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Zaragoza



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# ASYMPTOTIC SCALING IN YANG-MILLS AT LARGE- $N_c$

The lattice scale and the  $\Lambda_{\overline{\text{MS}}}$ -parameter at large- $N_c$   
from twisted volume reduction

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Autonomous University of Madrid

## THE LATTICE RUNNING COUPLING

### The $\Lambda_{\overline{\text{MS}}}$ -parameter

RG equations for the 't Hooft coupling  $\lambda = g^2 N_c$ :

$$\frac{d\lambda_s}{d \log(\mu^2)} = \beta_s(\lambda_s) \simeq -b_0 \lambda_s^2 - b_1 \lambda_s^3 - b_2^{(s)} \lambda_s^4 - \dots$$

and, upon integration

$$\frac{\Lambda_s}{\mu} = (b_0 \lambda_s)^{-\frac{b_1}{2b_0^2}} e^{-\frac{1}{2b_0 \lambda_s}} e^{-\int^\lambda dx \left[ \frac{1}{2\beta_s(x)} + \frac{1}{2b_0 x^2} - \frac{b_1}{2b_0 x} \right]}$$

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- Compute the non-perturbative running of the coupling at low energies  $\mu_{\text{had}}$  and match to PT at  $\mu_{\text{PT}} \gg \mu_{\text{had}}$

$$\frac{\Lambda_s}{\mu_{\text{had}}} = \frac{\Lambda_s}{\mu_{\text{pt}}} e^{-\int_{\lambda(\mu_{\text{pt}})}^{\lambda(\mu_{\text{had}})} \frac{dx}{2\beta_s(x)}}$$

(e.g. finite size scaling)

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Weak coupling is hard to simulate on the lattice and for feasible simulations

- $\mathcal{O}(a^2)$  corrections (scaling violations)
- Lattice scheme w/ Wilson action has large higher order terms in the  $\beta$ -function

$$\frac{\Lambda_{\overline{\text{MS}}}}{\Lambda_{\text{lat}}} \sim 38.853 e^{-\frac{3\pi^2}{11N_c^2}}$$

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(e.g. finite size scaling)

or

- Simulate large range of bare couplings  $b = 1/\lambda$ , use improved lattice couplings to improve convergence with PT

$$\lambda_I = \frac{b}{P(b)}, \quad \lambda_E = 8(1 - P(b)), \quad \lambda_{E'} = -8 \log P(b)$$

[Allton et al. JHEP 07 (2008) 21], [Gonzalez-Arroyo, Okawa, Phys. Let. B 718 (2013)]

## LARGE-N AND VOLUME REDUCTION

Gauge fields on a twisted lattice

[Gonzalez-Arroyo, Okawa,  
Phys. Rev. D 27 (1983),  
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$$S = bN \sum_n \sum_{\mu \neq \nu} \text{tr} (\mathbb{I} - U_\mu(n) U_\nu(n + \mu) U_\mu^\dagger(n + \nu) U_\nu^\dagger(n))$$

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Use reduction + twisted BC

$$\begin{aligned} U_\mu(n) &\rightarrow U_\mu \\ U_\mu(n + \nu) &\rightarrow \Gamma_\nu U_\mu \Gamma_\nu^\dagger \\ V_\mu &= U_\mu \Gamma_\mu \end{aligned}$$

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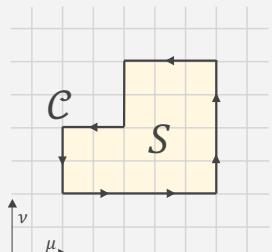
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Gauge observables

$$\begin{aligned} \langle \text{Tr } U(\mathcal{C}) \rangle_{N_c \rightarrow \infty} & \\ \approx & \\ z_{\mu\nu}(S) \langle \text{Tr } V_\mu \dots V_\nu \dots V_\mu^\dagger \dots \rangle_{\text{TEK } (N_c \sim \infty)} & \end{aligned}$$



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Finite- $N_c$  corrections

$1^4$ -lattice with tBC  
with  $N_c \sim 10^2 / 10^3$

$\approx$

"Effective"  
periodic lattice  
with  $L = \sqrt{N_c}$

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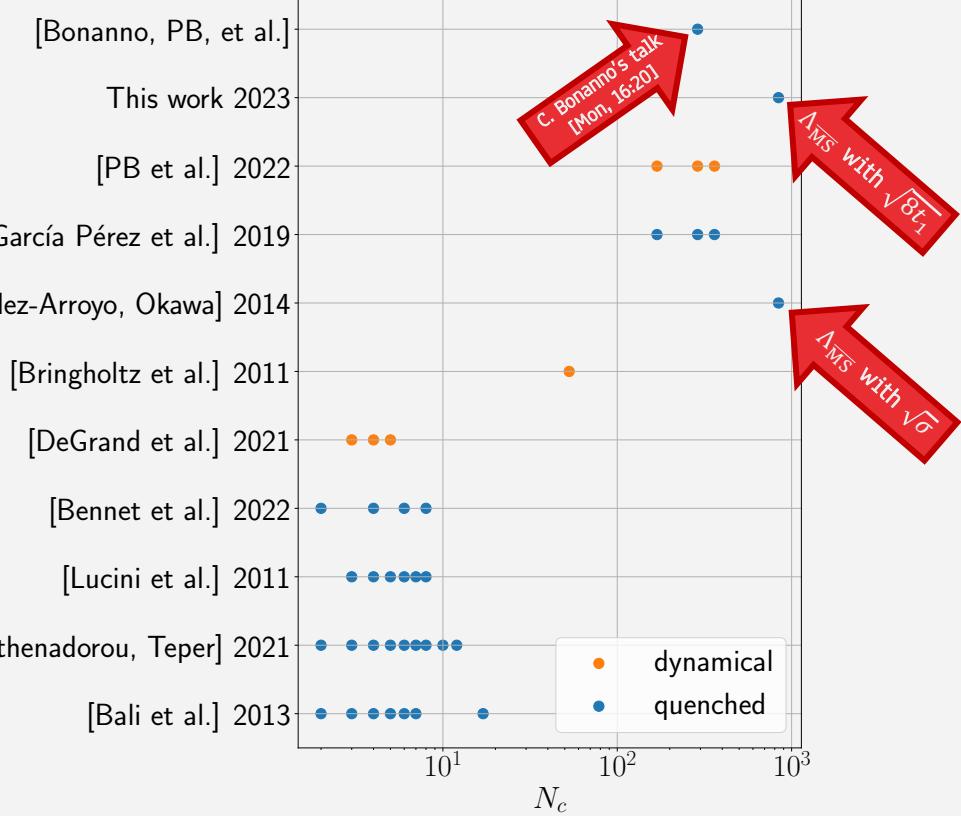
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## THE LATTICE SCALE

### The Wilson flow scale

- Flowed energy density

$$E = -\frac{1}{128} \sum_{\mu \neq \nu} z_{\nu\mu} \text{Tr} ( \text{Diagram} - h.c.)^2$$


- Integrate flow equations to get  $E(t)$
- Solve

$$\left. \left( \frac{t^2 E(t)}{N_c} \right) \right|_{t=t_1} = 0.05$$

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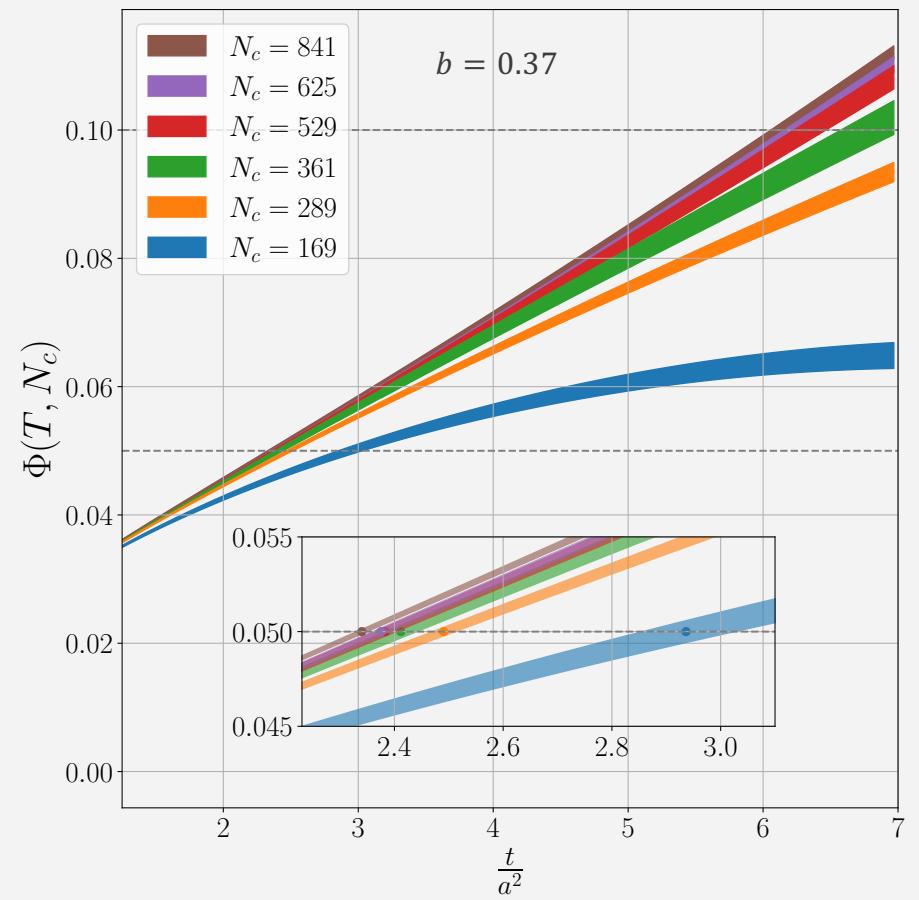
### The Wilson flow scale

