

Bayesian Interpretation of Backus-Gilbert methods

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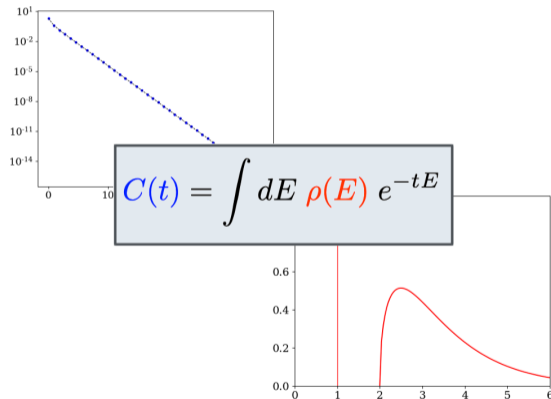
Inverse problem: generalities

Computing the **spectral density** $\rho(E)$ associated to a **lattice correlator** $C(t)$

Ill-posed in presence of a finite set of noisy data.

Regularisations are available: **Backus-Gilbert** & **Bayesian** methods have different philosophies but share similarities

$$\rho(E) = \lim_{t \rightarrow 0} \frac{1}{t} \sum_t g_t(\cdot; E) C(t)$$



Wish list

To obtain a function that is smooth even at finite volume:

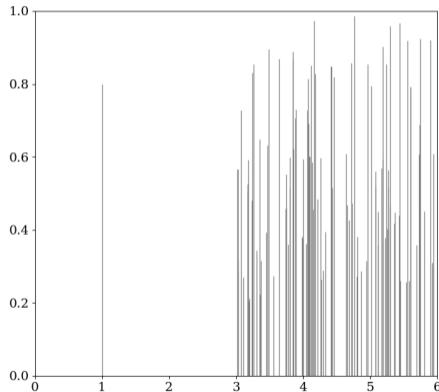
$$\langle f \rangle = \int dE S(E; t) \langle f(E) \rangle$$

For some applications, a **fixed smearing kernel** across lattice spacings, volumes, ... to control **systematics** of fits & extrapolations

Understand dependence of the result on **algorithmic inputs**, parameters, priors ...

Remark: linear combination of $C(t)$ is always smeared

$$\begin{aligned} \langle f(E) \rangle &= \sum_t g_t(E) C(t) \\ &= \sum_t g_t(E) \int dE' e^{-tE'} \langle f(E') \rangle \end{aligned}$$



Bayesian Inference with Gaussian Processes

Aim for a **probability distribution** over a functional space of possible **spectral densities**

Consider the stochastic field $R(E)$ **Gaussian-distributed** around the **prior** value $R^{\text{prior}}(E)$ with covariance $K^{\text{prior}}(E; E^0)$.

$$GP \sim \text{prior}(E); K^{\text{prior}}(E; E^0)$$

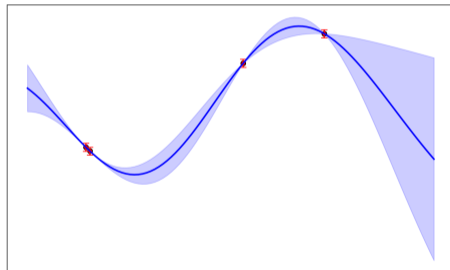
Similarly, assume that **observational noise** is Gaussian: $\epsilon(t)$

$$\mathbb{G}(\cdot; \text{Cov}_d) = \exp \left\{ -\frac{1}{2} \tilde{\cdot}^T \text{Cov}_d^{-1} \tilde{\cdot} \right\}$$

The stochastic variable associated to the correlator, C , is related to R and ϵ via

$$C(t) = \int dE e^{iEt} R(E) + \epsilon(t)$$

Valentine, Sambridge 19
Horak, Pawlowski, Rodríguez-Quintero, Turnwald, Urban 21
Del Debbio, Giani, Wilson 21



