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A staggered U(1) gauge theory inspired by self-adjoint extensions

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Kogut-Susskind Hamiltonian

[Kogut & Susskind PRD11 (1975) 395]

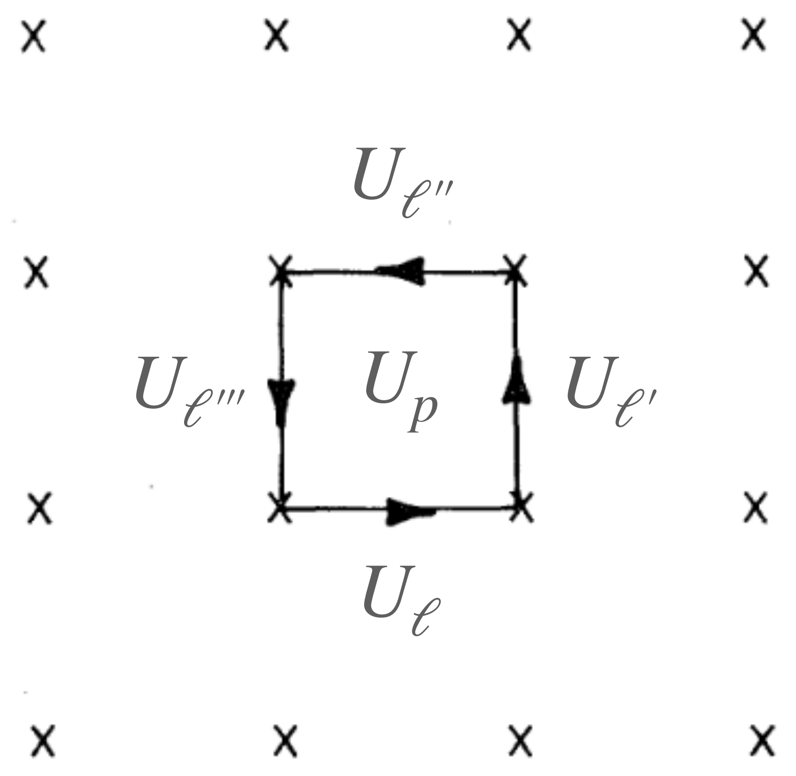
see also [D. Grabowska Tue 11:00]

Non-perturbative formulation of gauge theory
via spatial lattice and Hamiltonian picture

$$H = \frac{e^2}{2} \sum_{\text{links } \ell} E_\ell^2 + \frac{1}{2e^2} \sum_{\text{plaqs } p} B^2(U_p)$$

$$E = -i\partial/\partial\varphi_\ell$$

$$B^2(U_p) \overset{\text{e.g.}}{=} 1 - \text{Re}U_p = 1 - \cos\varphi_p$$



$$U_p = U_\ell U_{\ell'} U_{\ell''} U_{\ell'''} \in U(1)$$

$$\downarrow$$

$$\varphi_p = \varphi_\ell \varphi_{\ell'} \varphi_{\ell''} \varphi_{\ell'''} \in [0,2\pi]$$

Hilbert space: tensor product of
wavefunctions over $U(1)$-valued links

Trotterization

Familiar Euclidean lattice gauge theory path integral emerges by Trotterization

$$Z = \mathrm{Tr}[e^{-\beta H}] \approx \mathrm{Tr}[(e^{-\beta H_E/n} e^{-\beta H_B/n})^n]$$

Inserting complete sets of states $1 = \int dU \,|\, U\rangle\langle U \,|$, evaluating matrix elements

$$Z = \int d[U] e^{-S_{g,W}[U]}, \quad \text{where } S_{g,W}[U] = \frac{1}{e^2} \sum_{\text{plaqs } p} (1 - \mathrm{Re} U_p)$$

Isotropic **Trotterization is a useful construction.**

Introducing a “twist angle” Θ

Hilbert space: tensor product of wavefunctions over $U(1)$-valued links → Ambiguous!

$U(1)$ group structure requires that **probabilities** must be periodic

$$|\psi(\varphi = 0)|^2 = |\psi(\varphi = 2\pi)|^2$$

Consistent with an entire family of Hilbert spaces

$$\boxed{\psi(\varphi = 0) = e^{i\Theta}\, \psi(\varphi = 2\pi)}$$

$$\Updownarrow$$

$$E = -i\partial/\partial\varphi + \Theta/2\pi$$

Related to the mathematical concept of “self-adjoint extensions”

[Reed & Simon *“Fourier Analysis, Self-Adjointess: Volume 2”*, MMMP (1975)]
[Gieres RPP63 (2000) 1893]

Preserving symmetries

1. **Gauge symmetry:**
    - For $\Theta \neq 0$, the links provide a *projective* representation of the gauge group
2. **Charge conjugation:**
    - $\psi(\theta) \stackrel{C}{\to} \psi(2\pi - \theta)$
    - Preserved for $\Theta = 0$ (ordinary theory) and $\Theta = \pi$ (“staggered” theory)
3. **(Hyper-)Cubic rotations:**
    - Space-time rotation mixes E^2 and B^2 → choose $B^2(U_p)$ consistent with E^2
    - Straightforward in path integral picture…

