40th International Symposium on Lattice Field Theory
Fermilab, Batavia, Illinois
July 31st, 2023

ASYMPTOTIC SCALING IN YANG-MILLS AT LARGE-N_c

The lattice scale and the $\Lambda_{\overline{\mathrm{MS}}}$-parameter at large-$N_c$ from twisted volume reduction

Pietro Butti
IFT UAM-CSIC
University of Zaragoza
Antonio González-Arroyo
IFT UAM-CSIC
Autonomous University of Madrid

The $\Lambda_{\overline{\mathrm{MS}}}$-parameter

RG equations for the 't Hooft coupling $\lambda = g^2 N_c$:

$$\frac{\mathrm{d}\lambda_s}{\mathrm{d}\log(\mu^2)} = \beta_s(\lambda_s) \simeq -b_0\lambda_s^2 - b_1\lambda_s^3 - b_2^{(s)}\lambda_s^4 - \cdots$$

and, upon integration

$$\frac{\Lambda_s}{\mu} = (b_0\lambda_s)^{-\frac{b_1}{2b_0^2}}\, e^{-\frac{1}{2b_0\lambda_s}} e^{-\int^{\lambda}\mathrm{d}x\left[\frac{1}{2\beta_s(x)}+\frac{1}{2b_0x^2}-\frac{b_1}{2b_0x}\right]}$$

THE LATTICE RUNNING COUPLING

The $\Lambda_{\overline{\mathrm{MS}}}$-parameter

RG equations for the 't Hooft coupling $\lambda = g^2 N_c$:

$$\frac{\mathrm{d}\lambda_s}{\mathrm{d}\log(\mu^2)} = \beta_s(\lambda_s) \simeq -b_0\lambda_s^2 - b_1\lambda_s^3 - b_2^{(s)}\lambda_s^4$$

and, upon integration

$$\frac{\Lambda_s}{\mu} \simeq (b_0\lambda_s)^{-\frac{b_1}{2b_0^2}} \; e^{-\frac{1}{2b_0}\left(\frac{1}{\lambda_s} + c_1^{(s)}\lambda_s\right)}$$

$$c_1^{(s)} = \frac{b_2^{(s)}}{b_0} - \frac{b_1^2}{b_0^2}$$

THE LATTICE RUNNING COUPLING

The $\Lambda_{\overline{\text{MS}}}$-parameter

RG equations for the 't Hooft coupling $\lambda = g^2 N_c$:

$$\frac{\mathrm{d}\lambda_s}{\mathrm{d}\log(\mu^2)} = \beta_s(\lambda_s) \simeq -b_0\lambda_s^2 - b_1\lambda_s^3 - b_2^{(s)}\lambda_s^4$$

and, upon integration

$$\frac{\Lambda_s}{\mu} \simeq (b_0\lambda_s)^{-\frac{b_1}{2b_0^2}}\, e^{-\frac{1}{2b_0}\left(\frac{1}{\lambda_s}+c_1^{(s)}\lambda_s\right)} \qquad c_1^{(s)} = \frac{b_2^{(s)}}{b_0} - \frac{b_1^2}{b_0^2}$$

- Compute the non-perturbative running of the coupling at low energies μ_{had} and match to PT at $\mu_{\text{PT}} \gg \mu_{\text{had}}$

$$\frac{\Lambda_s}{\mu_{\text{had}}} = \frac{\Lambda_s}{\mu_{\text{pt}}}\, e^{-\int_{\lambda(\mu_{\text{pt}})}^{\lambda(\mu_{\text{had}})} \frac{\mathrm{d}x}{2\beta_s(x)}}$$

(e.g. finite size scaling)

THE LATTICE RUNNING COUPLING

The $\Lambda_{\overline{\mathrm{MS}}}$-parameter

RG equations for the 't Hooft coupling $\lambda = g^2 N_c$:

$$\frac{\mathrm{d}\lambda_s}{\mathrm{d}\log(\mu^2)} = \beta_s(\lambda_s) \simeq -b_0\lambda_s^2 - b_1\lambda_s^3 - b_2^{(s)}\lambda_s^4$$

and, upon integration

$$\frac{\Lambda_s}{\mu} \simeq (b_0\lambda_s)^{-\frac{b_1}{2b_0^2}} e^{-\frac{1}{2b_0}\left(\frac{1}{\lambda_s}+c_1^{(s)}\lambda_s\right)} \qquad c_1^{(s)} = \frac{b_2^{(s)}}{b_0} - \frac{b_1^2}{b_0^2}$$

Weak coupling is hard to simulate on the lattice and for feasible simulations

- $\mathcal{O}(a^2)$ corrections (scaling violations)
- Lattice scheme w/ Wilson action has large higher order terms in the β-function

$$\frac{\Lambda_{\overline{\mathrm{MS}}}}{\Lambda_{\mathrm{lat}}} \sim 38.853\, e^{-\frac{3\pi^2}{11N_c^2}}$$

- Compute the non-perturbative running of the coupling at low energies μ_{had} and match to PT at $\mu_{\mathrm{PT}} \gg \mu_{\mathrm{had}}$

$$\frac{\Lambda_s}{\mu_{\mathrm{had}}} = \frac{\Lambda_s}{\mu_{\mathrm{pt}}} e^{-\int_{\lambda(\mu_{\mathrm{pt}})}^{\lambda(\mu_{\mathrm{had}})} \frac{\mathrm{d}x}{2\beta_s(x)}}$$

(e.g. finite size scaling)

THE LATTICE RUNNING COUPLING

The $\Lambda_{\overline{\mathrm{MS}}}$-parameter

RG equations for the 't Hooft coupling $\lambda = g^2 N_c$:

$$\frac{\mathrm{d}\lambda_s}{\mathrm{d}\log(\mu^2)} = \beta_s(\lambda_s) \simeq -b_0\lambda_s^2 - b_1\lambda_s^3 - b_2^{(s)}\lambda_s^4$$

and, upon integration

$$\frac{\Lambda_s}{\mu} \simeq (b_0\lambda_s)^{-\frac{b_1}{2b_0^2}}\, e^{-\frac{1}{2b_0}\left(\frac{1}{\lambda_s}+c_1^{(s)}\lambda_s\right)} \qquad c_1^{(s)} = \frac{b_2^{(s)}}{b_0} - \frac{b_1^2}{b_0^2}$$

Weak coupling is hard to simulate on the lattice and for feasible simulations

- $\mathcal{O}(a^2)$ corrections (scaling violations)
- Lattice scheme w/ Wilson action has large higher order terms in the β-function

$$\frac{\Lambda_{\overline{\mathrm{MS}}}}{\Lambda_{\mathrm{lat}}} \sim 38.853\, e^{-\frac{3\pi^2}{11N_c^2}}$$

- Compute the non-perturbative running of the coupling at low energies μ_{had} and match to PT at $\mu_{\mathrm{PT}} \gg \mu_{\mathrm{had}}$

$$\frac{\Lambda_s}{\mu_{\mathrm{had}}} = \frac{\Lambda_s}{\mu_{\mathrm{pt}}}\, e^{-\int_{\lambda(\mu_{\mathrm{pt}})}^{\lambda(\mu_{\mathrm{had}})}\frac{\mathrm{d}x}{2\beta_s(x)}}$$

(e.g. finite size scaling)

or

- Simulate large range of bare couplings $b = 1/\lambda$, use improved lattice couplings to improve convergence with PT

$$\lambda_I = \frac{b}{P(b)}, \qquad \lambda_E = 8\big(1 - P(b)\big), \qquad \lambda_{E'} = -8\log P(b)$$

[Allton et al. JHEP 07 (2008) 21], [Gonzalez-Arroyo, Okawa, Phys. Let. B 718 (2013)]

LARGE-N AND VOLUME REDUCTION

Gauge fields on a twisted lattice

[Gonzalez-Arroyo, Okawa, Phys. Rev. D 27 (1983), Phys. Let. B 120 (1983)]

$$S = bN \sum_{n} \sum_{\mu \neq \nu} \mathrm{tr}\left(\mathbb{I} - U_\mu(n) U_\nu(n+\mu) U_\mu^\dagger(n+\nu) U_\nu^\dagger(n) \right)$$

LARGE-N AND VOLUME REDUCTION

Gauge fields on a twisted lattice

[Gonzalez-Arroyo, Okawa, Phys. Rev. D 27 (1983), Phys. Let. B 120 (1983)]

$$S = bN \sum_{n} \sum_{\mu \neq \nu} \mathrm{tr}\left(\mathbb{I} - U_\mu(n) U_\nu(n+\mu) U_\mu^\dagger(n+\nu) U_\nu^\dagger(n)\right)$$

Use reduction + **twisted BC**

$$U_\mu(n) \to U_\mu$$
$$U_\mu(n+\nu) \to \Gamma_\nu U_\mu \Gamma_\nu^\dagger$$
$$V_\mu = U_\mu \Gamma_\mu$$

$$\Gamma_\mu \Gamma_\nu = z_{\mu\nu} \Gamma_\nu \Gamma_\mu$$

$$z_{\mu\nu} = e^{\frac{2\pi i k}{\sqrt{N_c}} \epsilon_{\mu\nu}}$$

Gauge fields on a twisted lattice

[Gonzalez-Arroyo, Okawa, Phys. Rev. D 27 (1983), Phys. Let. B 120 (1983)]

$$S = bN \sum_{n} \sum_{\mu \neq \nu} \mathrm{tr}\left(\mathbb{I} - U_\mu(n) U_\nu(n+\mu) U_\mu^\dagger(n+\nu) U_\nu^\dagger(n)\right)$$

Use reduction + **twisted BC**

$$U_\mu(n) \to U_\mu$$
$$U_\mu(n+\nu) \to \Gamma_\nu U_\mu \Gamma_\nu^\dagger$$
$$V_\mu = U_\mu \Gamma_\mu$$

$$\Gamma_\mu \Gamma_\nu = z_{\mu\nu} \Gamma_\nu \Gamma_\mu$$

$$z_{\mu\nu} = e^{\frac{2\pi i k}{\sqrt{N_c}} \epsilon_{\mu\nu}}$$

$$S_{\mathrm{TEK}} = bN \sum_{\mu \neq \nu} \mathrm{tr}\left(\mathbb{I} - z_{\mu\nu} V_\mu V_\nu V_\mu^\dagger V_\nu^\dagger\right)$$

LARGE-N AND VOLUME REDUCTION

Gauge fields on a twisted lattice

[Gonzalez-Arroyo, Okawa, Phys. Rev. D 27 (1983), Phys. Let. B 120 (1983)]

$$S = bN \sum_n \sum_{\mu\neq\nu} \mathrm{tr}\left(\mathbb{I} - U_\mu(n)U_\nu(n+\mu)U_\mu^\dagger(n+\nu)U_\nu^\dagger(n)\right)$$

