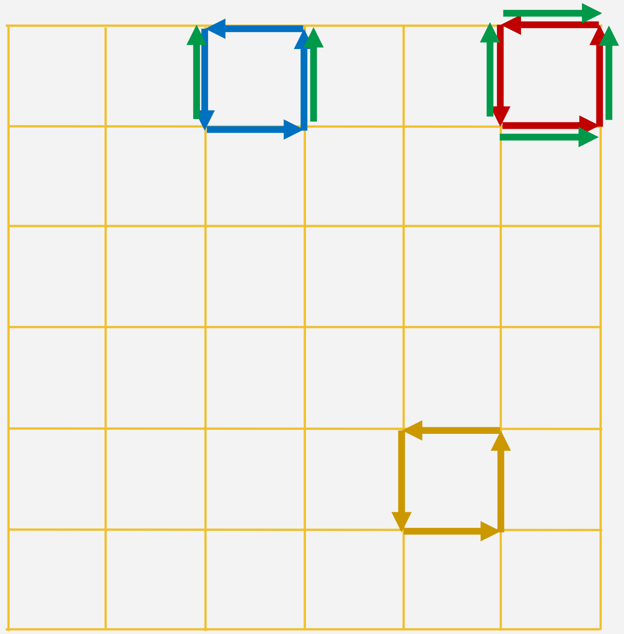
$$U'_\nu(x) = \Gamma_\nu \, U_\nu(x)$$

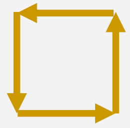
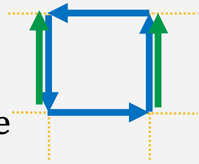
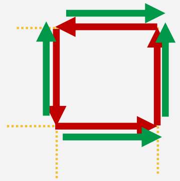


$$S_w = -\,bN_c \,\mathrm{Re}\,\mathrm{Tr}\left[\sum_{\text{inner}} \; + \sum_{\text{edge}} \; + \sum_{\text{corner}} \; \right]$$

TWISTED BC & VOLUME REDUCTION

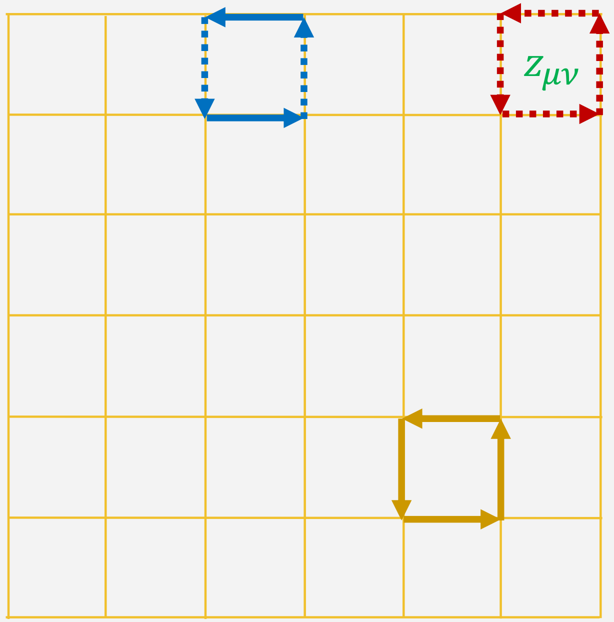
$$U'_\nu(x) = \Gamma_\nu\, U_\nu(x)$$

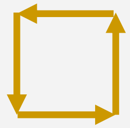
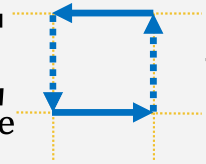
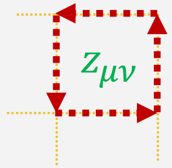


$S_w = -\,bN_c \mathrm{Re\,Tr}\Big[\sum_{\text{inner}}$  $+ \sum_{\text{edge}}$  $+ \sum_{\text{corner}}$  $\Big]$