Dual theory

$$\vartheta(z;\tau) = \sum_{n=-\infty}^{\infty} \exp\left(\pi i n^2 \tau + 2\pi i n z\right)$$

Jacobi theta function

The “staggered” theory faces a **severe sign problem.**

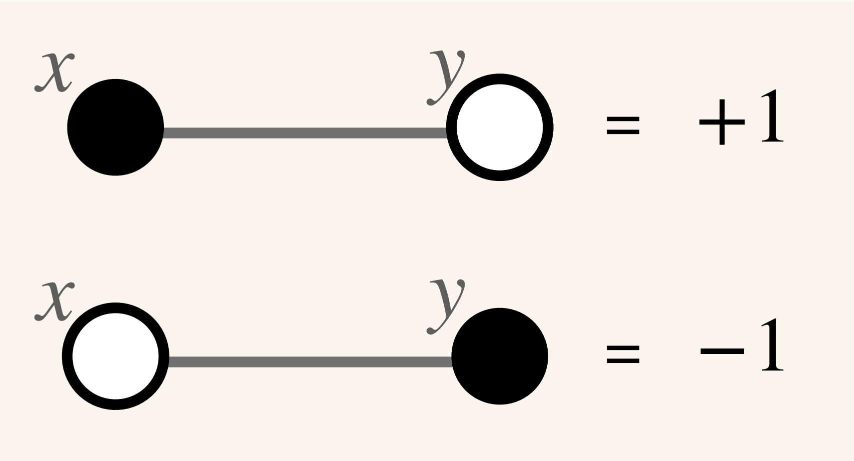
$$S[\varphi_p] = \sum_{\text{plaqs } p} \frac{1}{2e^2}\varphi_p^2 - \frac{i\Theta}{2\pi}\varphi_p - \ln \vartheta(i\varphi_p/e^2 + \Theta/2\pi; 2\pi i/e^2)$$

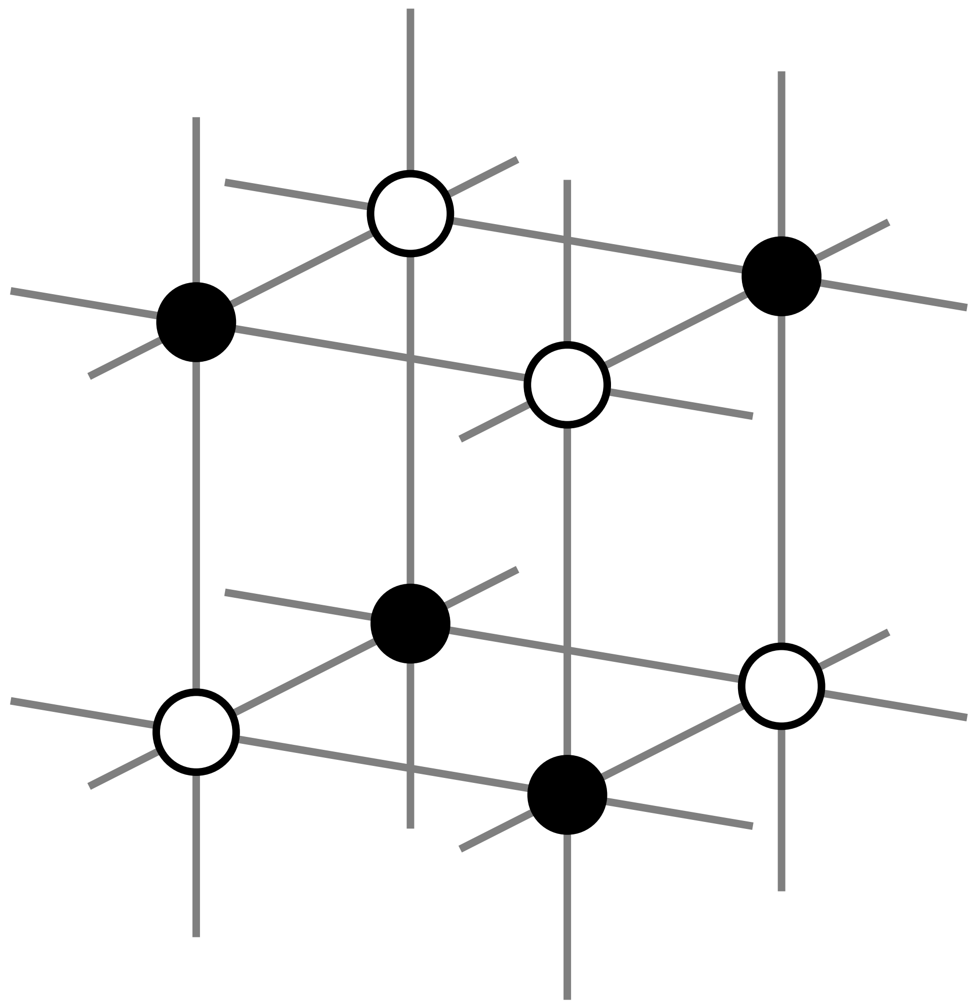
Circumvent this issue by dualization!