Use reduction + twisted BC

$$U_\mu(n) \to U_\mu$$
$$U_\mu(n+\nu) \to \Gamma_\nu U_\mu \Gamma_\nu^\dagger$$
$$V_\mu = U_\mu \Gamma_\mu$$

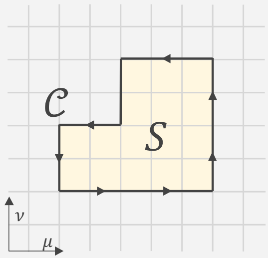
$$\Gamma_\mu\Gamma_\nu = z_{\mu\nu}\Gamma_\nu\Gamma_\mu$$

$$z_{\mu\nu} = e^{\frac{2\pi i k}{\sqrt{N_c}}\epsilon_{\mu\nu}}$$

$$S_{\mathrm{TEK}} = bN \sum_{\mu\neq\nu} \mathrm{tr}\left(\mathbb{I} - z_{\mu\nu}V_\mu V_\nu V_\mu^\dagger V_\nu^\dagger\right)$$

Gauge observables

$$\langle \mathrm{Tr}\, U(\mathcal{C})\rangle_{\substack{N_c\to\infty \\ V\to\infty}}$$
$$\approx$$
$$z_{\mu\nu}(S)\left\langle \mathrm{Tr}\, V_\mu \ldots V_\nu \ldots V_\mu^\dagger \ldots \right\rangle_{\mathrm{TEK}\,(\mathrm{N_c}\sim\infty)}$$



LARGE-N AND VOLUME REDUCTION

Gauge fields on a twisted lattice

[Gonzalez-Arroyo, Okawa, Phys. Rev. D 27 (1983), Phys. Let. B 120 (1983)]

$$S = bN \sum_{n} \sum_{\mu \neq \nu} \mathrm{tr}\left(\mathbb{I} - U_\mu(n) U_\nu(n+\mu) U_\mu^\dagger(n+\nu) U_\nu^\dagger(n)\right)$$

Use reduction + **twisted BC**

$$U_\mu(n) \to U_\mu$$
$$U_\mu(n+\nu) \to \Gamma_\nu U_\mu \Gamma_\nu^\dagger$$
$$V_\mu = U_\mu \Gamma_\mu$$

$$\Gamma_\mu \Gamma_\nu = z_{\mu\nu} \Gamma_\nu \Gamma_\mu$$

$$z_{\mu\nu} = e^{\frac{2\pi i k}{\sqrt{N_c}} \epsilon_{\mu\nu}}$$

$$S_{\mathrm{TEK}} = bN \sum_{\mu \neq \nu} \mathrm{tr}\left(\mathbb{I} - z_{\mu\nu} V_\mu V_\nu V_\mu^\dagger V_\nu^\dagger\right)$$

Finite-N_c corrections

1^4-lattice with tBC with $N_c \sim 10^2/10^3$ $\approx$ "Effective" periodic lattice with $L = \sqrt{N_c}$

LARGE-N AND VOLUME REDUCTION

Gauge fields on a twisted lattice

[Gonzalez-Arroyo, Okawa, Phys. Rev. D 27 (1983), Phys. Let. B 120 (1983)]

$$S = bN \sum_{n} \sum_{\mu \neq \nu} \mathrm{tr}\left(\mathbb{I} - U_\mu(n) U_\nu(n+\mu) U_\mu^\dagger(n+\nu) U_\nu^\dagger(n)\right)$$

Use reduction + **twisted BC**

$$U_\mu(n) \to U_\mu$$
$$U_\mu(n+\nu) \to \Gamma_\nu U_\mu \Gamma_\nu^\dagger$$
$$V_\mu = U_\mu \Gamma_\mu$$

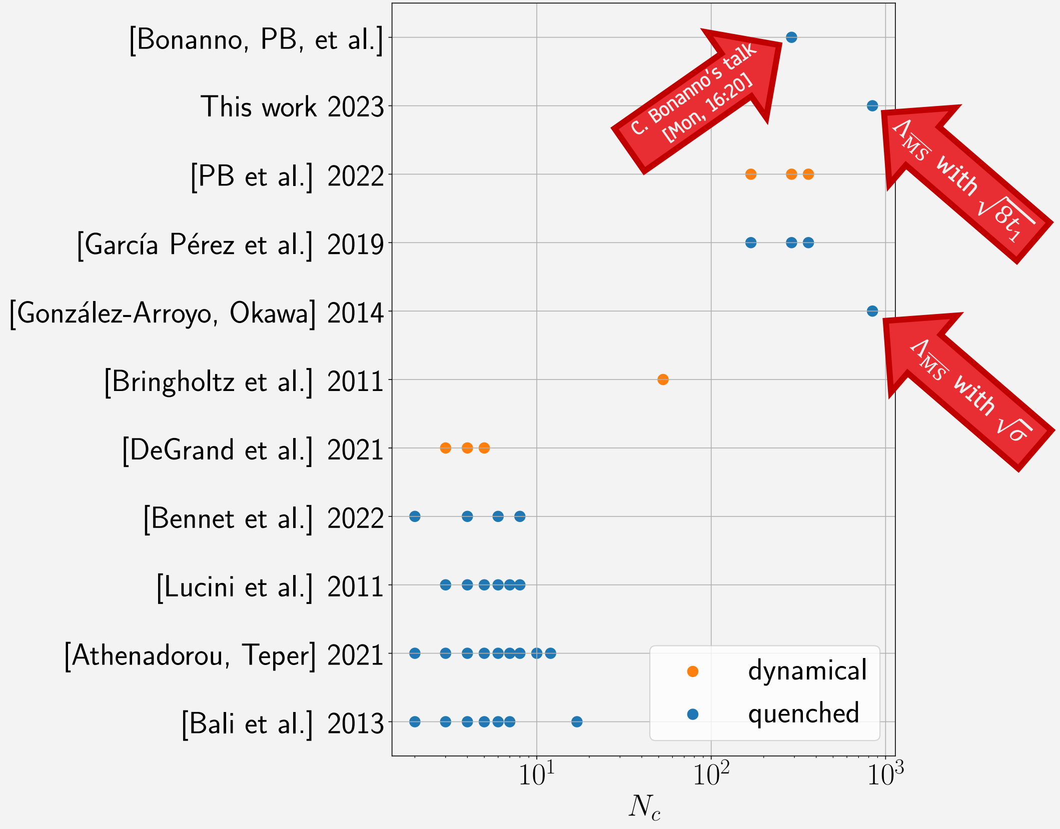
$$\Gamma_\mu \Gamma_\nu = z_{\mu\nu} \Gamma_\nu \Gamma_\mu$$

$$z_{\mu\nu} = e^{\frac{2\pi i k}{\sqrt{N_c}} \epsilon_{\mu\nu}}$$

$$S_{\mathrm{TEK}} = bN \sum_{\mu \neq \nu} \mathrm{tr}\left(\mathbb{I} - z_{\mu\nu} V_\mu V_\nu V_\mu^\dagger V_\nu^\dagger\right)$$

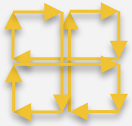
Finite-N_c corrections

1^4-lattice with tBC with $N_c \sim 10^2/10^3$ $\approx$ "Effective" periodic lattice with $L = \sqrt{N_c}$



The Wilson flow scale

- Flowed energy density

$E = -\frac{1}{128}\sum_{\mu\neq\nu} z_{\nu\mu}\,\mathrm{Tr}\,($  $-\mathrm{h.c.})^2$

- Integrate flow equations to get $E(t)$
- Solve

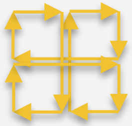
$$\left.\left\langle \frac{t^2 E\,(t)}{N_c} \right\rangle\right|_{t=t_1} = 0.05$$

THE LATTICE SCALE

The Wilson flow scale

- Flowed energy density

$$E = -\frac{1}{128}\sum_{\mu\neq\nu} z_{\nu\mu}\mathrm{Tr}\,\Big($$

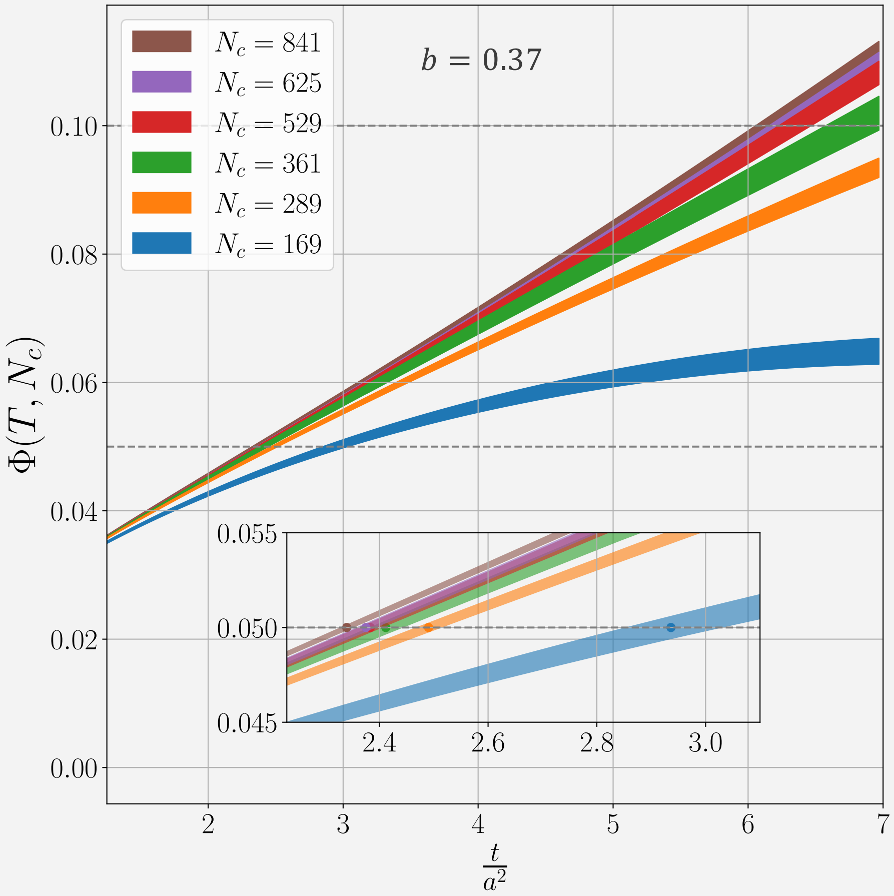


$$-\mathrm{h.c.}\Big)^2$$

- Integrate flow equations to get $E(t)$

- Solve

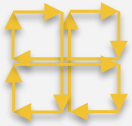
$$\left.\left\langle \frac{t^2 E(t)}{N_c} \right\rangle\right|_{t=t_1} = 0.05$$