Bayesian Inference with Gaussian Processes

The joint, **posterior distribution** for post at some energy has centre and variance (set $\text{prior} = 0$)

$$\text{post}(I) = \prod_{t=1}^{t_{\max}} g_t^{\text{GP}}(I) C(t)$$

$$K^{\text{post}}(I; I) = @K^{\text{prior}}(I; I) \prod_{t=1}^{t_{\max}} g_t^{\text{GP}}(I) F_t(I) A$$

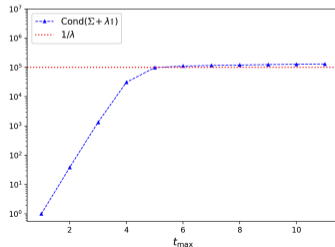
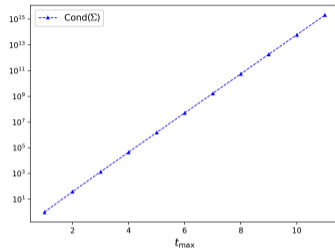
The coefficients are

$$g_t^{\text{GP}}(I) = \prod_{r=1}^{t_{\max}} \frac{1}{1 + \text{Cov}_d} F_r(I)$$

Ingredients:

$$tr = \int dE_1 \int dE_2 e^{iE_1} K^{\text{prior}}(E_1; E_2) e^{iE_2} \quad \text{ill cond}$$

$$F_t(I) = \int dE K^{\text{prior}}(I; E) e^{iE}$$



Bayesian Inference with Gaussian Processes

Let the model prior for the spectral density have covariance

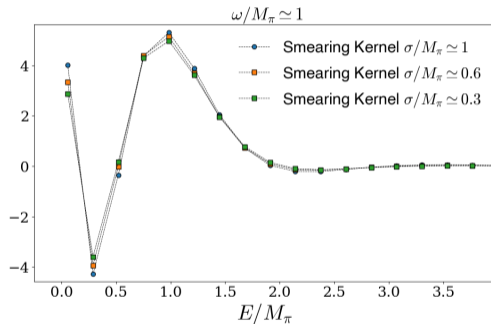
$$K(E; E^0) = \frac{e^{-(E - E^0)^2 / 2\sigma^2}}{\sigma\sqrt{2\pi}}; \quad \text{prior} = 0$$

The model covariance acts as a smearing kernel,

$$S^{\text{GP}}(E; !) = \sum_{t=1}^{N_{\text{ax}}} g_t^{\text{GP}}(; !) e^{-tE}$$

If we are interested in removing the smearing, there should be a limit in which S approaches a δ -function

$$(!) = \int dE S^{\text{GP}}(E; !) \delta(E)$$



Prior dependence

Allow the model prior to vary

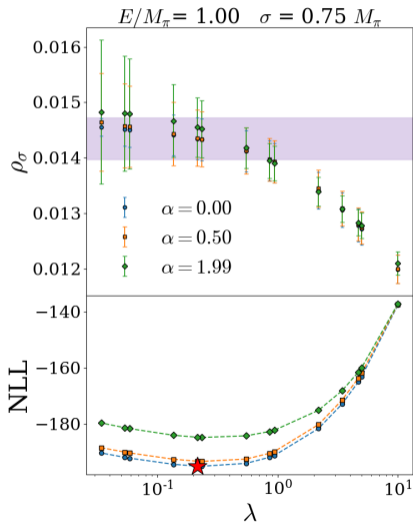
$$K^{\text{prior}}(E; E^0) = \frac{e^{-(E - E^0)^2 / 2 \sigma^2}}{\int e^{-(E - E^0)^2 / 2 \sigma^2}} e^{-E};$$

By changing α and λ we can explore the dependence on the posterior from the prior

There is a region in which the dependence on the prior is absorbed in the statistical error