2+1D: $U(1)$ gauge theory ↔ $\mathbb{Z}$ spin model

$$Z = \int d[\varphi_p] e^{-S[\varphi_p]} = [\prod_{x\in\Lambda^*} \sum_{h_x}] e^{-S[h_x]}$$

$$S[h_x] = \sum_{\langle xy \rangle} \frac{e^2}{2}(h_x - h_y \pm \frac{\Theta}{2\pi})^2$$





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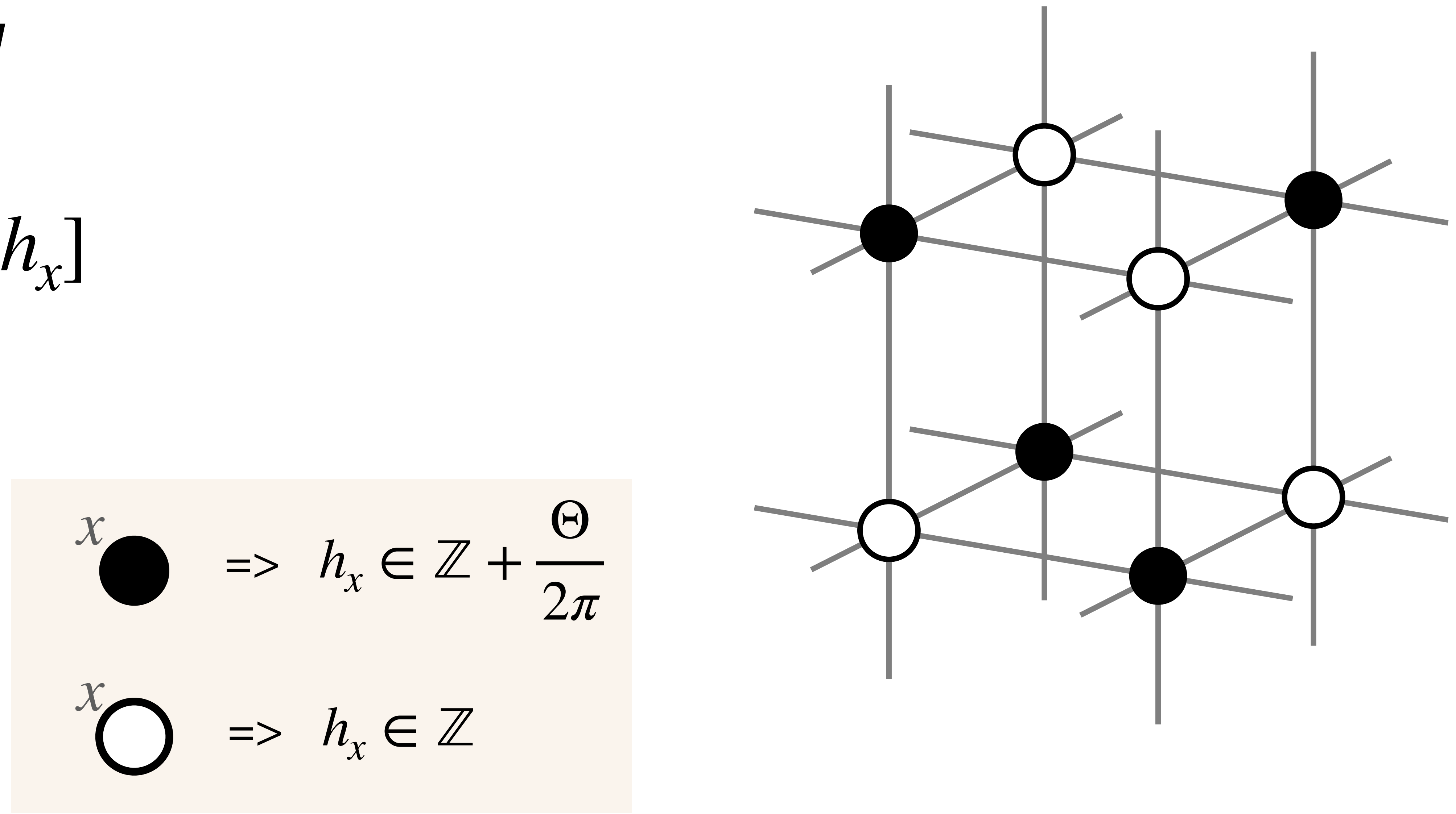


Symmetries in the dual theory

1. Charge conjugation $C \;\cong\; \mathbb{Z}_2$

$$h_x \to -h_x$$

2. One-site translation $S \;\cong\; \mathbb{Z}_2$

$$h_x \to h_{x+\hat{\mu}}$$

3. Offsets $\cong\; \mathbb{Z}$

$$h_x \to h_x + n$$

To close the symmetry group, must combine with (a) parity + (b) charge conjugation + (c) quarter integer offset