The Wilson flow scale

- Flowed energy density

$$E = -\frac{1}{128}\sum_{\mu\neq\nu} z_{\nu\mu}\,\mathrm{Tr}\,\Big(\quad -\mathrm{h.c.}\Big)^2$$

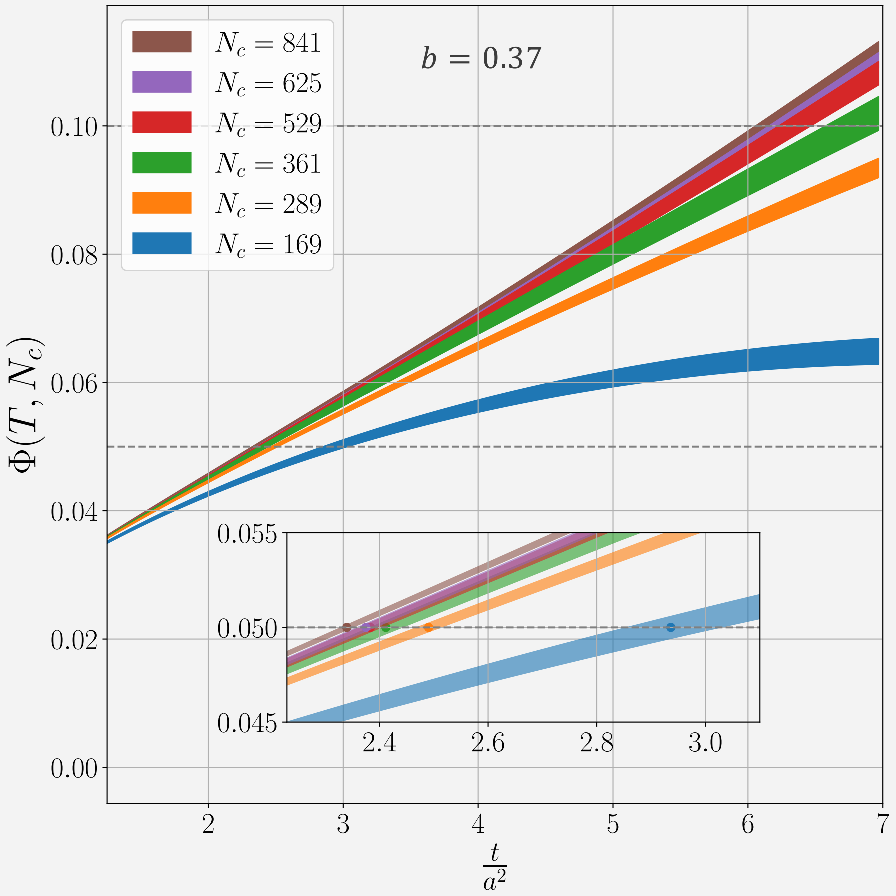


- Integrate flow equations to get $E(t)$

- Solve

$$\left.\left\langle \frac{t^2 E(t)}{N_c} \right\rangle\right|_{t=t_1} = 0.05$$

- Treat finite "volume" effects



The Wilson flow scale

- (twisted) flowed running coupling

$$\lambda_{\mathrm{gf}}\left(\mu=\frac{1}{\sqrt{8t}}\right)\equiv\frac{128\pi^2\,N_c^2}{N_c^2-1}\left\langle\frac{t^2E(t)}{N_c}\right\rangle$$

- On a $V=1^4$ torus with twisted BC at finite N_c
  [García Pérez, Ibañez, *JHEP* 03 (2019) 200]

$$\hat{\lambda}(\mu)\equiv\mathcal{N}^{-1}\left(\frac{\sqrt{8t}}{\sqrt{N_c}}\right)\left\langle\frac{t^2E(t)}{N_c}\right\rangle\xrightarrow{N_c\to\infty}\lambda_{\mathrm{gf}}(\mu)$$

- Build up a new flow observable from $\hat{\lambda}$, free from finite volume effects ($\mathcal{N}$) at leading order in PT
  [PB, García Pérez, González-Arroyo, Ishikawa, Okawa JHEP 07 (2022) 074]

$$\hat{\Phi}(t)\equiv\frac{{}^{3}/{}_{128\pi^2}}{\mathcal{N}_{\mathrm{lat}}\left(\sqrt{8t}/\sqrt{N_c}\right)}\left\langle\frac{t^2E(t)}{N_c}\right\rangle\qquad T\in\left[1.25,\gamma^2\frac{N_c}{8}\right]$$

The Wilson flow scale

- (twisted) flowed running coupling

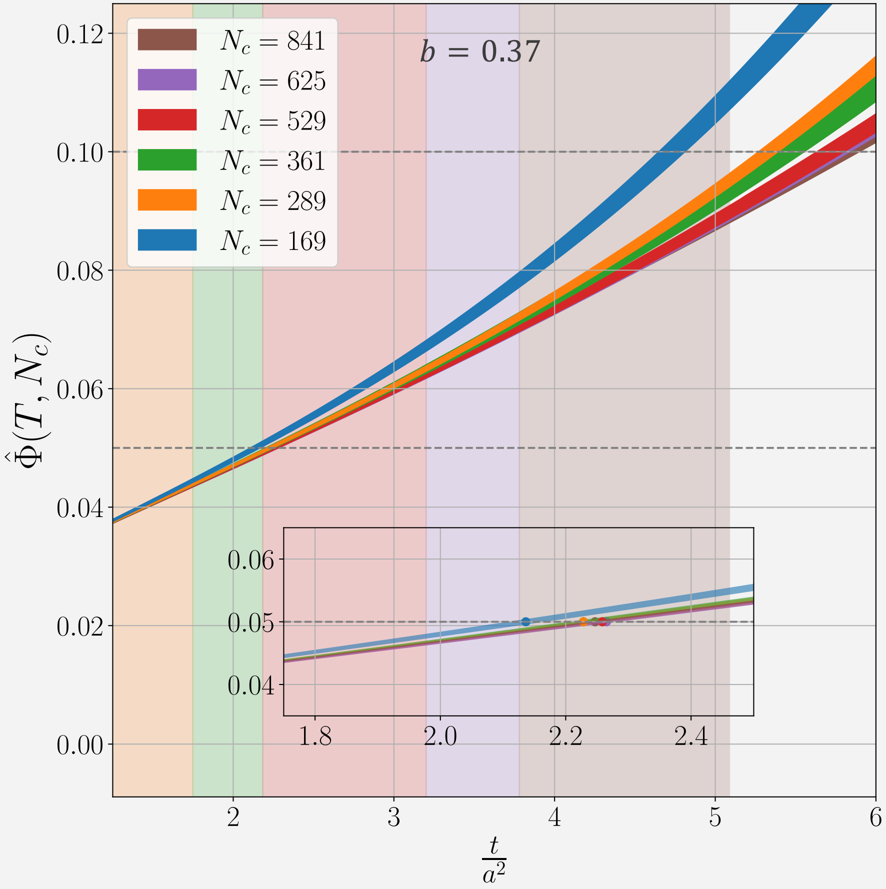
$$\lambda_{\rm gf}\left(\mu = \frac{1}{\sqrt{8t}}\right) \equiv \frac{128\pi^2 N_c^2}{N_c^2 - 1}\left\langle \frac{t^2 E(t)}{N_c} \right\rangle$$

- On a $V = 1^4$ torus with twisted BC at finite N_c
  [García Pérez, Ibañez, *JHEP* 03 (2019) 200]

$$\hat{\lambda}(\mu) \equiv \mathcal{N}^{-1}\left(\frac{\sqrt{8t}}{\sqrt{N_c}}\right)\left\langle \frac{t^2 E(t)}{N_c} \right\rangle \xrightarrow{N_c \to \infty} \lambda_{\rm gf}(\mu)$$

- Build up a new flow observable from $\hat{\lambda}$, free from finite volume effects ($\mathcal{N}$) at leading order in PT
  [PB, García Pérez, González-Arroyo, Ishikawa, Okawa JHEP 07 (2022) 074]

$$\hat{\Phi}(t) \equiv \frac{{}^{3}/_{128\pi^2}}{\mathcal{N}_{\rm lat}\left(\sqrt{8t}/\sqrt{N_c}\right)}\left\langle \frac{t^2 E(t)}{N_c} \right\rangle \qquad T \in \left[1.25, \gamma^2 \frac{N_c}{8}\right]$$



The Wilson flow scale

- (twisted) flowed running coupling

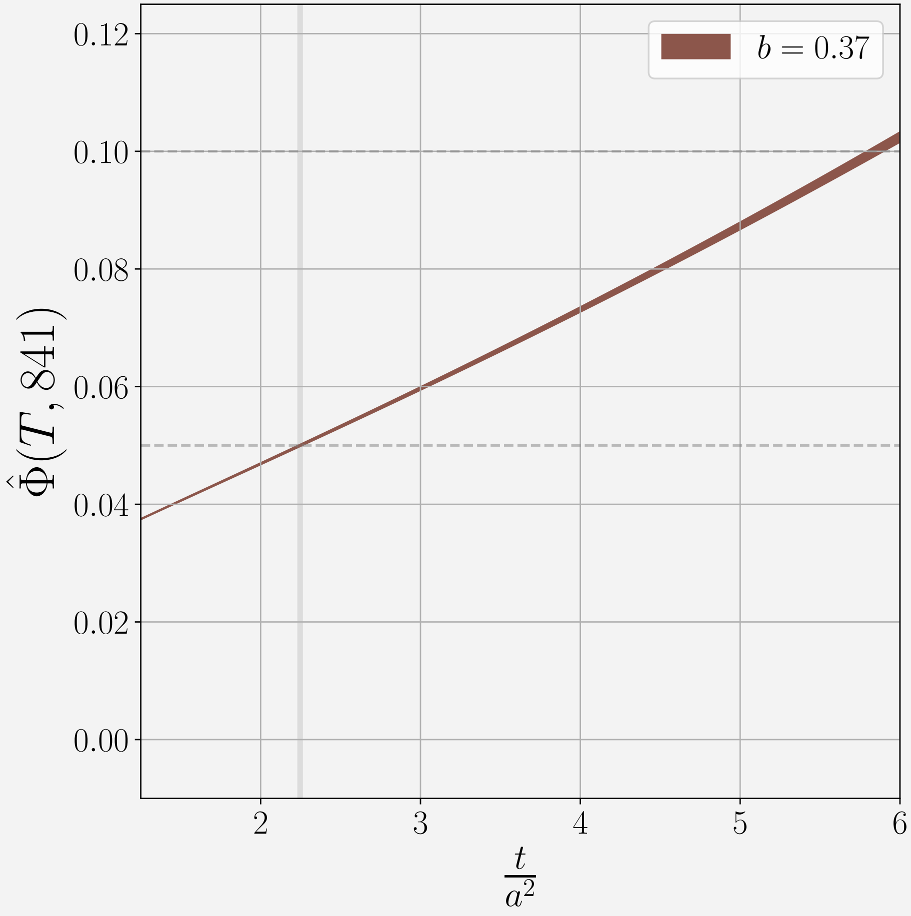
$$\lambda_{\rm gf}\left(\mu = \frac{1}{\sqrt{8t}}\right) \equiv \frac{128\pi^2\, N_c^2}{N_c^2 - 1} \left\langle \frac{t^2 E(t)}{N_c} \right\rangle$$