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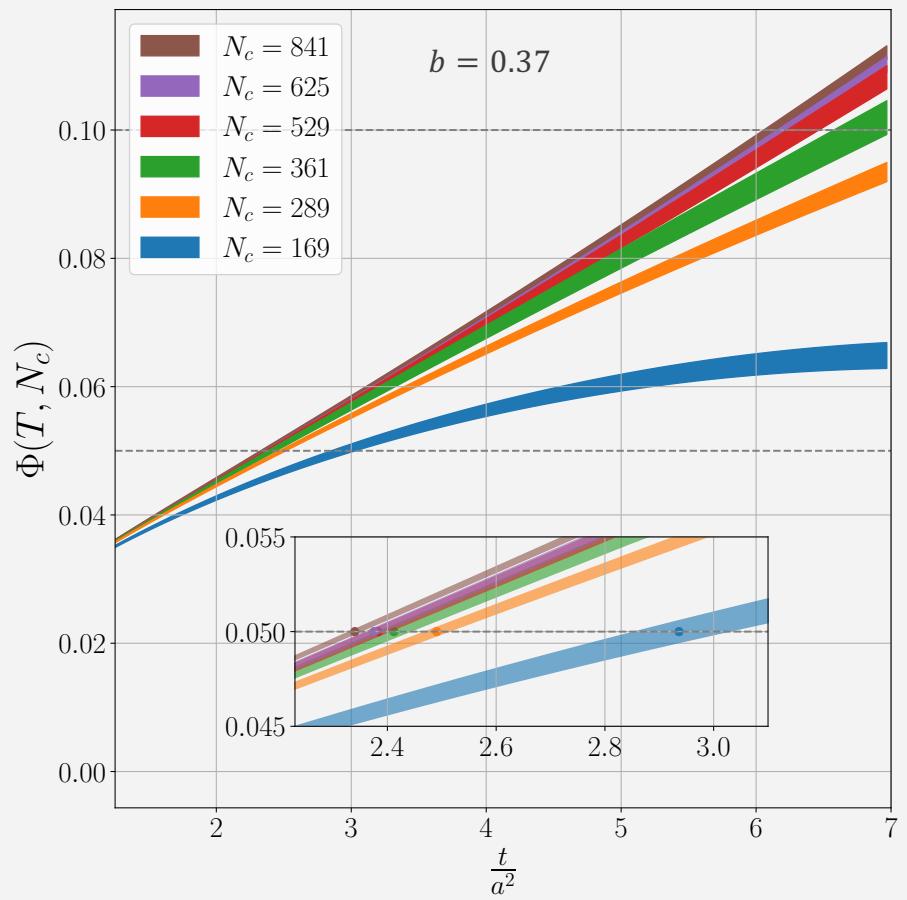
$$E = -\frac{1}{128} \sum_{\mu \neq \nu} z_{\nu\mu} \text{Tr} \left( \begin{array}{ccccc} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{array} \right) - h.c.)^2$$

- Integrate flow equations to get  $E(t)$

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- Treat finite “volume” effects



## THE LATTICE SCALE

### The Wilson flow scale

- (twisted) flowed running coupling

$$\lambda_{\text{gf}} \left( \mu = \frac{1}{\sqrt{8t}} \right) \equiv \frac{128\pi^2 N_c^2}{N_c^2 - 1} \left\langle \frac{t^2 E(t)}{N_c} \right\rangle$$

- On a  $V = 1^4$  torus with twisted BC at finite  $N_c$   
[García Pérez, Ibañez, JHEP 03 (2019) 200]

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- Build up a new flow observable from  $\hat{\lambda}$ , free from finite volume effects ( $\mathcal{N}$ ) at leading order in PT

[PB, García Pérez, González-Arroyo, Ishikawa, Okawa JHEP 07 (2022) 074]

$$\hat{\Phi}(t) \equiv \frac{3/128\pi^2}{\mathcal{N}_{\text{lat}}(\sqrt{8t}/\sqrt{N_c})} \left\langle \frac{t^2 E(t)}{N_c} \right\rangle \quad T \in \left[ 1.25, \gamma^2 \frac{N_c}{8} \right]$$