Numerical results

Efficient cluster update method

Dual theory ensembles

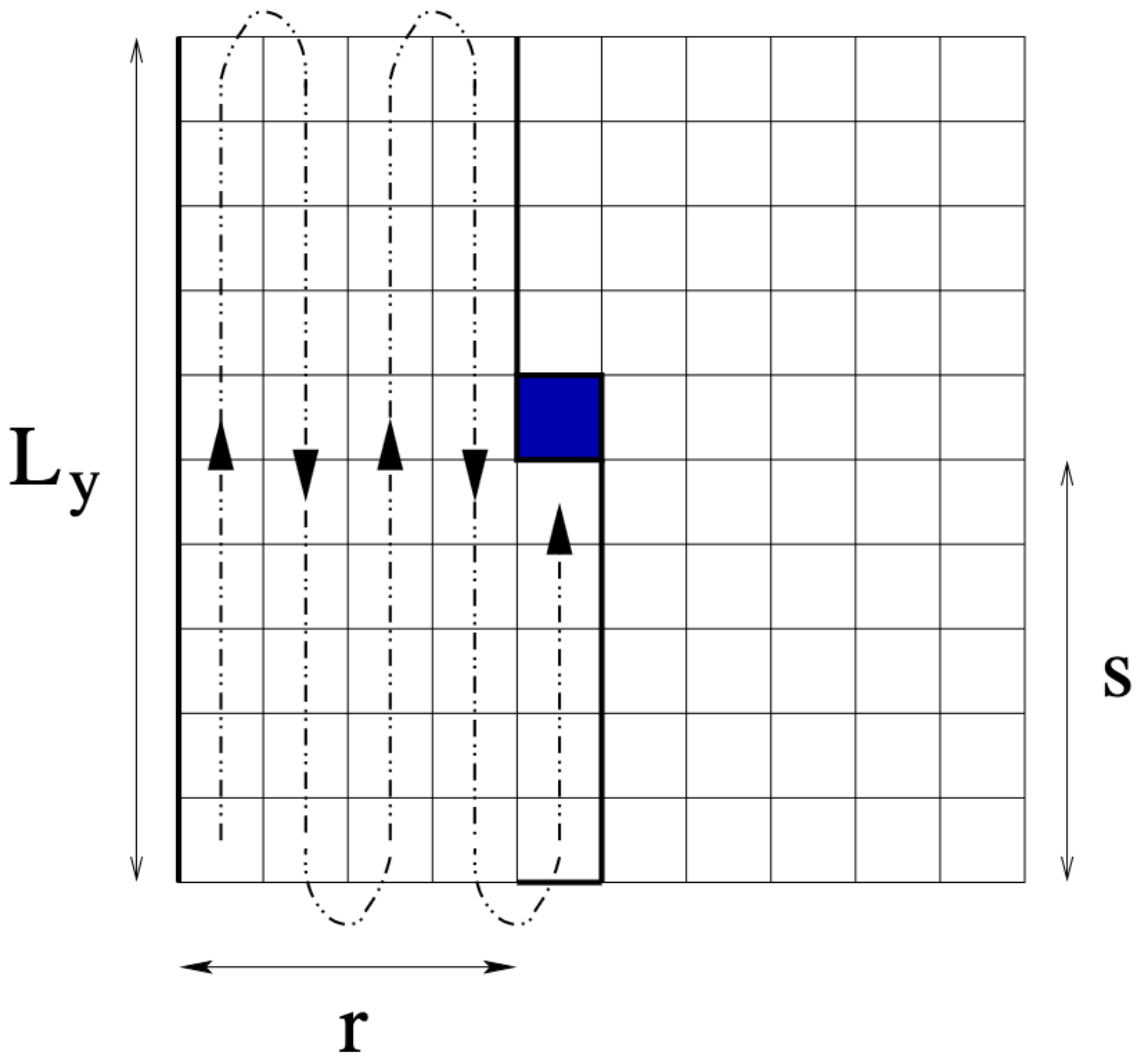
- $V/a^3 = 32^3$ to $V/a^3 = 256^3$
- Bare coupling $e^2 \in [0.25, 2.0]$

Modified Snake algorithm for string tension

[de Forcrand, d'Elia, Pepe PRL86 (2001) 1438]

[de Forcrand & Noth PRD72 (2005) 114501]