- On a $V = 1^4$ torus with twisted BC at finite N_c
  [García Pérez, Ibañez, *JHEP* 03 (2019) 200]

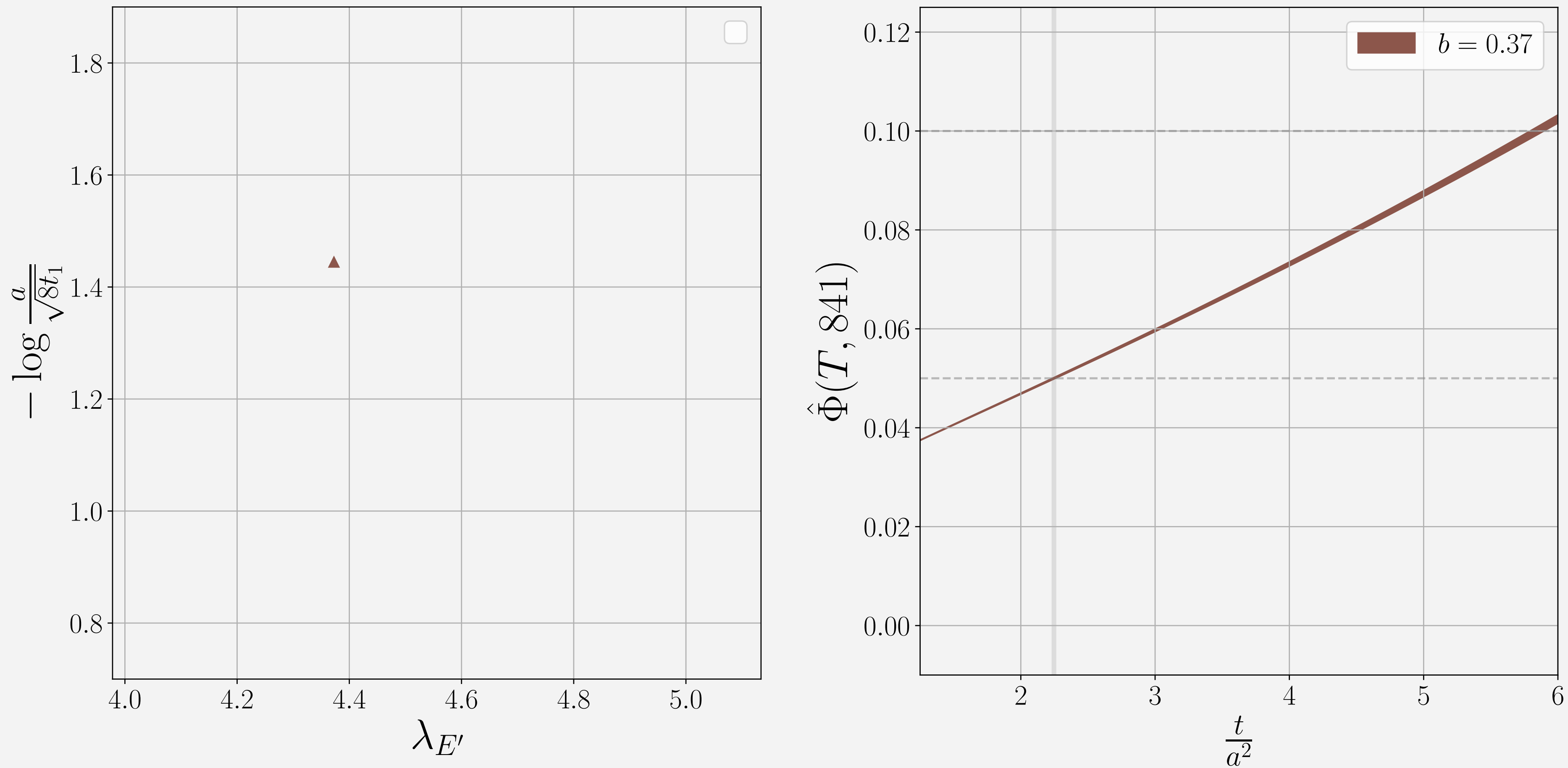
$$\hat{\lambda}(\mu) \equiv \mathcal{N}^{-1}\left(\frac{\sqrt{8t}}{\sqrt{N_c}}\right) \left\langle \frac{t^2 E(t)}{N_c} \right\rangle \xrightarrow{N_c \to \infty} \lambda_{\rm gf}(\mu)$$

- Build up a new flow observable from $\hat{\lambda}$, free from finite volume effects ($\mathcal{N}$) at leading order in PT
  [PB, García Pérez, González-Arroyo, Ishikawa, Okawa JHEP 07 (2022) 074]

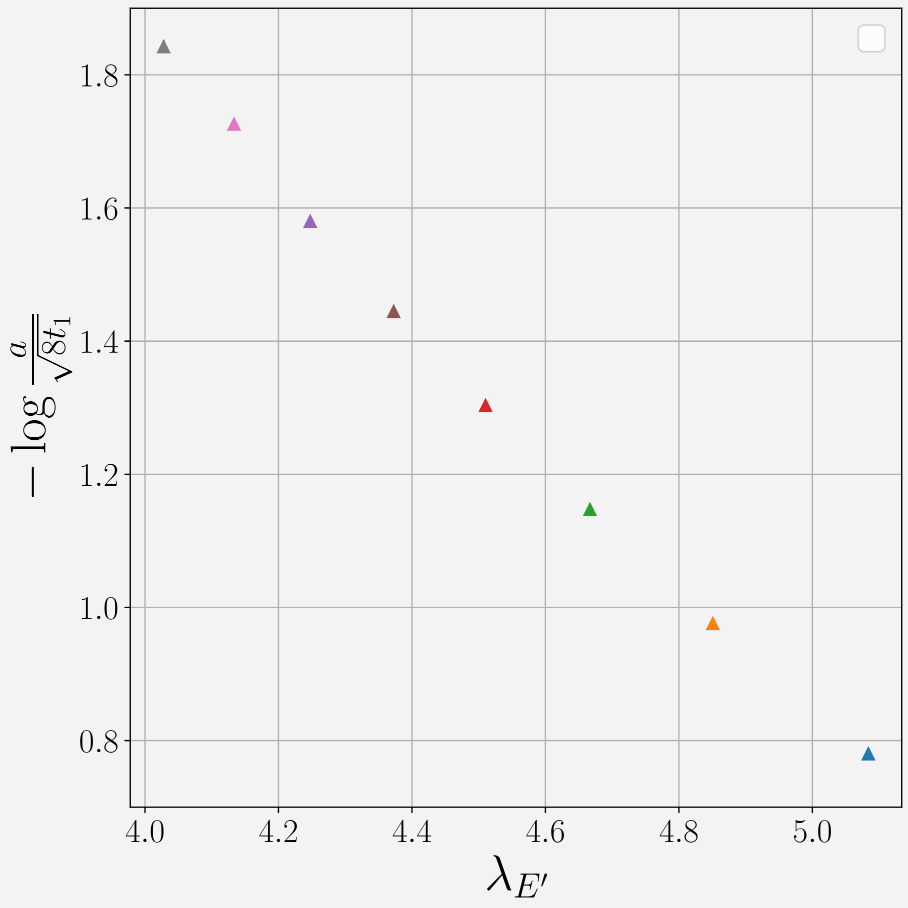
$$\hat{\Phi}(t) \equiv \frac{{}^{3}/_{128\pi^2}}{\mathcal{N}_{\rm lat}\left(\sqrt{8t}/\sqrt{N_c}\right)} \left\langle \frac{t^2 E(t)}{N_c} \right\rangle \qquad T \in \left[1.25, \gamma^2 \frac{N_c}{8}\right]$$