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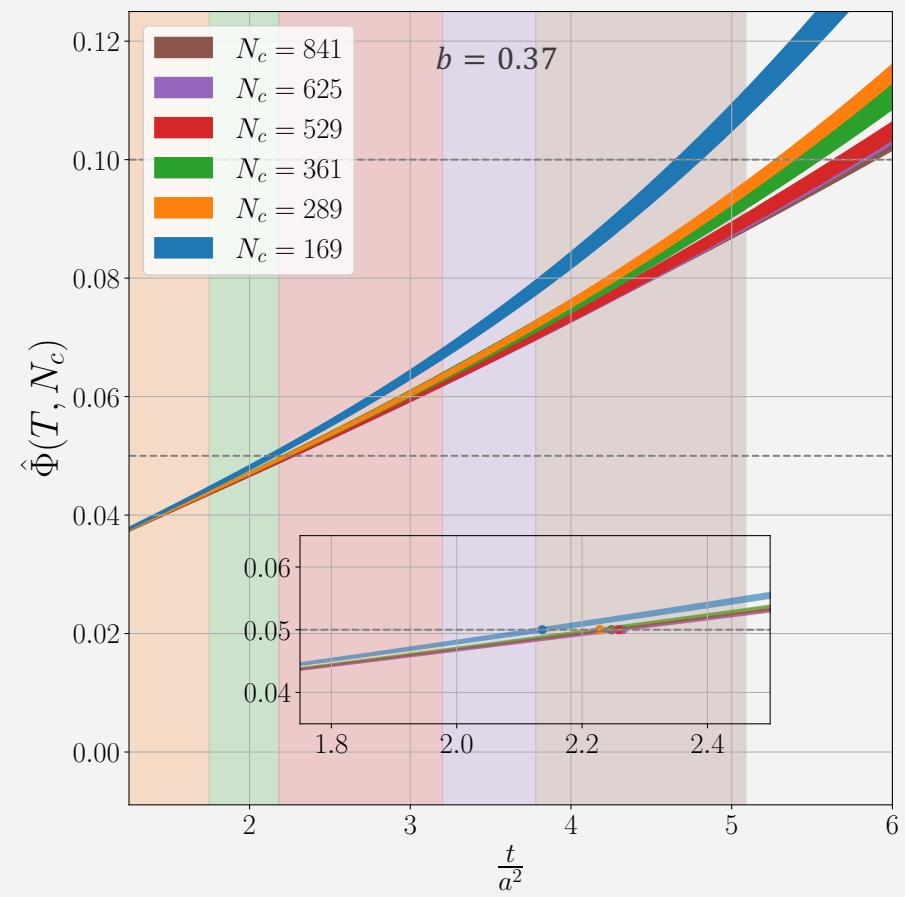
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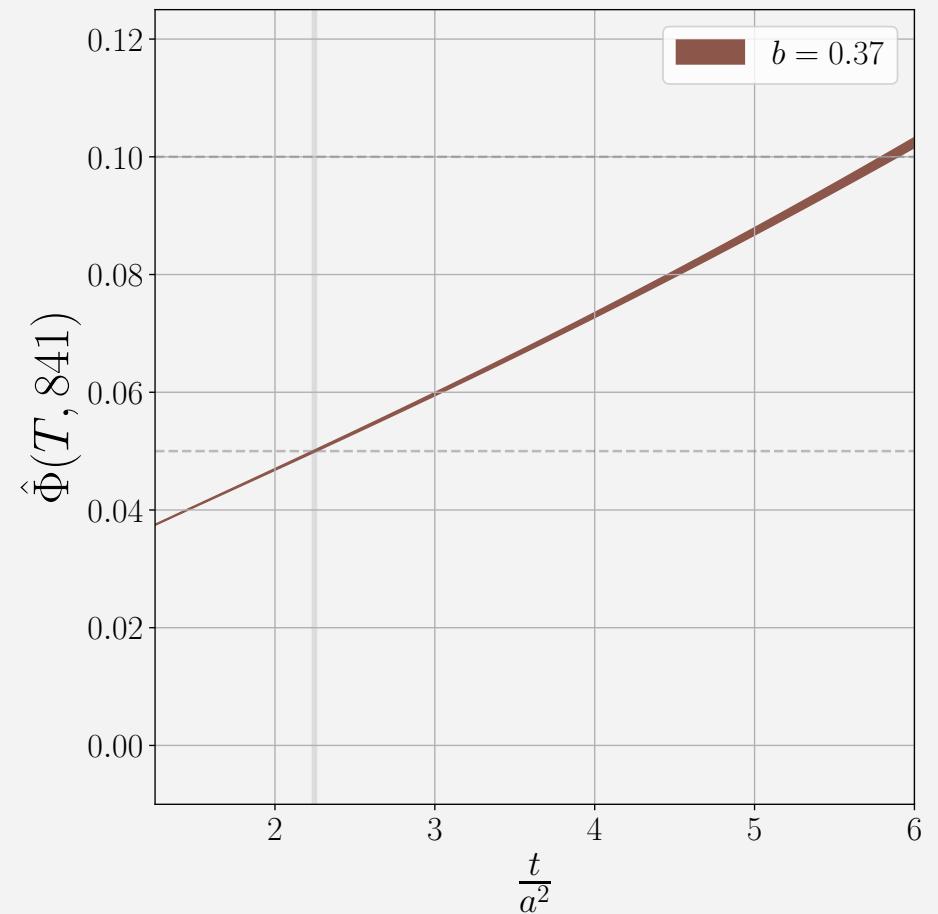
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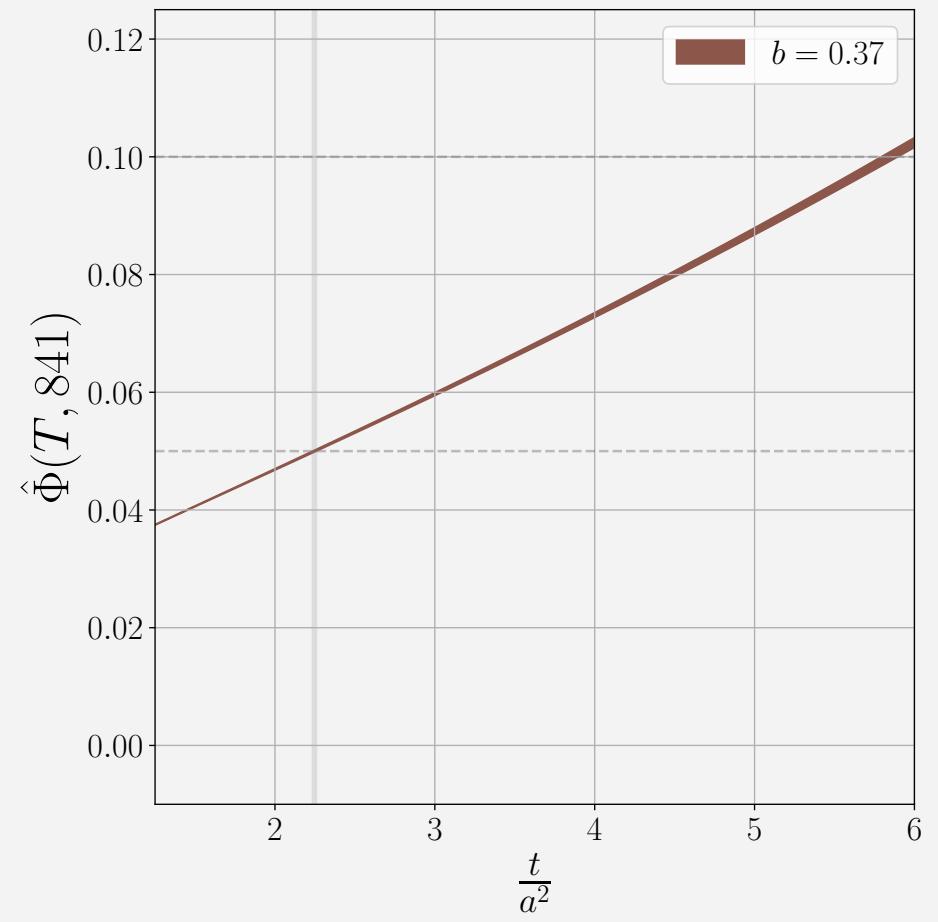
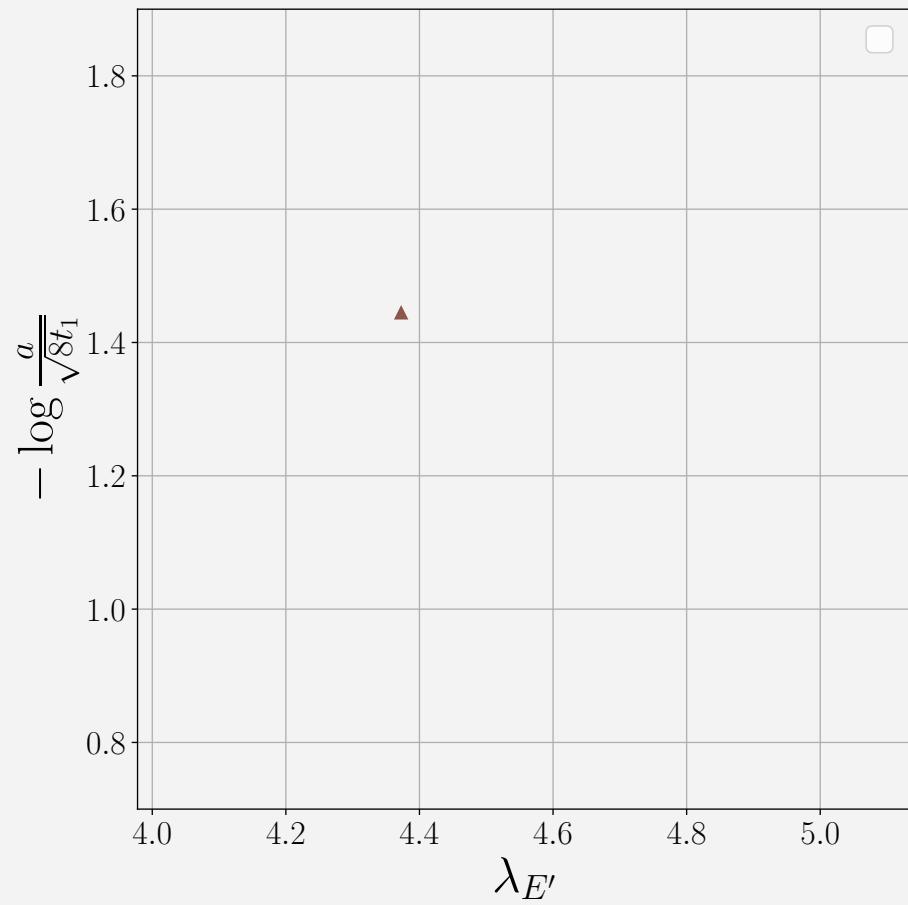