- Heatbath resampling snake "head"
- Improved estimator for string force



Mass gap

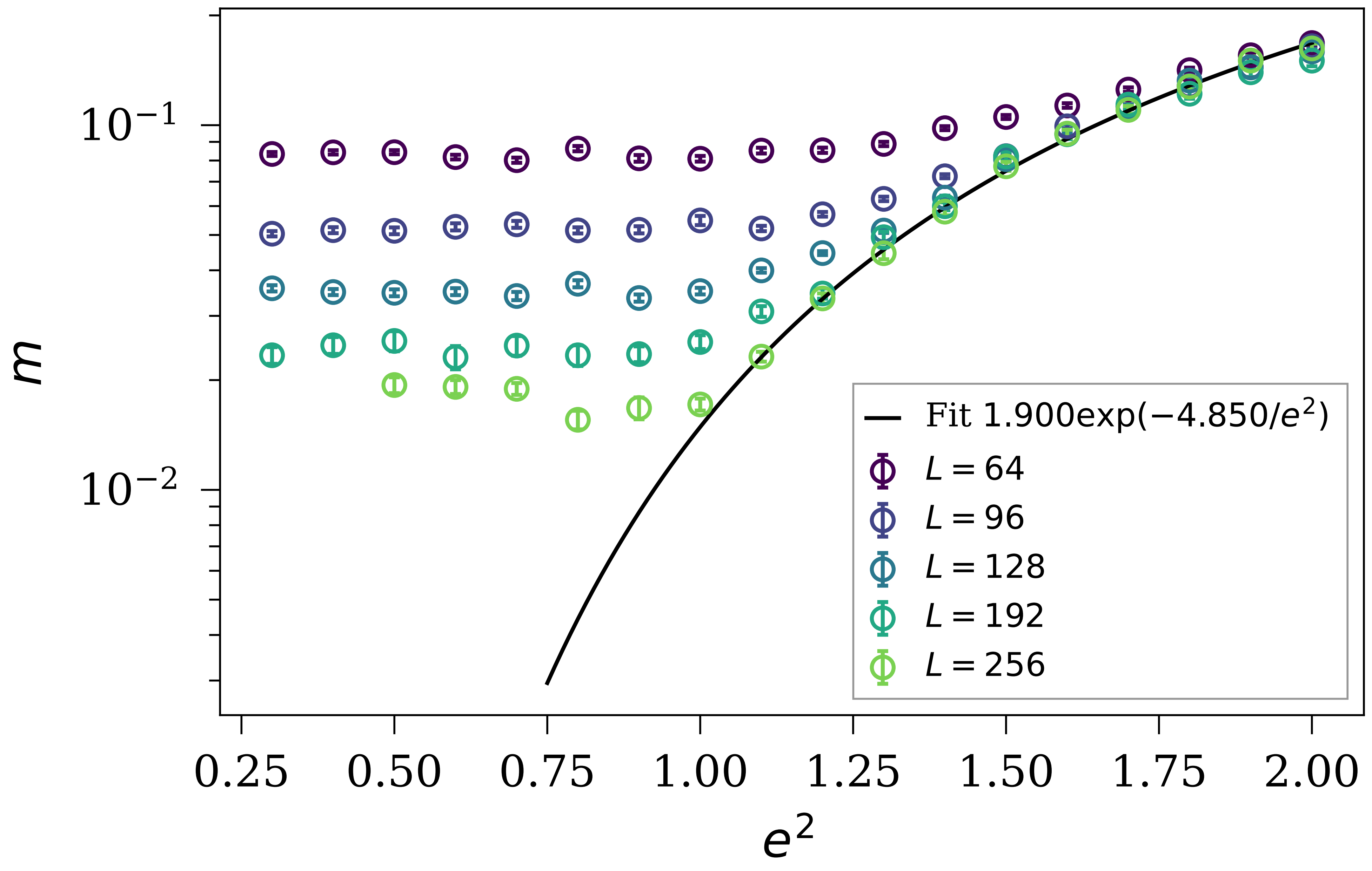
Two-point connected correlation function

$$C(x,y) = \left\langle (h_x - h_y)^2 \right\rangle - \left\langle (h_0 - h_z)^2 |_{z\to\infty} \right\rangle$$

Single exponential fits sufficient to extract masses when volume is large enough.

Infinite-volume curve collapse, fit to this curve inspired by unstaggered theory analytical results [Göpfert & Mack CMP82 (1982) 545]