Choice of the parameters: minimise Negative Log Likelihood

$$\log P(\text{data}|\text{parameters})$$



Backus Gilbert

(Hansen Lupo Tantalò 19) Target a spectral density **smeared with a chosen kernel**

Recipe for g_t : make them such that

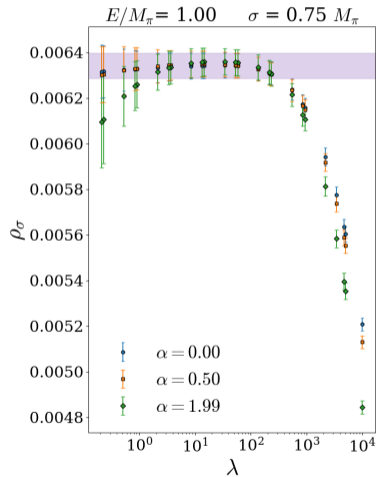
$$\int_{t=1}^{\infty} g_t(t) e^{-tE} = S(E) \quad \Rightarrow \quad \int_{t=1}^{\infty} g_t(t) c(t) = c(E)$$

How? By minimising

$$\int_0^{\infty} dE e^{-E} \int_{t=1}^{\infty} g_t e^{-tE} S(t; E)^2 + \underbrace{\int_{t=1}^{\infty} g_t^2}_{\text{Regularises}}$$

Provides solution

$\alpha \in (0; 1)$ and $\lambda < 2$ are algorithmical **input parameters**.



Backus Gilbert

Our prescription leads to

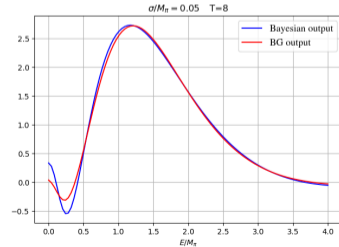
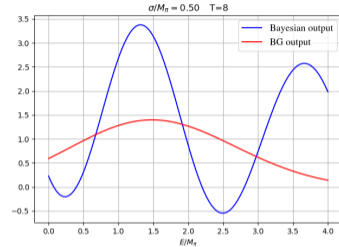
$$g^{BG}(\cdot;!) = \frac{1}{\sigma + \sigma' \text{Cov}_d} F_r(\cdot;!); \quad \sigma = \sigma'(1 - \dots)$$

The ingredients are

$$\sigma_{tr} = \int dE e^{tE} e^{rE} e^E; \quad \text{ill-cond}$$

$$F_t(\cdot;!) = \int dE e^{tE} e^E S(E;!)$$

When $! \neq 0$, $g^{GP} \neq g^{BG}$ but for $\neq 0$ they differ



Backus-Gilbert as a Gaussian Process

Compute the posterior probability distribution for a spectral density smeared with a **fixed kernel** $G(E; E^0) = \exp(-(E - E^0)^2 / 2\sigma^2)$

Let the model **prior** for the spectral density have **diagonal** covariance

$$K(E; E^0) = \frac{(E - E^0)^2}{2\sigma^2} e^{-\frac{(E - E^0)^2}{2\sigma^2}}$$

By computing the posterior probability, one gets coefficients g^{GPsmr}

$$g^{\text{GPsmr}}(\omega; !) = g^{\text{BG}}(\omega; !) \quad \text{even at finite}$$

The only difference is in the **error** (bootstrap for Backus-Gilbert methods)

$$g^{\text{GPsmr}}(\omega; !)^2 = \frac{1}{2} \int dE \sum_i g_i^{\text{BG}}(\omega; !)^2 e^{-iE} G(E; !)^2$$

Conclusions

Same regularisation of the problem via Cov_d

Algorithmic parameters of BG can be understood as Bayesian priors, and vice versa

There is a region in which the result **does not depend on the inputs / priors** within statistical error. In the same region, the **NLL finds its minimum**.

The statistical error of BG is **of the same order of magnitude** of the Bayesian error GP_{smr}

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