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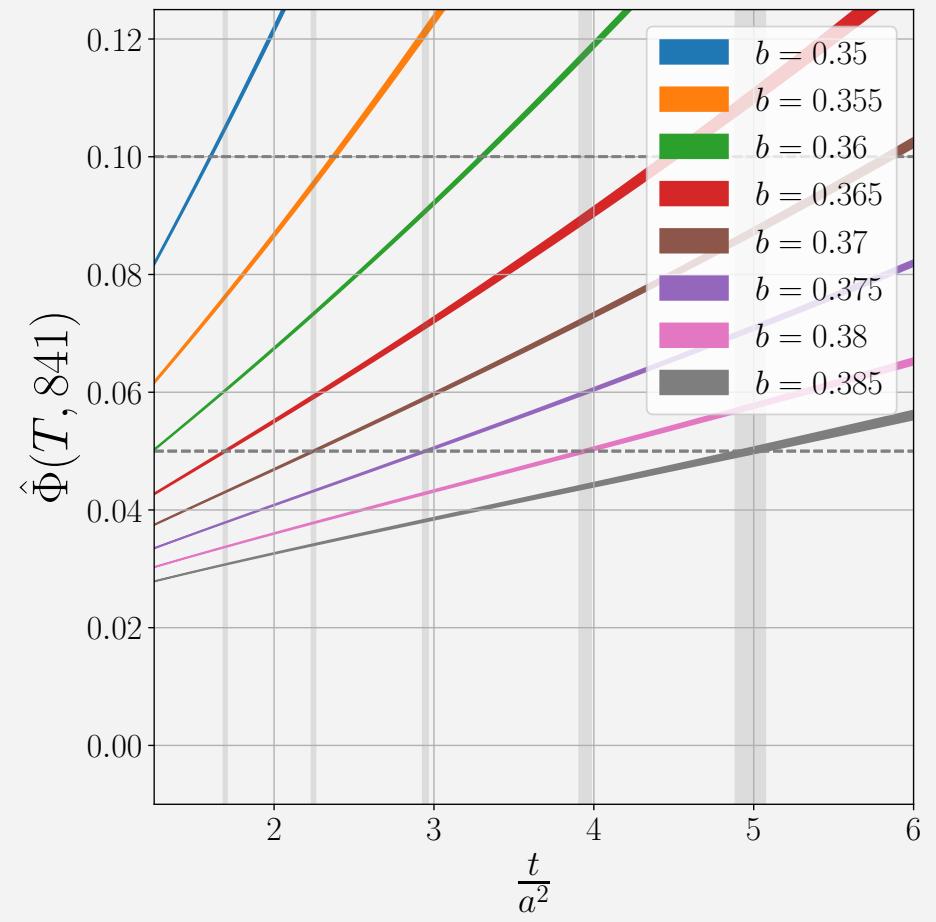
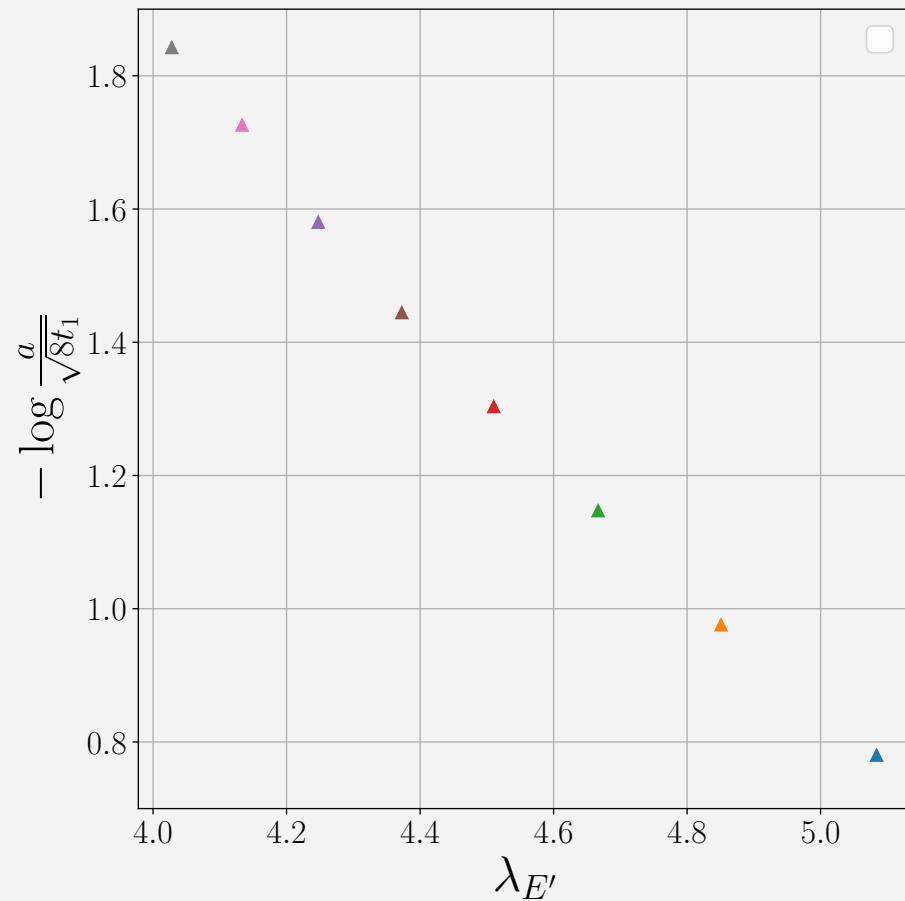
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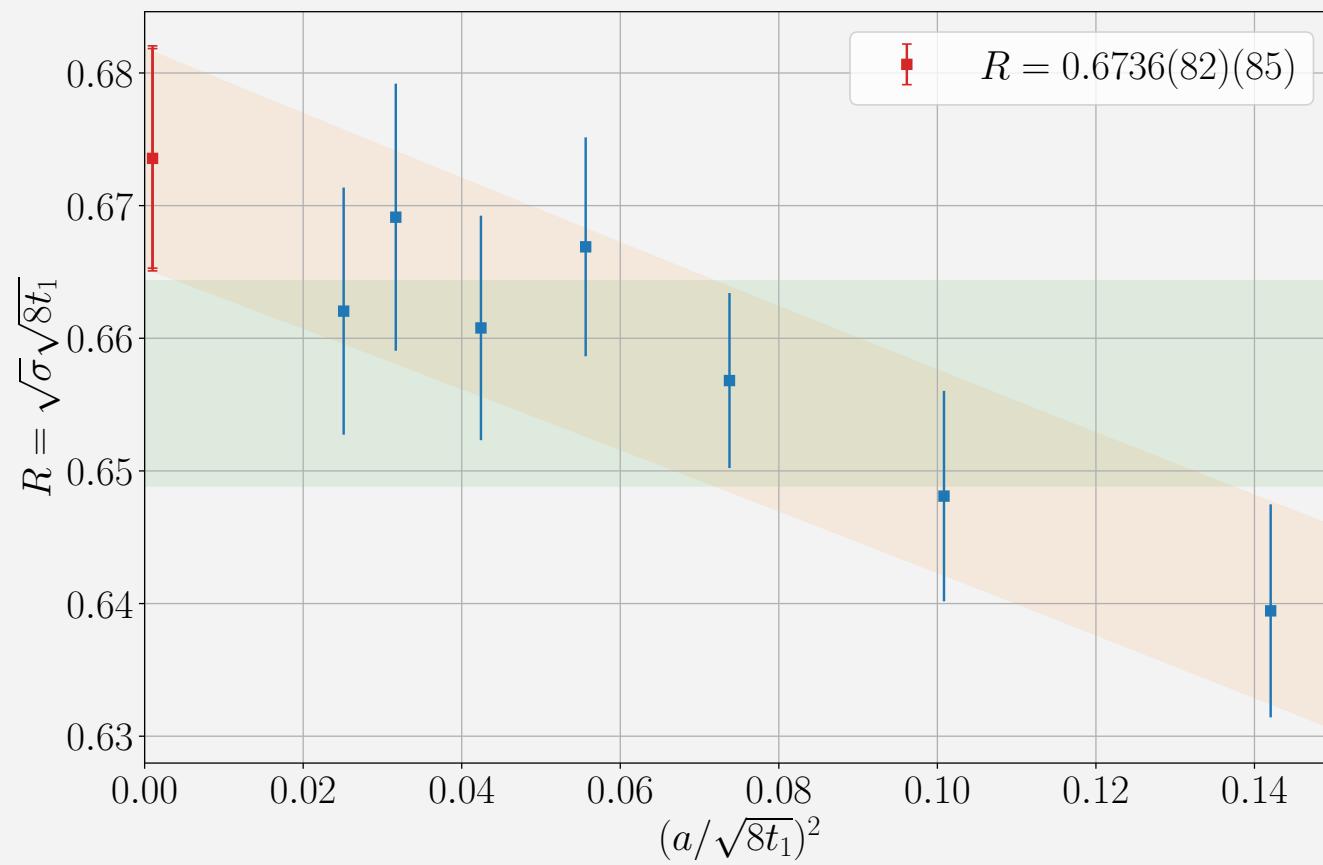
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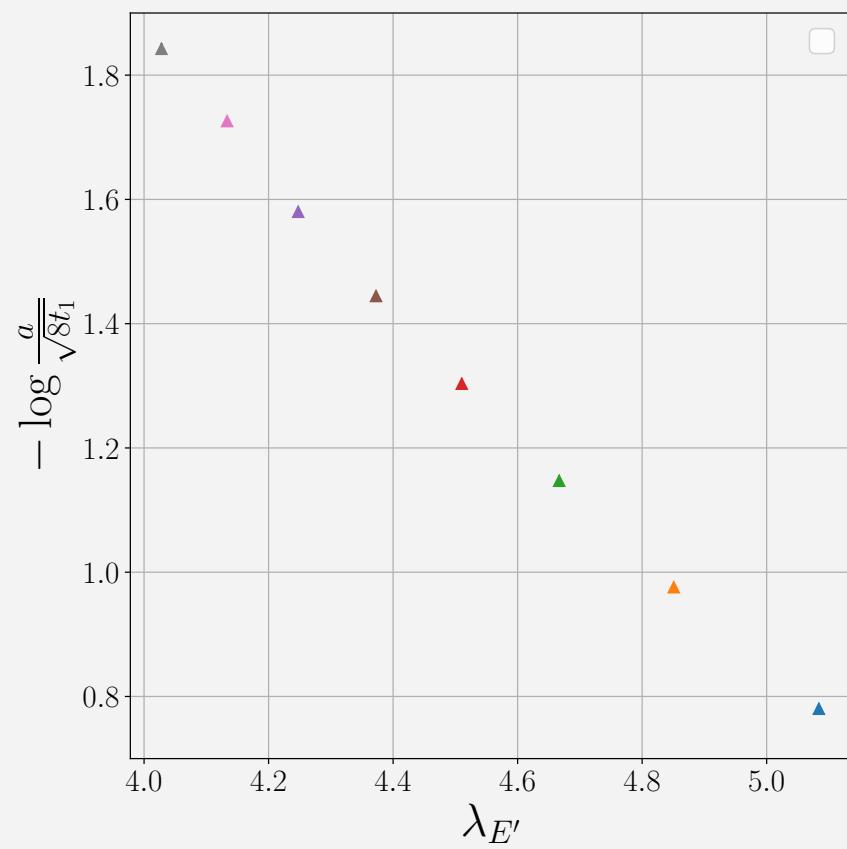
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## SCALING