$$m(e^2) \sim 1.900 e^{-4.850/e^2}$$



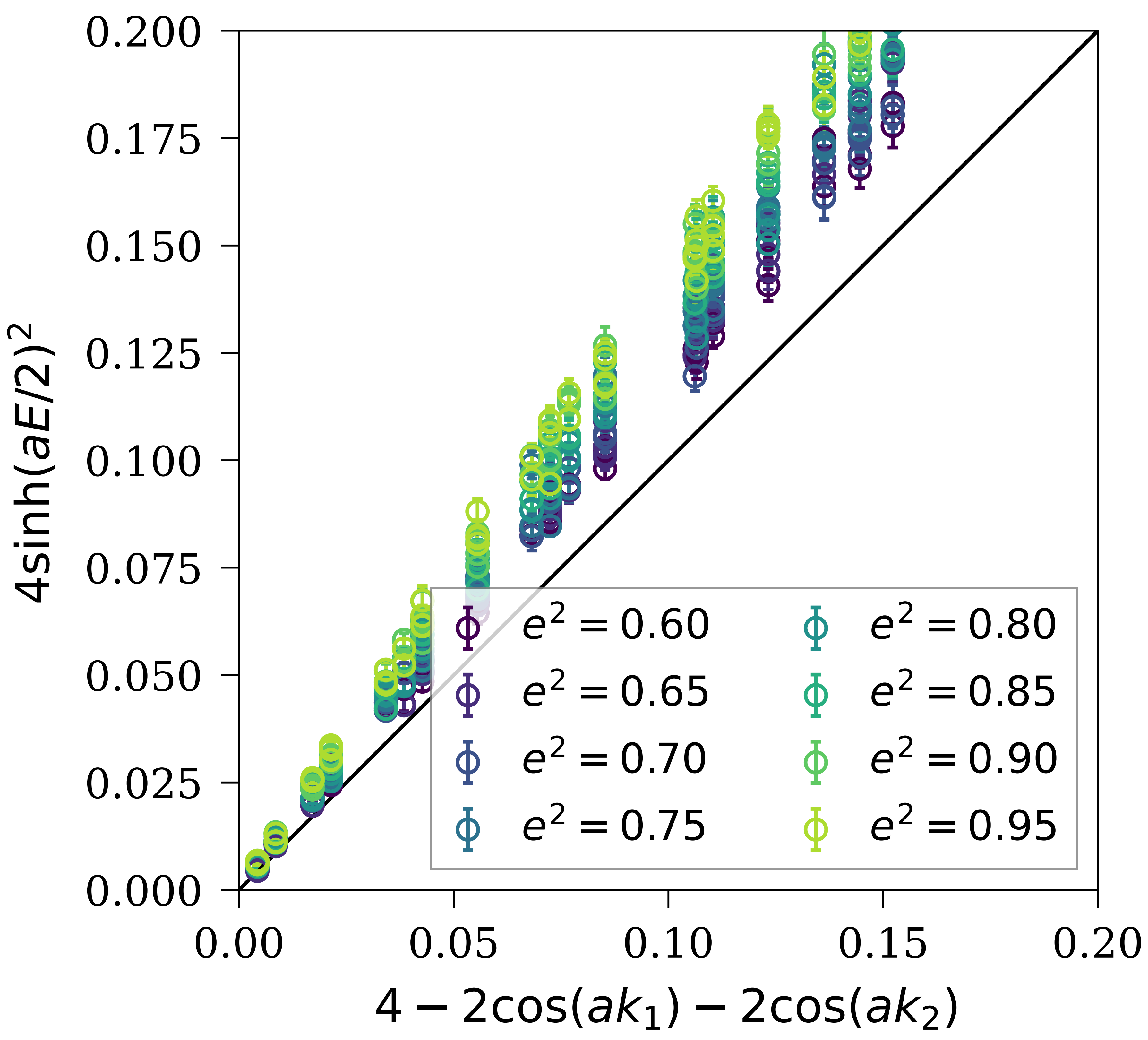
Relativistic dispersion relation

Momentum-projected two-point function for a variety of $\vec{k} = (k_1, k_2)$

Lattice relativistic dispersion relation

$$4\sinh(aE/2)^2 = 4 - 2\sum_i \cos(ak_i)$$

… emerging for small lattice spacings



Confinement and string tension