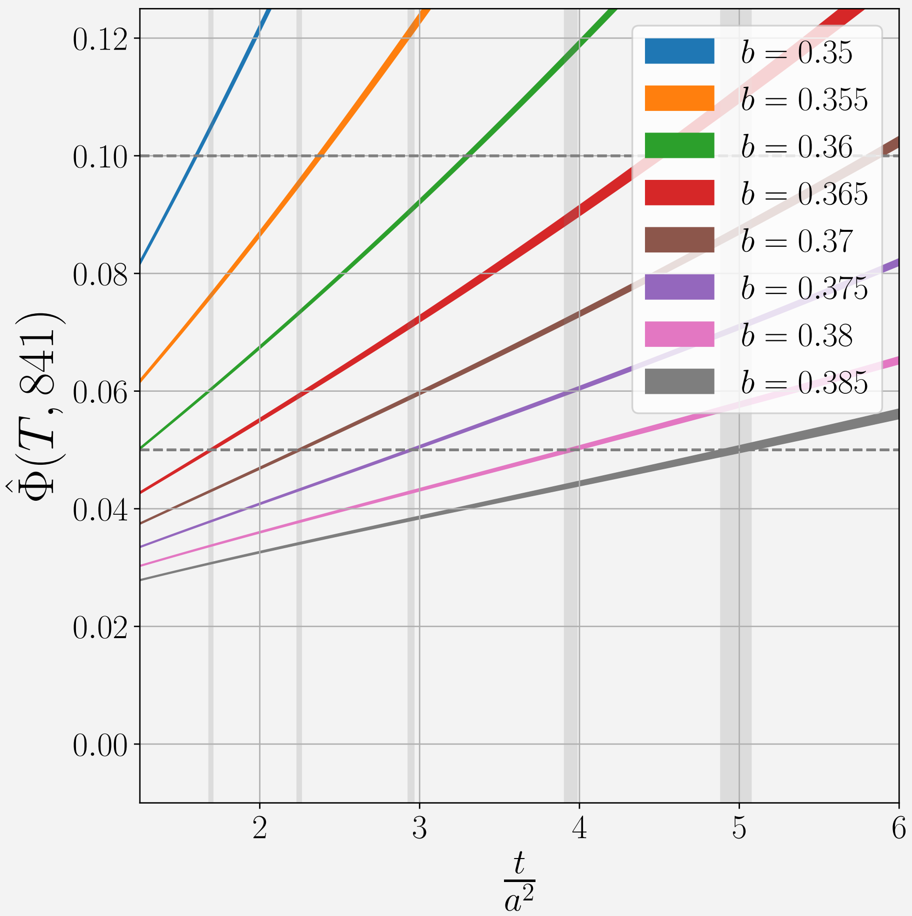


THE LATTICE SCALE

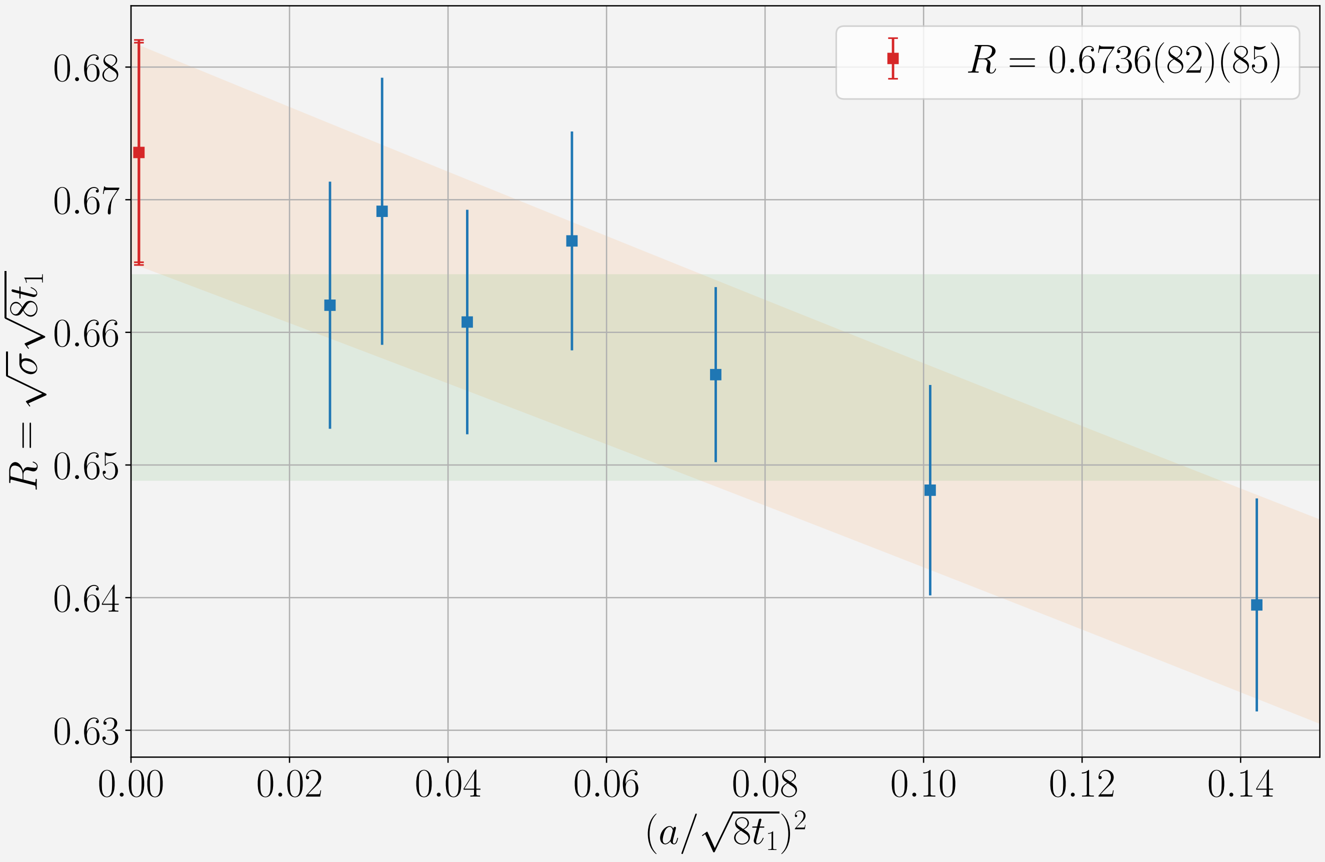


THE LATTICE SCALE



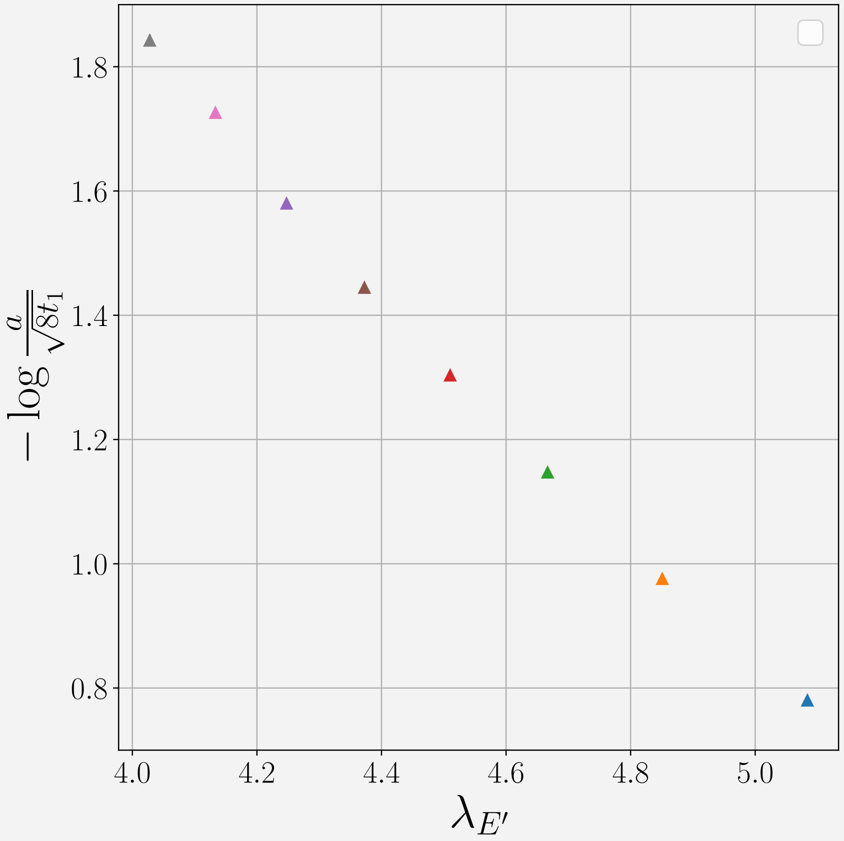


SCALING



$R = 0.6736(82)(85)$
0.68
0.67
0.66
0.65
0.64
0.63
$R = \sqrt{\sigma}\sqrt{8t_1}$
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
$(a/\sqrt{8t_1})^2$

SCALING



ASYMPTOTIC SCALING

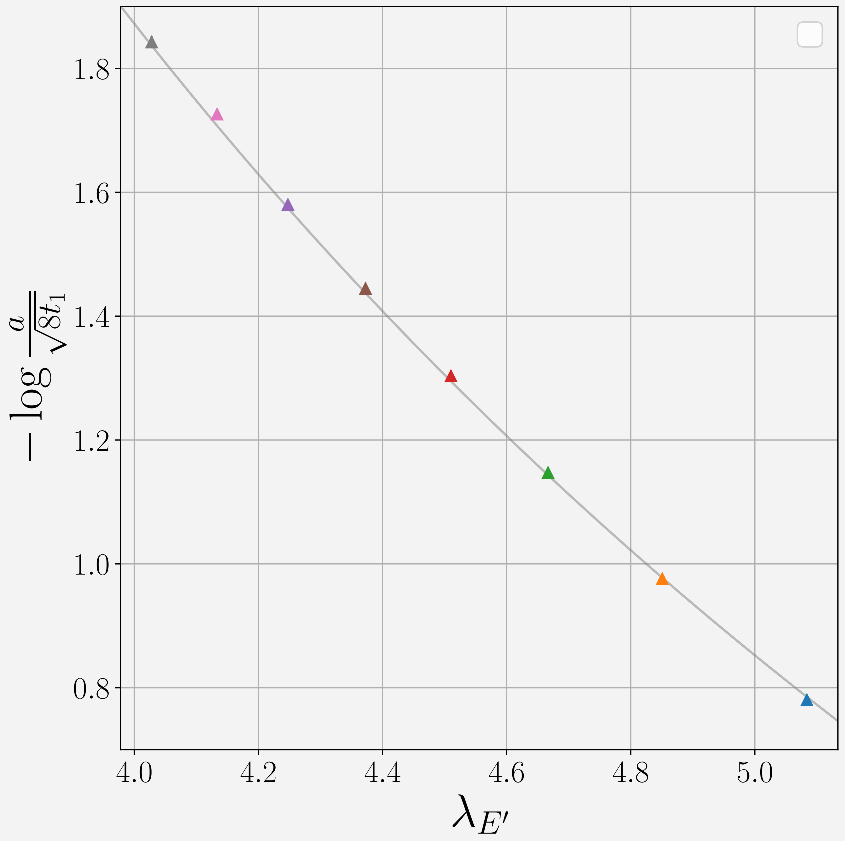
Fit PT to data

- Integrate the perturbative β-function at $\mathcal{O}(\lambda^4)$

$$-\log\frac{a}{\sqrt{8t_1}} = \log\Lambda_s\sqrt{8t_1} + \frac{1}{2b_0\lambda_s} + \frac{b_1}{2b_0^2}\log(b_0\lambda_s) + \frac{c_1^{(s)}}{2b_0}\lambda_s$$

- Try to fit against one improved coupling

$$\lambda_I = \frac{\lambda}{P(\lambda)}, \qquad \lambda_E = 8\big(1 - P(\lambda)\big), \qquad \lambda_{E'} = -8\log P(\lambda)$$