TWISTED BC & VOLUME REDUCTION

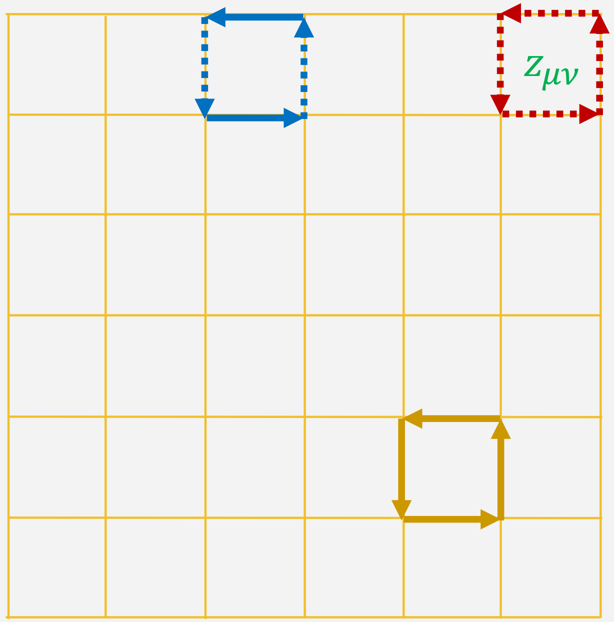
$$U'_\nu(x) = \Gamma_\nu \, U_\nu(x)$$

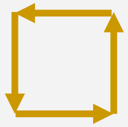
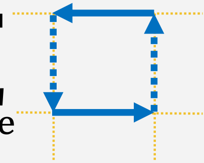
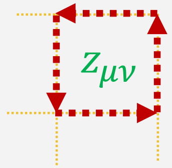


$S_w = -\, bN_c \mathrm{Re\,Tr}\Big[\sum_{\text{inner}}$  $+ \sum_{\text{edge}}$  $+ \sum_{\text{corner}}$ 

$\Big]$

TWISTED BC & VOLUME REDUCTION



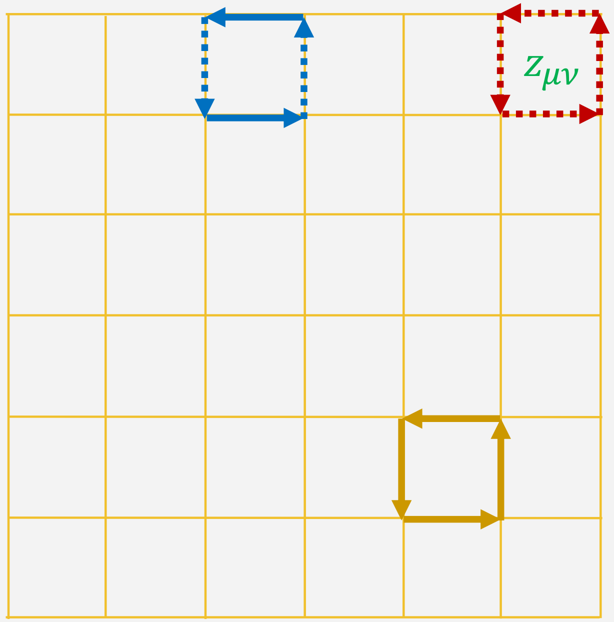
$S_w = -\,bN_c \mathrm{Re\,Tr}\Big[\sum_{\text{inner}}$  $+ \sum_{\text{edge}}$  $+ \sum_{\text{corner}}$ 

$\Big]$

$$U'_\nu(x) = \Gamma_\nu \, U_\nu(x)$$

$$= -bN_c \sum_{n,\mu\neq\nu} z_{\mu\nu}(n)\ \mathrm{Re\ Tr}\left[U_\mu(n) U_\nu(n+\hat{\mu}) U^\dagger_\mu(n+\hat{\nu}) U^\dagger_\nu(n)\right]$$

TWISTED BC & VOLUME REDUCTION



$$S_w = -\,bN_c \mathrm{Re\,Tr}\left[\sum_{\text{inner}} \;+\; \sum_{\text{edge}} \;+\; \sum_{\text{corner}} z_{\mu\nu} \right]$$

$$U'_\nu(x) = \Gamma_\nu\, U_\nu(x)$$

$$= -bN_c \sum_{n,\mu\neq\nu} z_{\mu\nu}(n)\ \mathrm{Re\ Tr}\left[U_\mu(n)U_\nu(n+\hat{\mu})U_\mu^\dagger(n+\hat{\nu})U_\nu^\dagger(n)\right]$$

$$U_\mu(n) \to U_\mu$$

$$S_{\mathrm{TEK}} = -bN_c \sum_{n,\mu\neq\nu} z_{\mu\nu}\ \mathrm{Re\ Tr}\left[U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\right]$$