Snake algorithm yields force

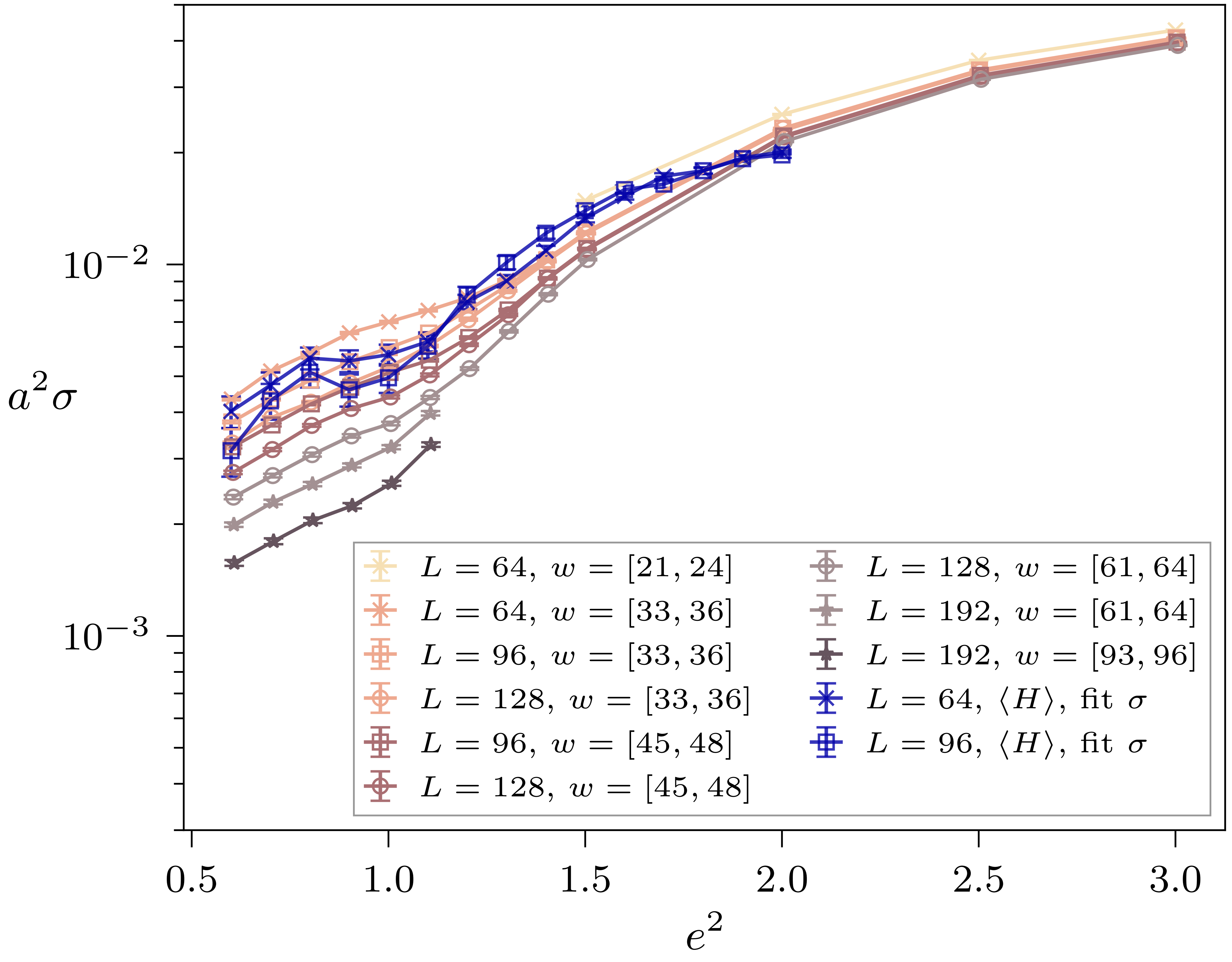
$$F(r) = V(r+1) - V(r)$$

for select distances r.

String tension from $F(r) \overset{r\to\infty}{=} \sigma$

- We extract effective σ from windows $r \in w = [r_i, r_f]$

Infinite-volume curve collapse visible for large enough volume. Extrapolation / fit remains to do!