THE $\Lambda_{\overline{\rm MS}}$ -PARAMETER

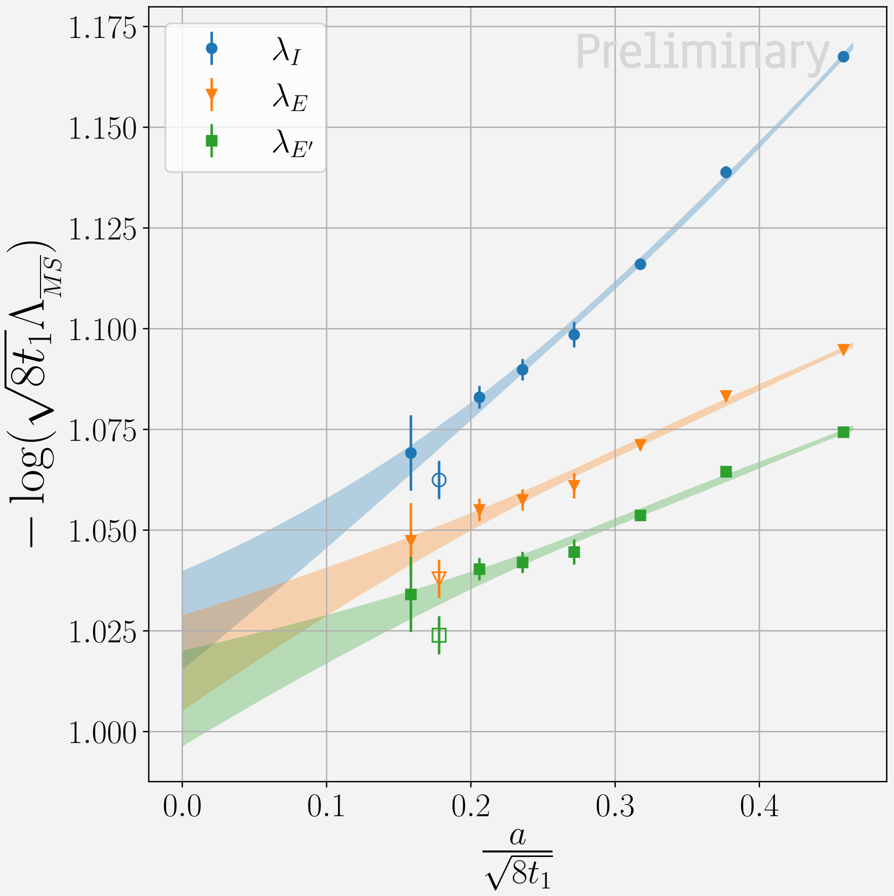
Fit PT to data

- Integrate the perturbative β-function at $\mathcal{O}(\lambda^4)$

$$-\log\frac{a}{\sqrt{8t_1}} = \log\Lambda_s\sqrt{8t_1} + \underbrace{\frac{1}{2b_0\lambda_s} + \frac{b_1}{2b_0^2}\log(b_0\lambda_s) + \frac{c_1^{(s)}}{2b_0}\lambda_s}_{\log f(\lambda_s)}$$

- Subtract $\log f(\lambda_s)$ to data, convert to $\overline{\rm MS}$, take the limit

$$\Lambda_{\overline{\rm MS}}\sqrt{8t_1} = \lim_{\lambda_s\to 0}\frac{\Lambda_{\overline{\rm MS}}}{\Lambda_s}f(\lambda_s)\frac{\sqrt{8t_1}}{a}$$



THE $\Lambda_{\overline{MS}}$ -PARAMETER

Fit PT to data

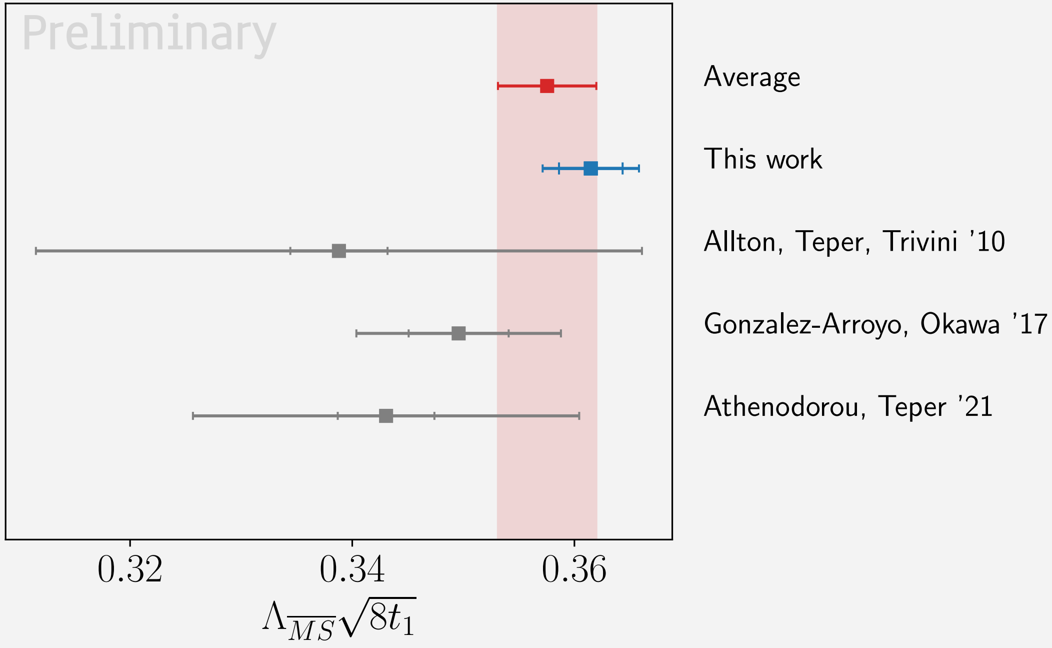
- Integrate the perturbative β-function at $\mathcal{O}(\lambda^4)$

$$-\log\frac{a}{\sqrt{8t_1}} = \log\Lambda_s\sqrt{8t_1} + \underbrace{\frac{1}{2b_0\lambda_s} + \frac{b_1}{2b_0^2}\log(b_0\lambda_s) + \frac{c_1^{(s)}}{2b_0}\lambda_s}_{\log f(\lambda_s)}$$

- Subtract $\log f(\lambda_s)$ to data, convert to $\overline{\mathrm{MS}}$, take the limit

$$\Lambda_{\overline{\mathrm{MS}}}\sqrt{8t_1} = \lim_{\lambda_s\to 0}\frac{\Lambda_{\overline{\mathrm{MS}}}}{\Lambda_s}f(\lambda_s)\frac{\sqrt{8t_1}}{a}$$

- Convert to common units to compare



THANK YOU

OVERVIEW

1 THE Λ-PARAMETER

2 LARGE-N_c LIMIT AND REDUCED MODELS

3 WILSON FLOW SCALE

4 RESULTS

BACKUP MATERIAL

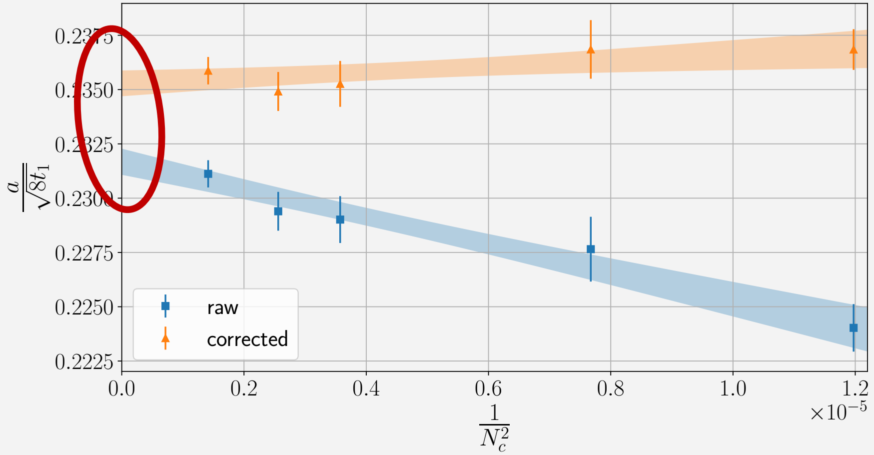
THE LATTICE SCALE

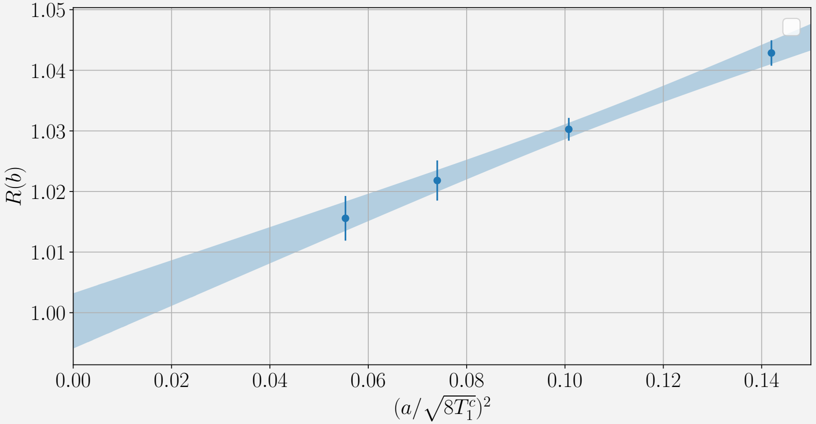
The Wilson flow scale

- Build up a new flow observable which defines a coupling free from finite volume ($\mathcal{N}$) effect at leading order

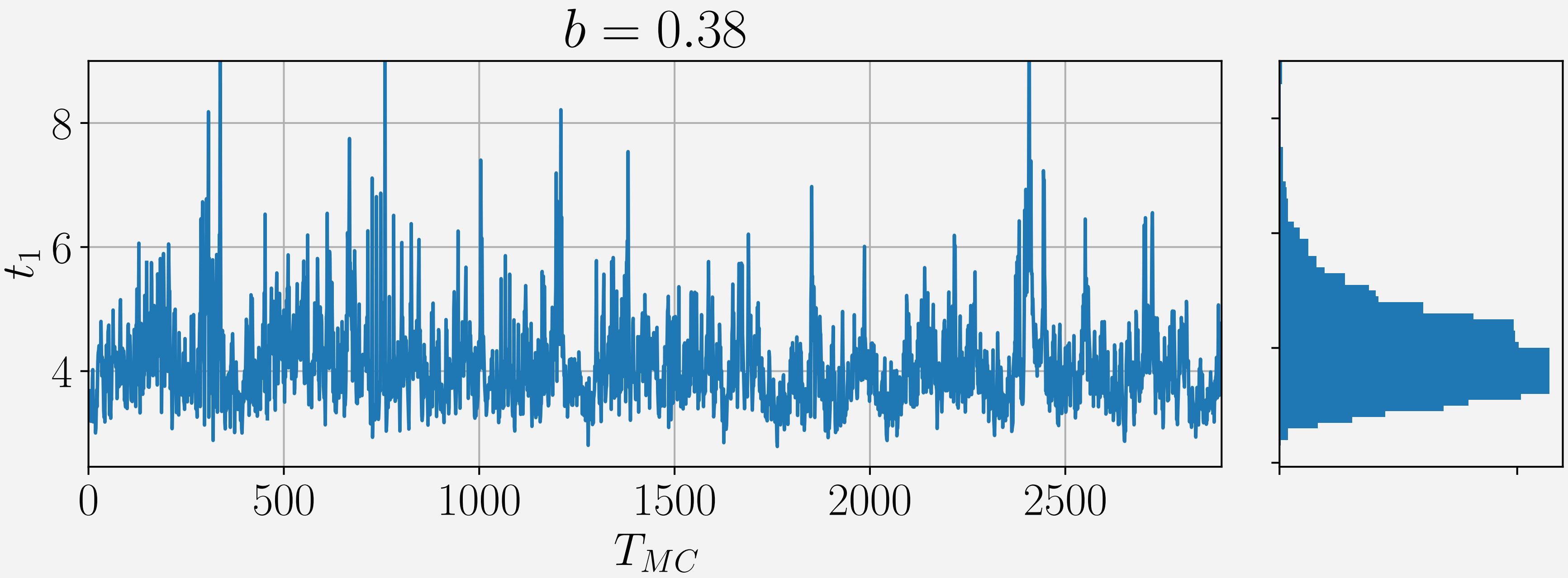
$$\widehat{\Phi}(t) \equiv \frac{3/128\pi^2}{\mathcal{N}_{\text{lat}}\left(\sqrt{8t}/\sqrt{N_c}\right)} \left\langle \frac{t^2 E(t)}{N_c} \right\rangle \qquad T \in \left[1.25, \gamma^2 \frac{N_c}{8}\right]$$