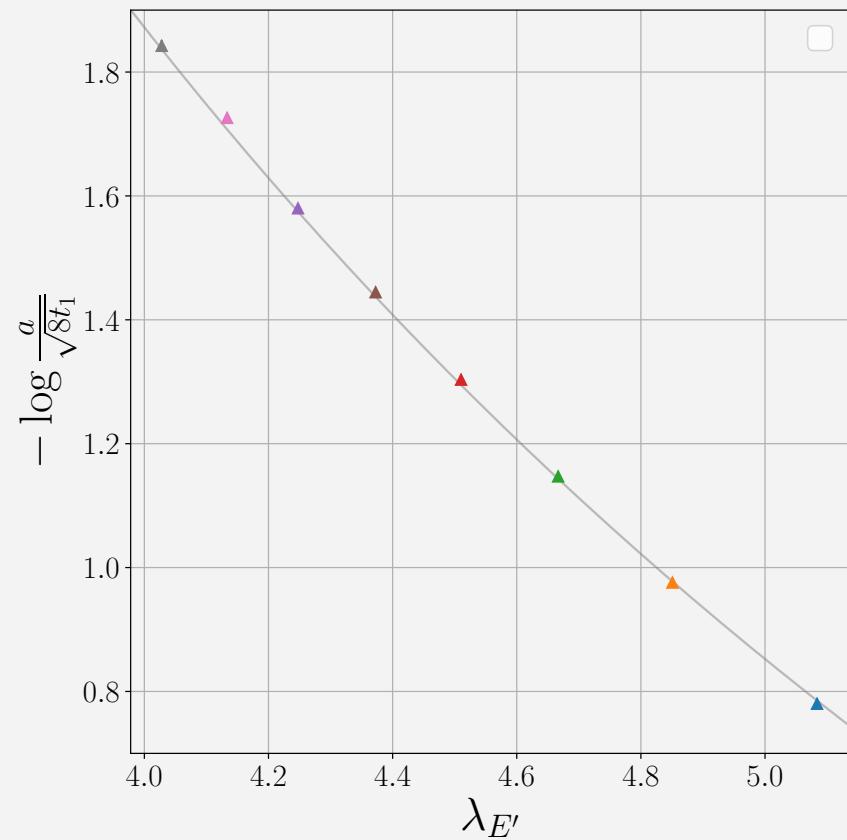
## SCALING



## ASYMPTOTIC SCALING

### Fit PT to data

- Integrate the perturbative  $\beta$ -function at  $\mathcal{O}(\lambda^4)$ 
$$-\log \frac{a}{\sqrt{8t_1}} = \log \Lambda_s \sqrt{8t_1} + \frac{1}{2b_0 \lambda_s} + \frac{b_1}{2b_0^2} \log(b_0 \lambda_s) + \frac{c_1^{(s)}}{2b_0} \lambda_s$$
- Try to fit against one improved coupling
$$\lambda_I = \frac{\lambda}{P(\lambda)}, \quad \lambda_E = 8(1 - P(\lambda)), \quad \lambda_{E'} = -8 \log P(\lambda)$$



## THE $\Lambda_{\overline{\text{MS}}}$ -PARAMETER

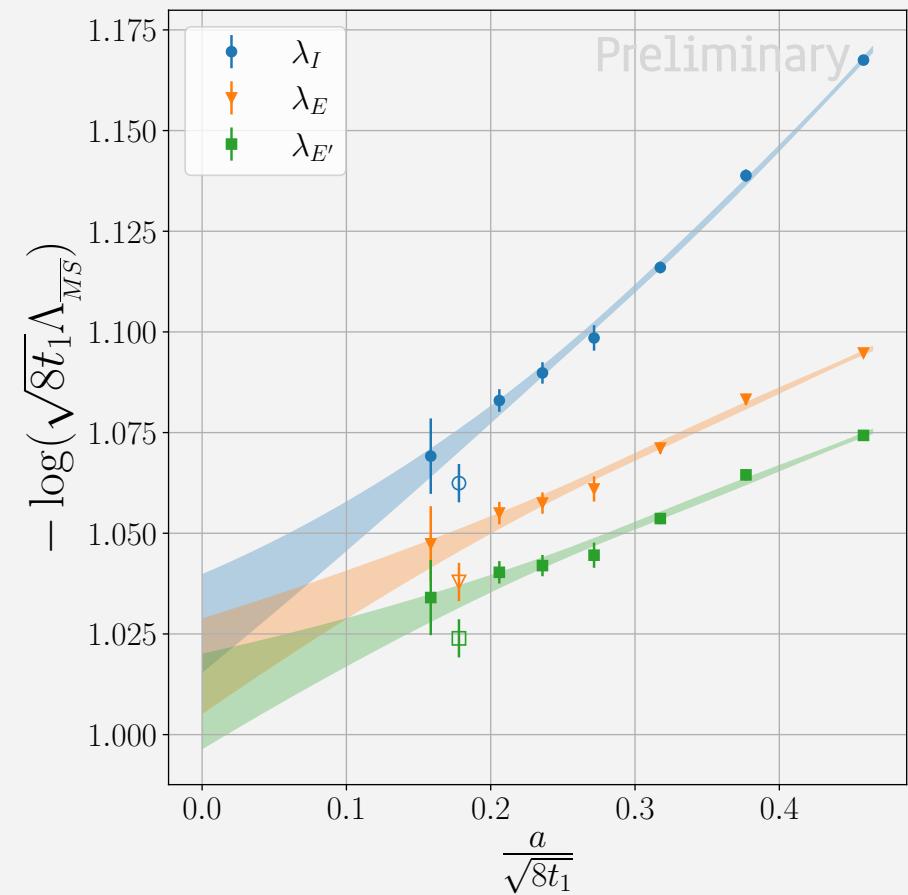
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- Subtract  $\log f(\lambda_s)$  to data, convert to  $\overline{\text{MS}}$ , take the limit

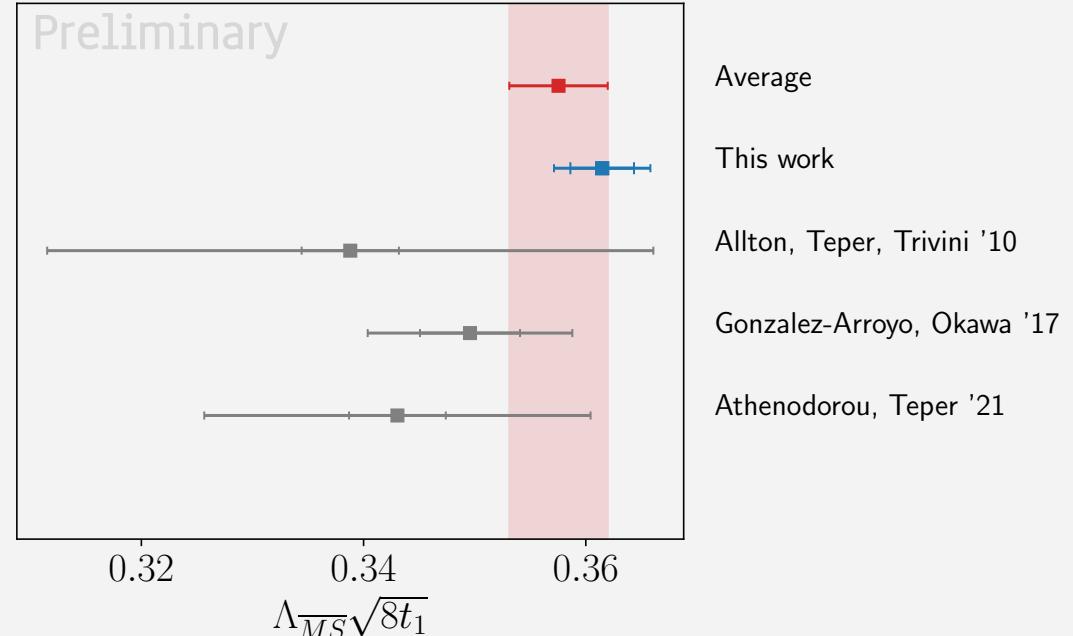
$$\Lambda_{\overline{\text{MS}}} \sqrt{8t_1} = \lim_{\lambda_s \rightarrow 0} \frac{\Lambda_{\overline{\text{MS}}}}{\Lambda_s} f(\lambda_s) \frac{\sqrt{8t_1}}{a}$$



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- Subtract  $\log f(\lambda_s)$  to data, convert to  $\overline{MS}$ , take the limit
- $$\Lambda_{\overline{MS}} \sqrt{8t_1} = \lim_{\lambda_s \rightarrow 0} \frac{\Lambda_{\overline{MS}}}{\Lambda_s} f(\lambda_s) \frac{\sqrt{8t_1}}{a}$$
- Convert to common units to compare



# **THANK YOU**

## OVERVIEW

1  
THE  $\Lambda$ -PARAMETER

2  
LARGE- $N_c$  LIMIT AND  
REDUCED MODELS

3  
WILSON FLOW SCALE

4  
RESULTS

## **BACKUP MATERIAL**

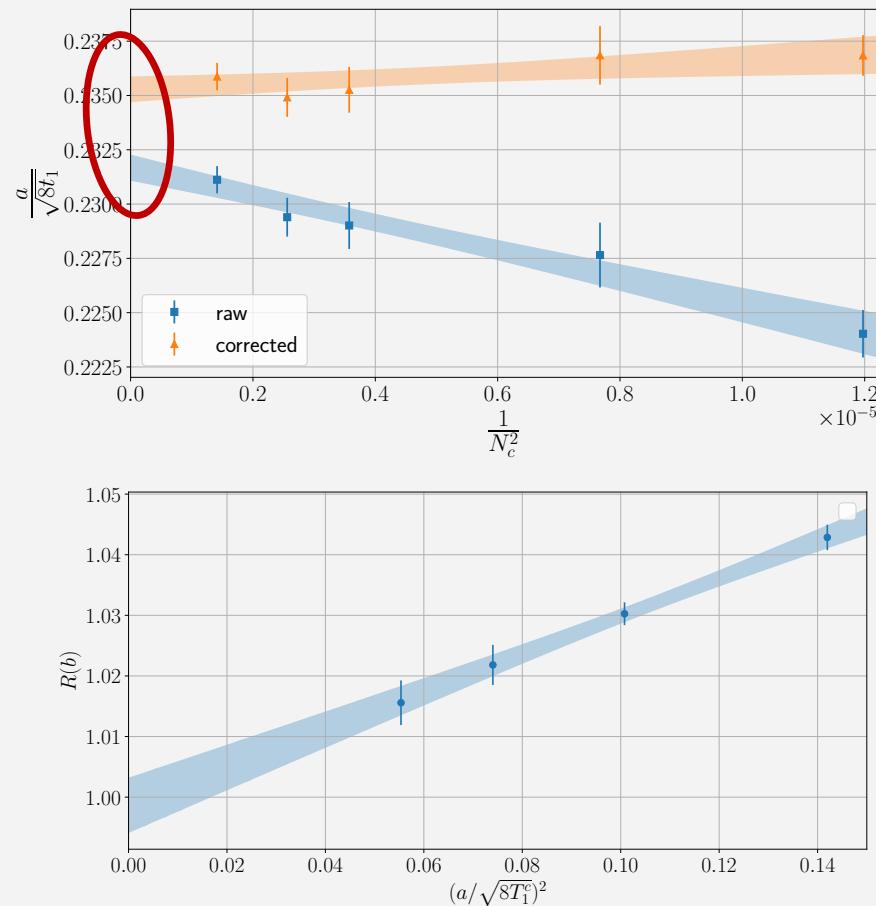
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### The Wilson flow scale

- Build up a new flow observable which defines a coupling free from finite volume ( $\mathcal{N}$ ) effect at leading order

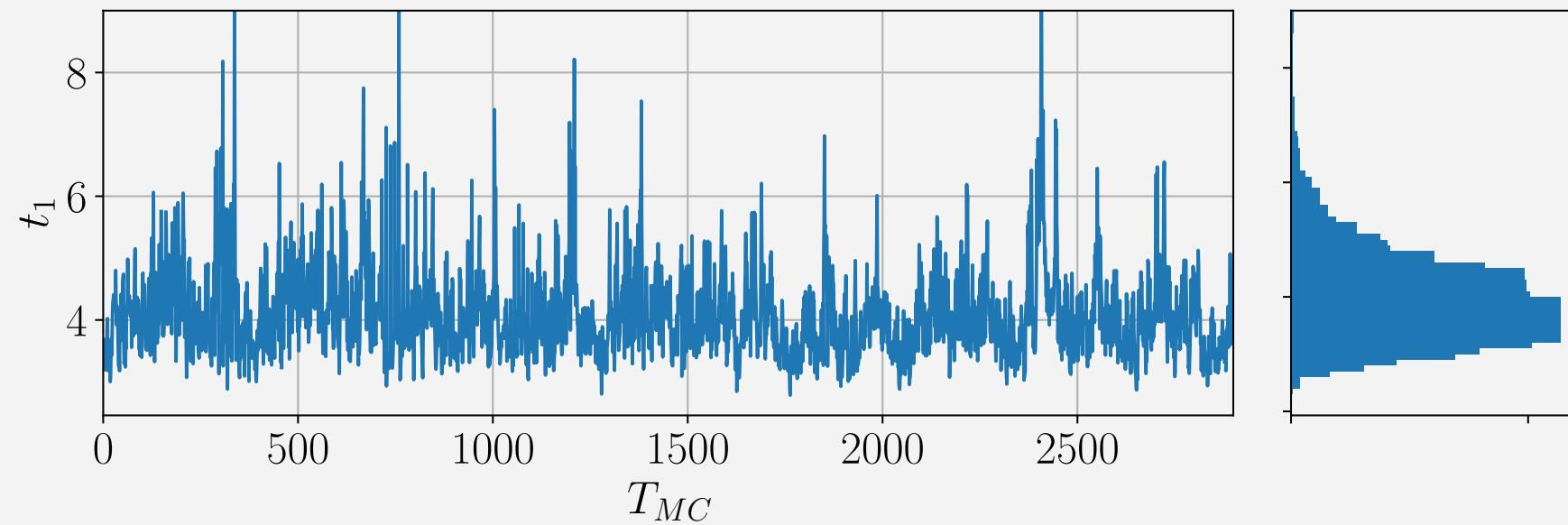
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- Controls both finite-volume / lattice artefacts
- Remnant finite-volume effect are  $< 3\%$  in the scaling window