- Controls both finite-volume / lattice artefacts
- Remnant finite-volume effect are $< 3‰$ in the scaling window





$b = 0.38$

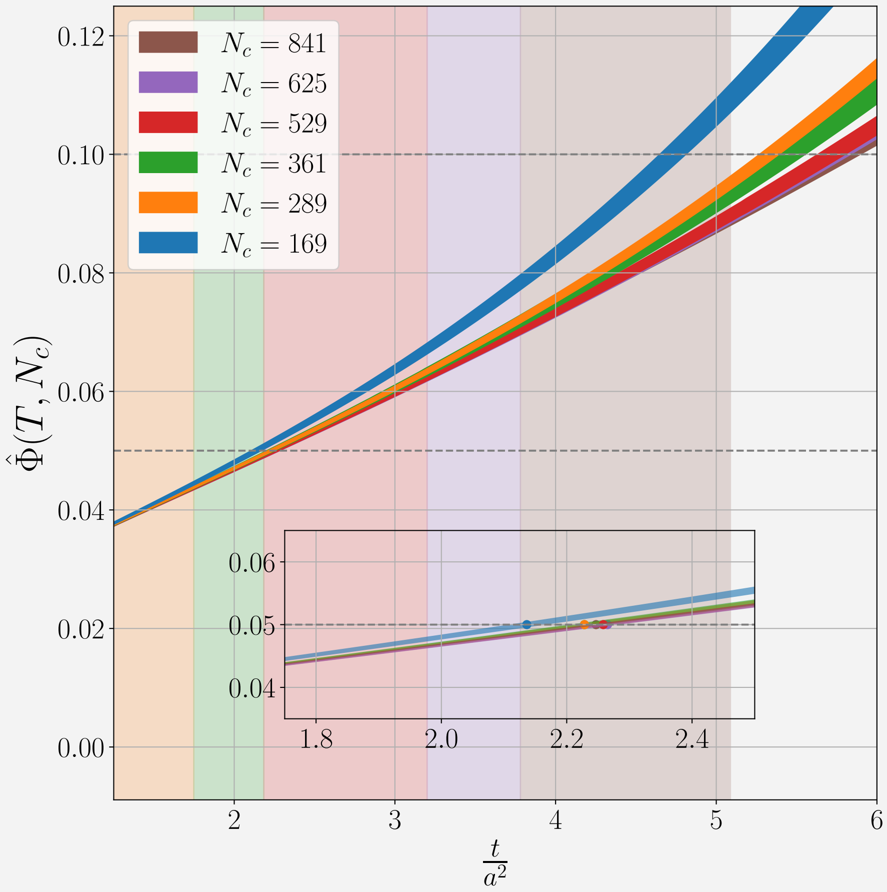


The Wilson flow scale

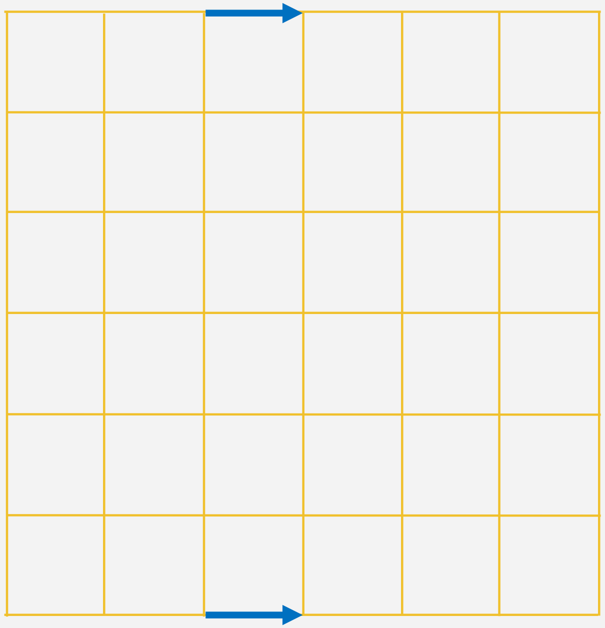
- Build up a new flow observable which defines a coupling free from finite volume ($\mathcal{N}$) effect at leading order

$$\widehat{\Phi}(t) \equiv \frac{3/128\pi^2}{\mathcal{N}_{\text{lat}}\left(\sqrt{8t}/\sqrt{N_c}\right)}\left\langle\frac{t^2 E(t)}{N_c}\right\rangle \qquad T \in \left[1.25, \gamma^2 \frac{N_c}{8}\right]$$

- Controls both finite-volume / lattice artefacts
- Remnant finite-volume effect are $< 3‰$ in the scaling window



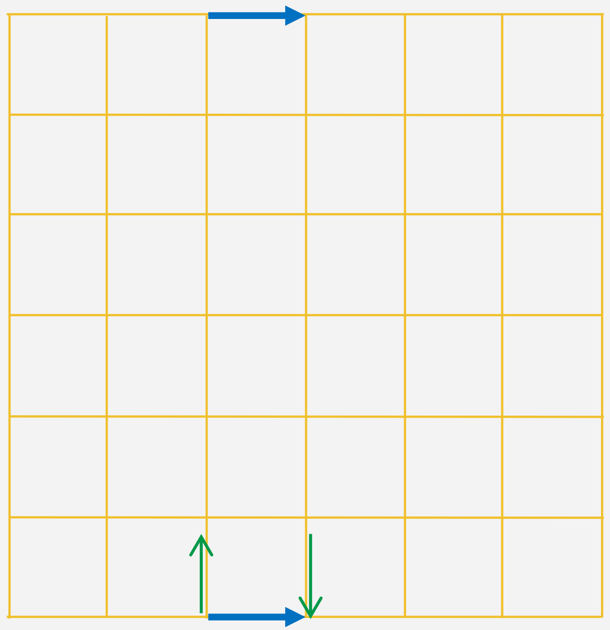
TWISTED BC & VOLUME REDUCTION



PERIODIC boundary conditions

$$U_\mu(n + L\,\hat{\nu}) \;=\; U_\mu(n)$$

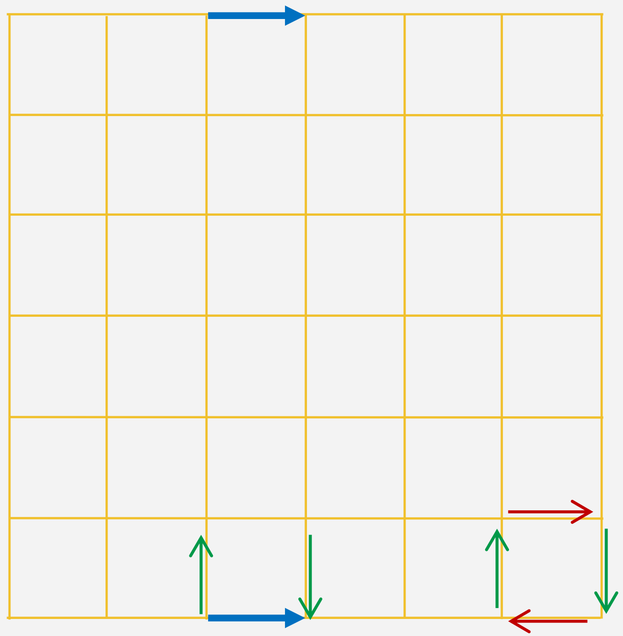
TWISTED BC & VOLUME REDUCTION



TWISTED boundary conditions

$$U_\mu(n + L\,\hat{\nu}) = \Gamma_\nu\, U_\mu(n)\, \Gamma_\nu^\dagger$$

TWISTED BC & VOLUME REDUCTION

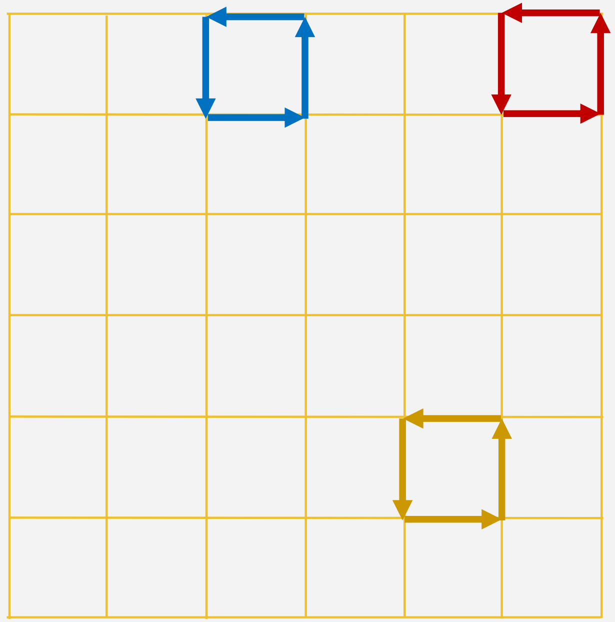


$$\Gamma_\mu \Gamma_\nu = e^{\frac{2\pi i}{N_c}\epsilon_{\mu\nu}} \; \Gamma_\nu \Gamma_\mu$$