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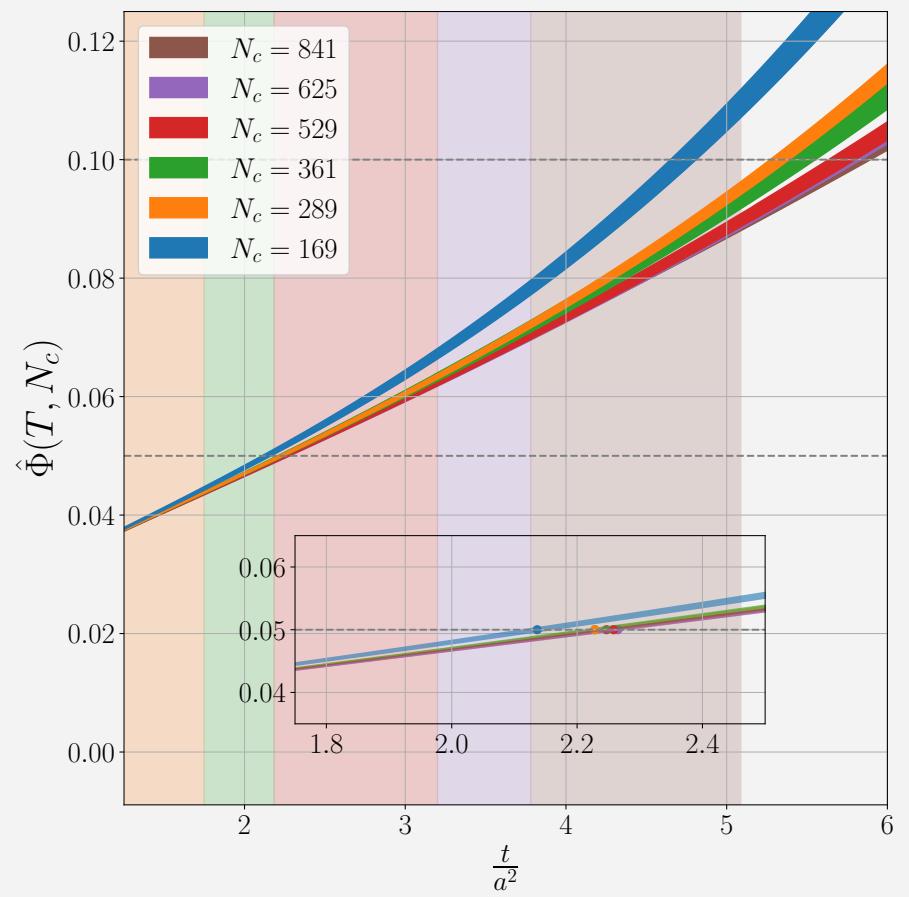
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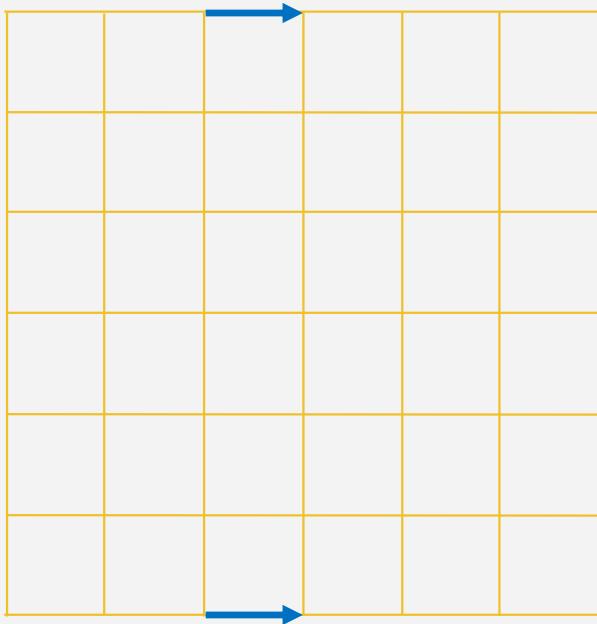
- Build up a new flow observable which defines a coupling free from finite volume ( $\mathcal{N}$ ) effect at leading order

$$\hat{\Phi}(t) \equiv \frac{3/128\pi^2}{\mathcal{N}_{\text{lat}}(\sqrt{8t}/\sqrt{N_c})} \left( \frac{t^2 E(t)}{N_c} \right) \quad T \in \left[ 1.25, \gamma^2 \frac{N_c}{8} \right]$$

- Controls both finite-volume / lattice artefacts
- Remnant finite-volume effect are  $< 3\%$  in the scaling window



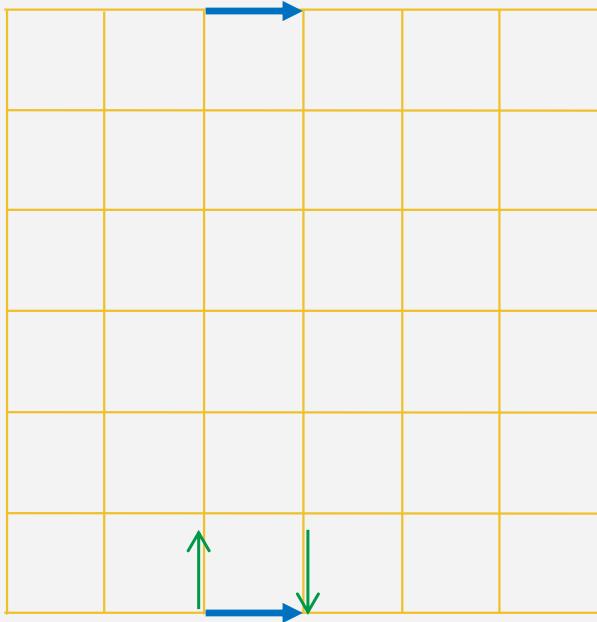
## TWISTED BC & VOLUME REDUCTION



PERIODIC boundary conditions

$$U_\mu(n + L \hat{v}) = U_\mu(n)$$

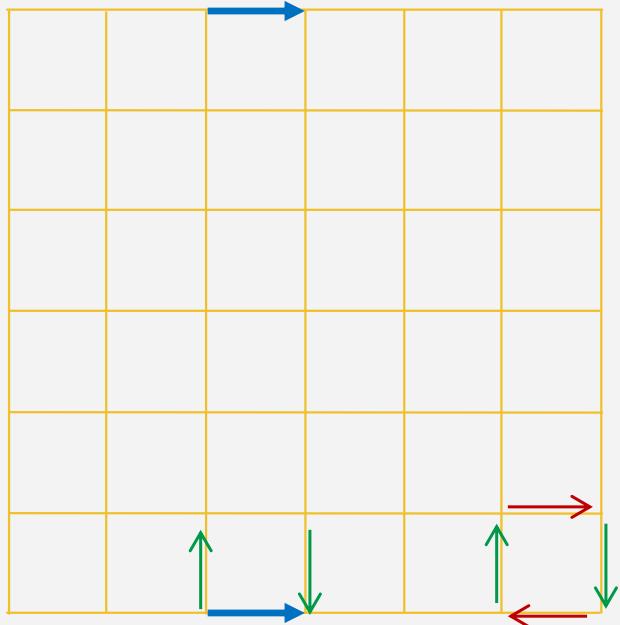
## TWISTED BC & VOLUME REDUCTION



TWISTED boundary conditions

$$U_\mu(n + L \hat{v}) = \Gamma_\nu U_\mu(n) \Gamma_\nu^\dagger$$

## TWISTED BC & VOLUME REDUCTION

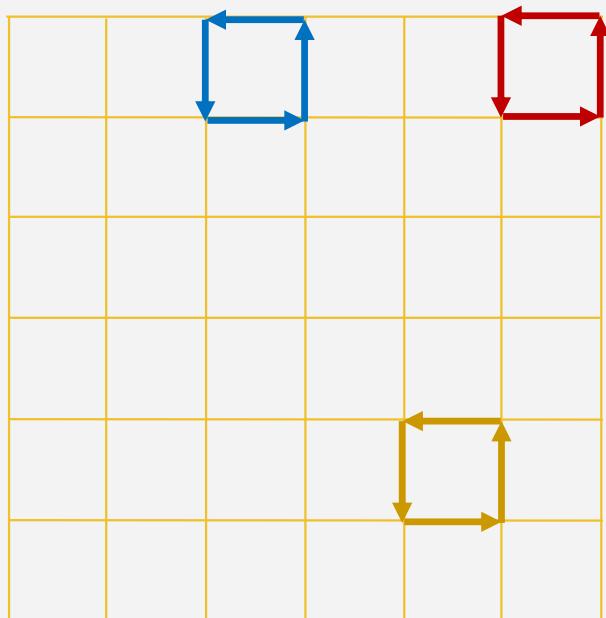


TWISTED boundary conditions

$$U_\mu(n + L \hat{v}) = \Gamma_\nu U_\mu(n) \Gamma_\nu^\dagger$$

$$\Gamma_\mu \Gamma_\nu = e^{\frac{2\pi i}{N_c} \epsilon_{\mu\nu}} \Gamma_\nu \Gamma_\mu$$

## TWISTED BC & VOLUME REDUCTION

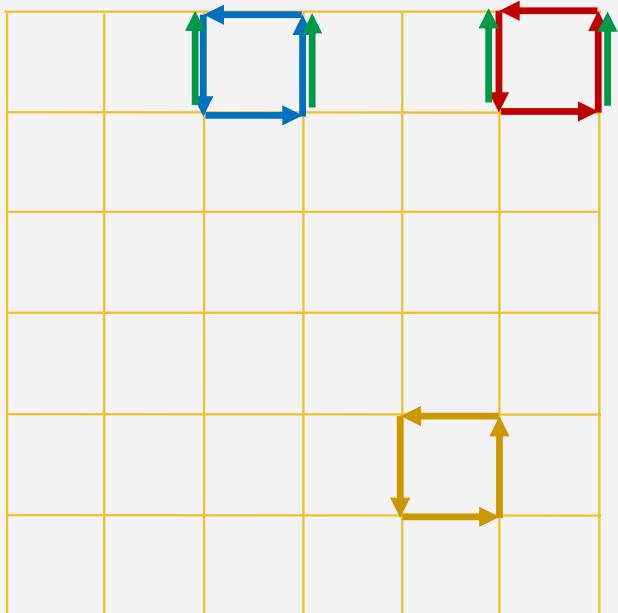


$$S_w = - b N_c \text{Re} \text{Tr} \left[ \sum_{\text{inner}} + \sum_{\text{edge}} + \sum_{\text{corner}} \right]$$

The equation is accompanied by three small diagrams. The first diagram shows a square with all four edges having arrows pointing clockwise. The second diagram shows a square with two opposite edges having arrows pointing clockwise and the other two having arrows pointing counter-clockwise. The third diagram shows a square with all four edges having arrows pointing counter-clockwise.