TWISTED boundary conditions

$$U_\mu(n + L\,\hat{\nu}) \;=\; \Gamma_\nu\, U_\mu(n)\, \Gamma_\nu^\dagger$$

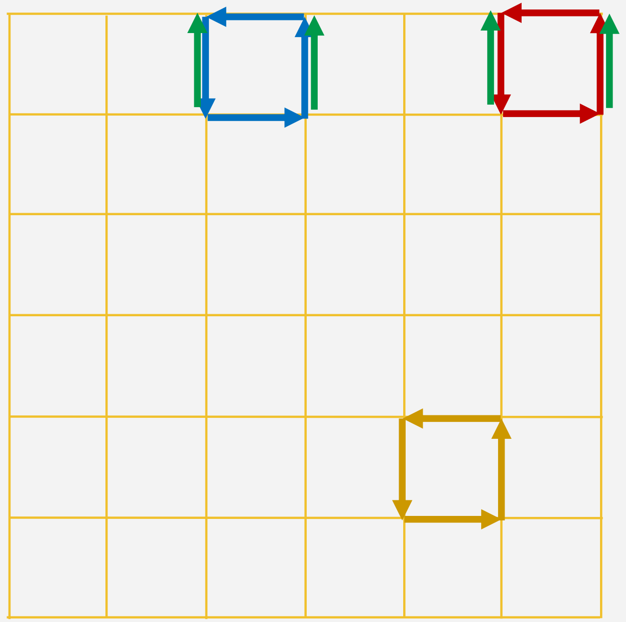
TWISTED BC & VOLUME REDUCTION

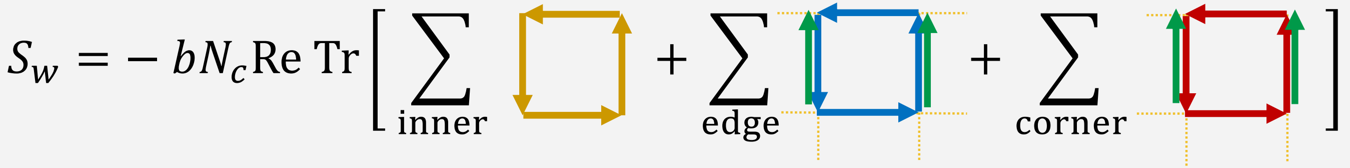


$$S_w = -\,bN_c \mathrm{Re}\,\mathrm{Tr}\left[\sum_{\text{inner}} \square + \sum_{\text{edge}} \square + \sum_{\text{corner}} \square\right]$$

TWISTED BC & VOLUME REDUCTION

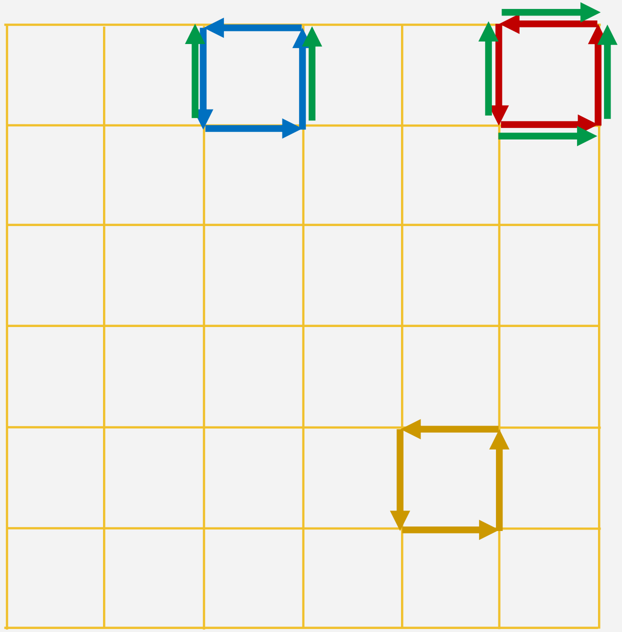
$$U'_\nu(x) = \Gamma_\nu \, U_\nu(x)$$

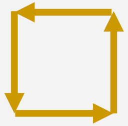
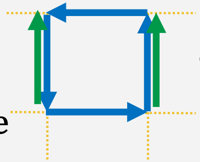
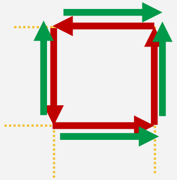




TWISTED BC & VOLUME REDUCTION

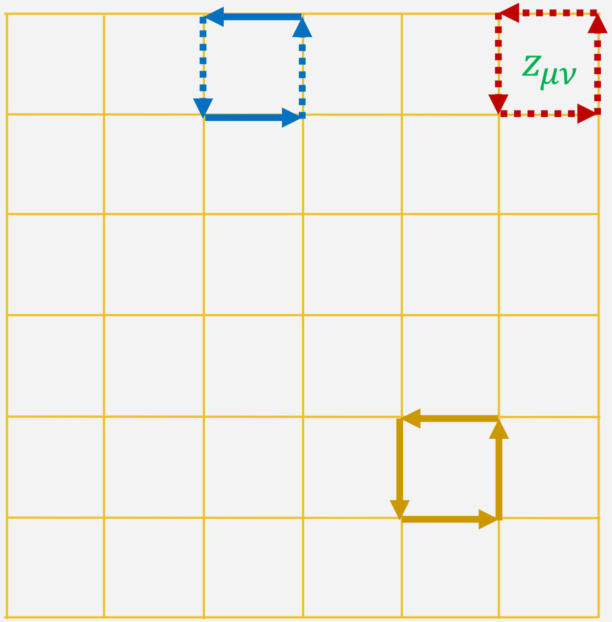
$$U'_\nu(x) = \Gamma_\nu\, U_\nu(x)$$

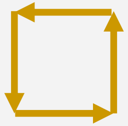
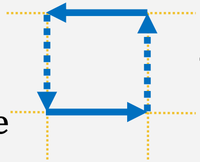
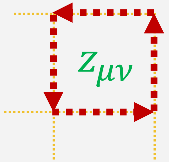


$S_w = -\,bN_c \mathrm{Re\ Tr}\Big[\sum_{\text{inner}}$  $+\sum_{\text{edge}}$  $+\sum_{\text{corner}}$  $\Big]$

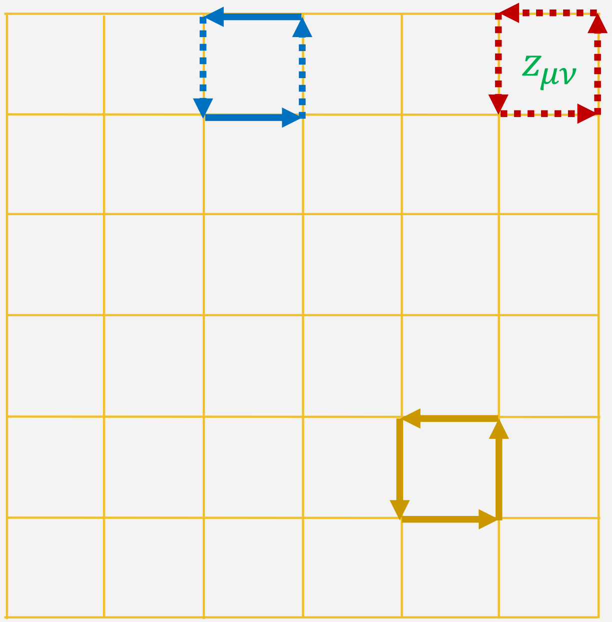
TWISTED BC & VOLUME REDUCTION

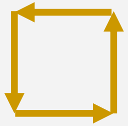
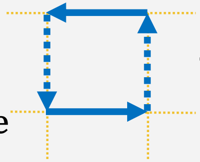
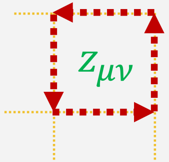
$$U'_\nu(x) = \Gamma_\nu \, U_\nu(x)$$



$S_w = -\, bN_c \mathrm{Re}\,\mathrm{Tr}\Big[\sum_{\text{inner}}$  $+ \sum_{\text{edge}}$  $+ \sum_{\text{corner}}$  $\Big]$

TWISTED BC & VOLUME REDUCTION

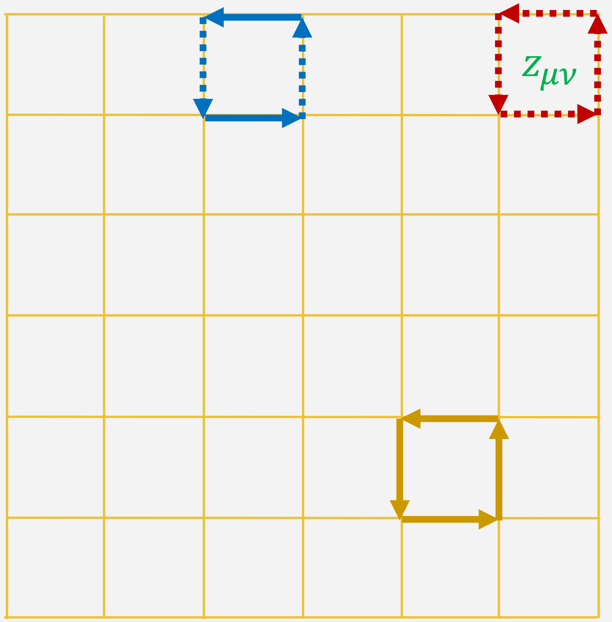


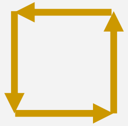
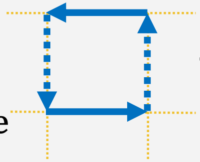
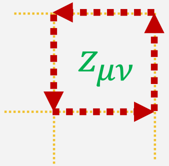
$S_w = -\, bN_c \mathrm{Re\ Tr} \Big[\sum_{\text{inner}}$  $+ \sum_{\text{edge}}$  $+ \sum_{\text{corner}}$  $\Big]$

$$U'_\nu(x) = \Gamma_\nu\, U_\nu(x)$$

$$= -bN_c \sum_{n,\mu\neq\nu} z_{\mu\nu}(n)\ \mathrm{Re\ \ Tr}\left[U_\mu(n)U_\nu(n+\hat{\mu})U^\dagger_\mu(n+\hat{\nu})U^\dagger_\nu(n)\right]$$

TWISTED BC & VOLUME REDUCTION



$S_w = -\, bN_c \mathrm{Re\ Tr} \Big[\sum_{\text{inner}}$  $+ \sum_{\text{edge}}$  $+ \sum_{\text{corner}}$  $\Big]$

$U'_\nu(x) = \Gamma_\nu\, U_\nu(x)$

$$= -bN_c \sum_{n,\mu\neq\nu} z_{\mu\nu}(n)\ \mathrm{Re\ \ Tr}\left[U_\mu(n) U_\nu(n+\hat{\mu}) U_\mu^\dagger(n+\hat{\nu}) U_\nu^\dagger(n)\right]$$

$U_\mu(n) \to U_\mu$

$$S_{\mathrm{TEK}} = -bN_c \sum_{n,\mu\neq\nu} z_{\mu\nu}\ \mathrm{Re\ \ Tr}\left[U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\right]$$