## TWISTED BC & VOLUME REDUCTION

$$U'_\nu(x) = \Gamma_\nu U_\nu(x)$$

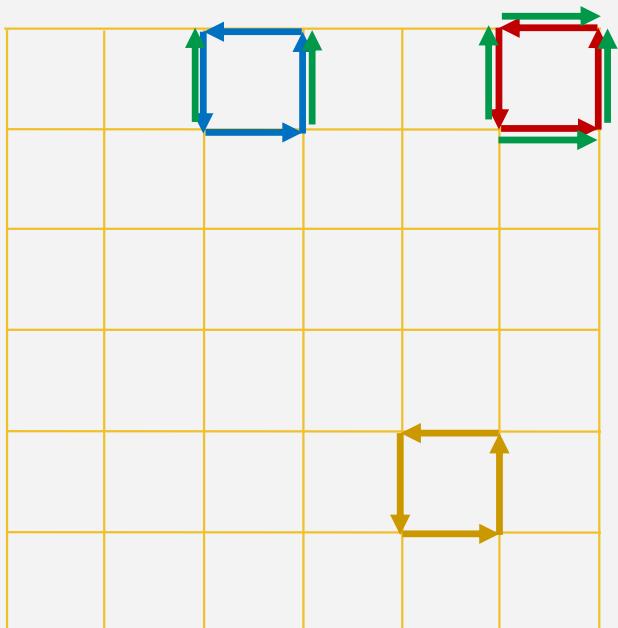


$$S_w = - b N_c \text{Re} \text{Tr} \left[ \sum_{\text{inner}} \text{ } \text{ } \text{ } + \sum_{\text{edge}} \text{ } \text{ } \text{ } + \sum_{\text{corner}} \text{ } \text{ } \text{ } \right]$$

Diagram showing three types of boundary contributions: inner (a small square), edge (a rectangle with one side twisted), and corner (a square with two adjacent sides twisted).

## TWISTED BC & VOLUME REDUCTION

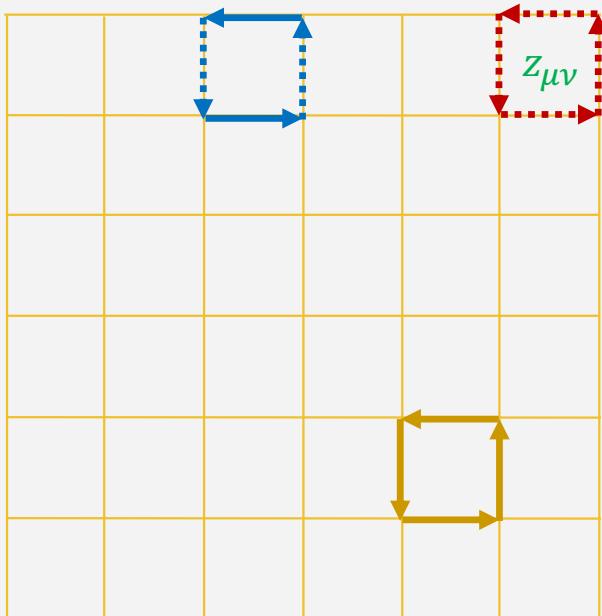
$$U'_\nu(x) = \Gamma_\nu U_\nu(x)$$



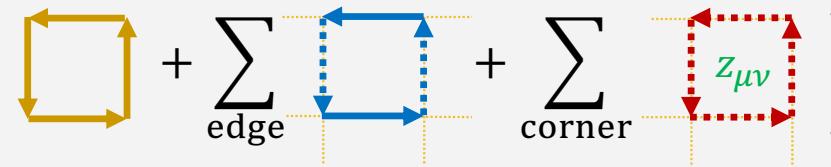
$$S_w = - b N_c \operatorname{Re} \operatorname{Tr} \left[ \sum_{\text{inner}} \square + \sum_{\text{edge}} \square + \sum_{\text{corner}} \square \right]$$

## TWISTED BC & VOLUME REDUCTION

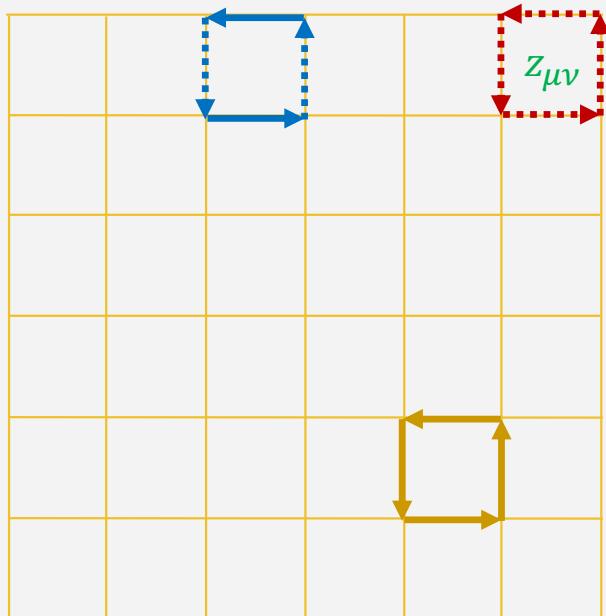
$$U'_\nu(x) = \Gamma_\nu U_\nu(x)$$



$$S_w = - b N_c \text{Re} \text{Tr} \left[ \sum_{\text{inner}} \text{ } \text{ } \text{ } + \sum_{\text{edge}} \text{ } \text{ } \text{ } + \sum_{\text{corner}} \text{ } \text{ } \text{ } \right]$$



## TWISTED BC & VOLUME REDUCTION

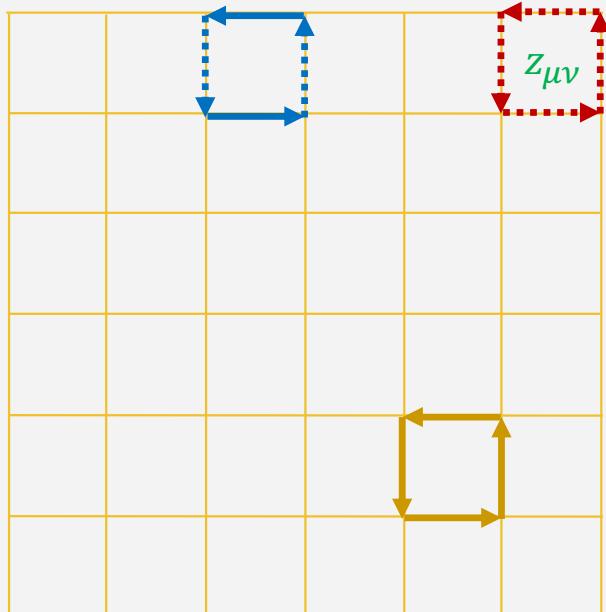


$$S_w = - b N_c \operatorname{Re} \operatorname{Tr} \left[ \sum_{\text{inner}} \text{ (dashed blue square loop)} + \sum_{\text{edge}} \text{ (dashed blue square loop)} + \sum_{\text{corner}} \text{ (dashed red square loop)} \right]$$

$$U'_\nu(x) = \Gamma_\nu U_\nu(x)$$

$$= -b N_c \sum_{n, \mu \neq \nu} z_{\mu\nu}(n) \operatorname{Re} \operatorname{Tr} [U_\mu(n) U_\nu(n + \hat{\mu}) U_\mu^\dagger(n + \hat{\nu}) U_\nu^\dagger(n)]$$

## TWISTED BC & VOLUME REDUCTION



$$S_w = -bN_c \operatorname{Re} \operatorname{Tr} \left[ \sum_{\text{inner}} \text{ (Diagram of a square loop)} + \sum_{\text{edge}} \text{ (Diagram of a rectangle with one edge highlighted)} + \sum_{\text{corner}} \text{ (Diagram of a corner with a red dashed box labeled } Z_{\mu\nu}) \right]$$

$$U'_\nu(x) = \Gamma_\nu U_\nu(x)$$

$$= -bN_c \sum_{n,\mu \neq \nu} z_{\mu\nu}(n) \operatorname{Re} \operatorname{Tr} [U_\mu(n) U_\nu(n + \hat{\mu}) U_\mu^\dagger(n + \hat{\nu}) U_\nu^\dagger(n)]$$

$$U_\mu(n) \rightarrow U_\mu$$

$$S_{\text{TEK}} = -bN_c \sum_{n,\mu \neq \nu} z_{\mu\nu} \operatorname{Re} \operatorname{Tr} [U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger]$$