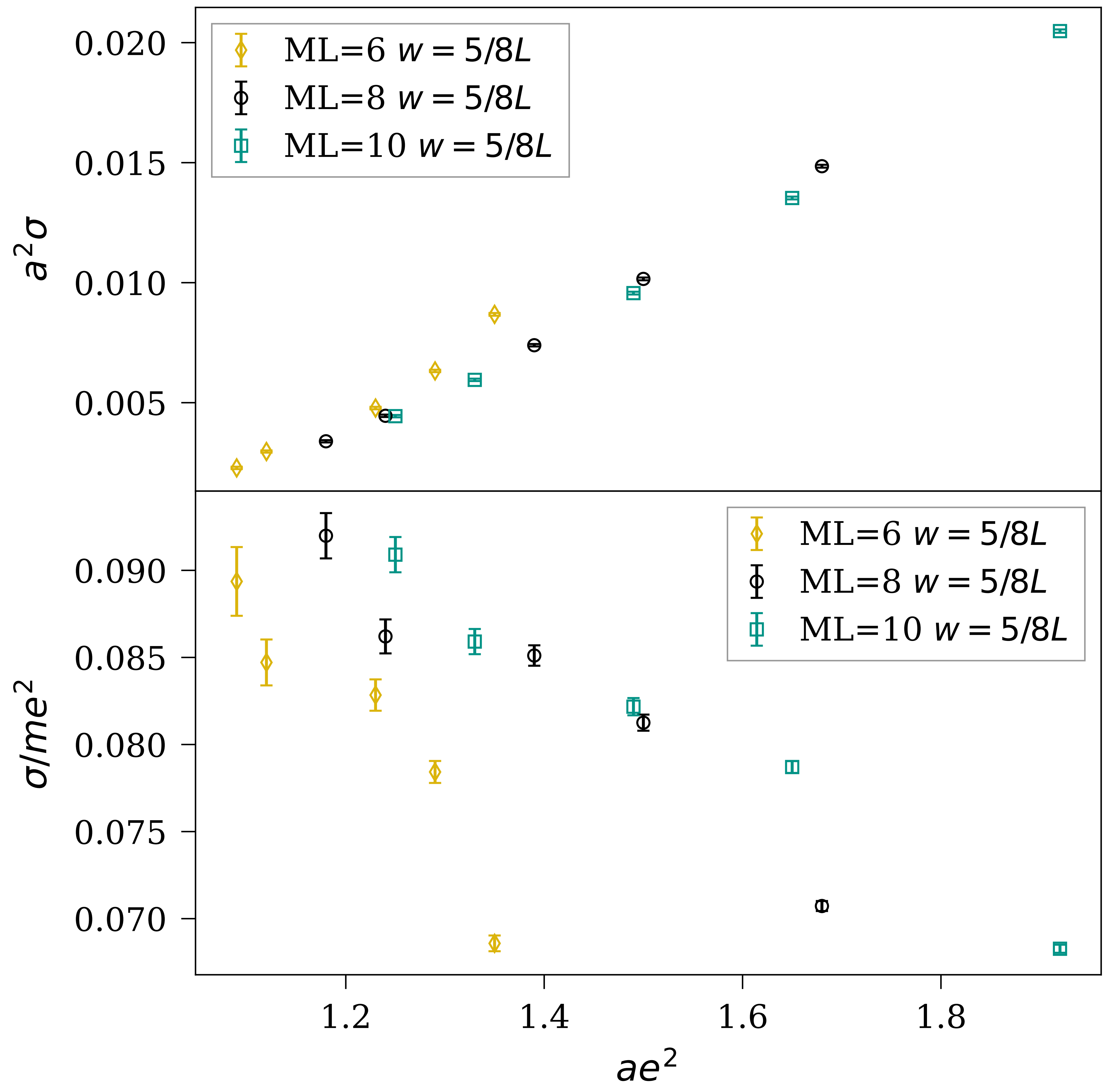
Scaling

Unstaggered theory features peculiar scaling between m and σ:

$$\sigma = \text{const.} \times me^2$$

Taking this as an ansatz, measure σ/me^2 for *fixed* physical volume mL

- Expect scaling for each choice of box size
- Difference in scaling only up to FSE

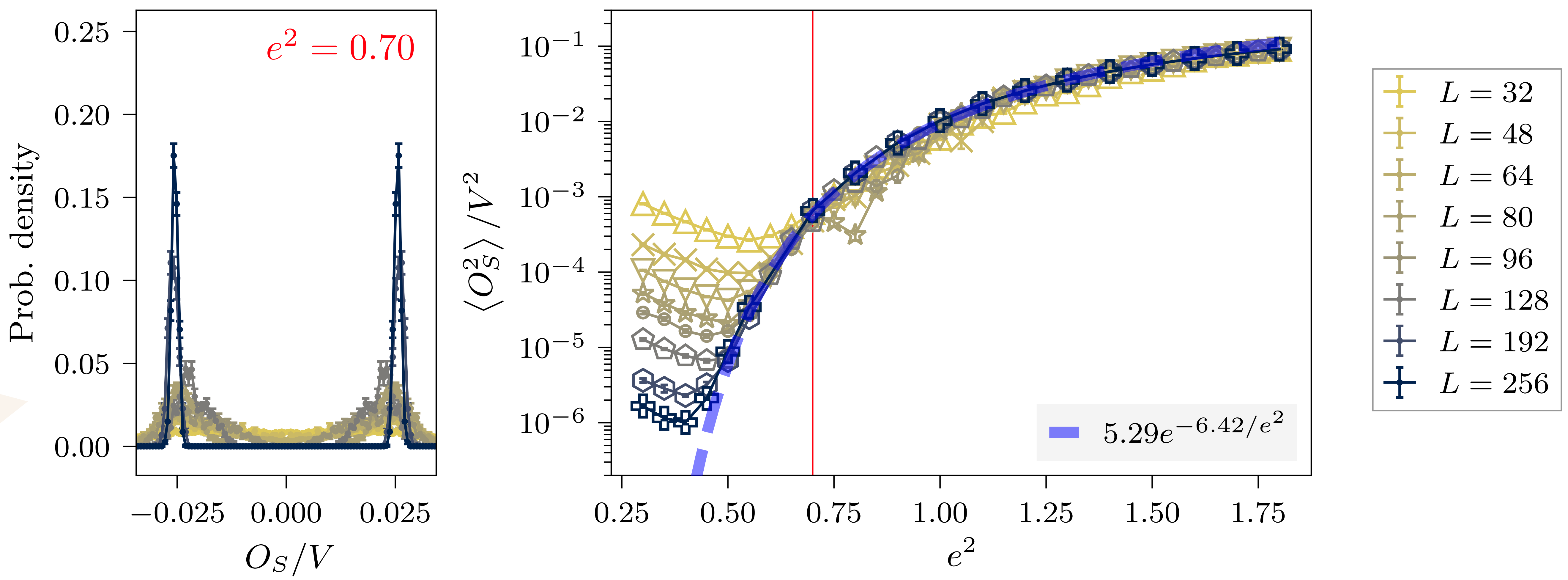


Order parameters

$$O_S = \sum_{\text{cubes } c} \sum_{x \in c} (-1)^x \left(h_x - \bar{h}_c\right)^2$$

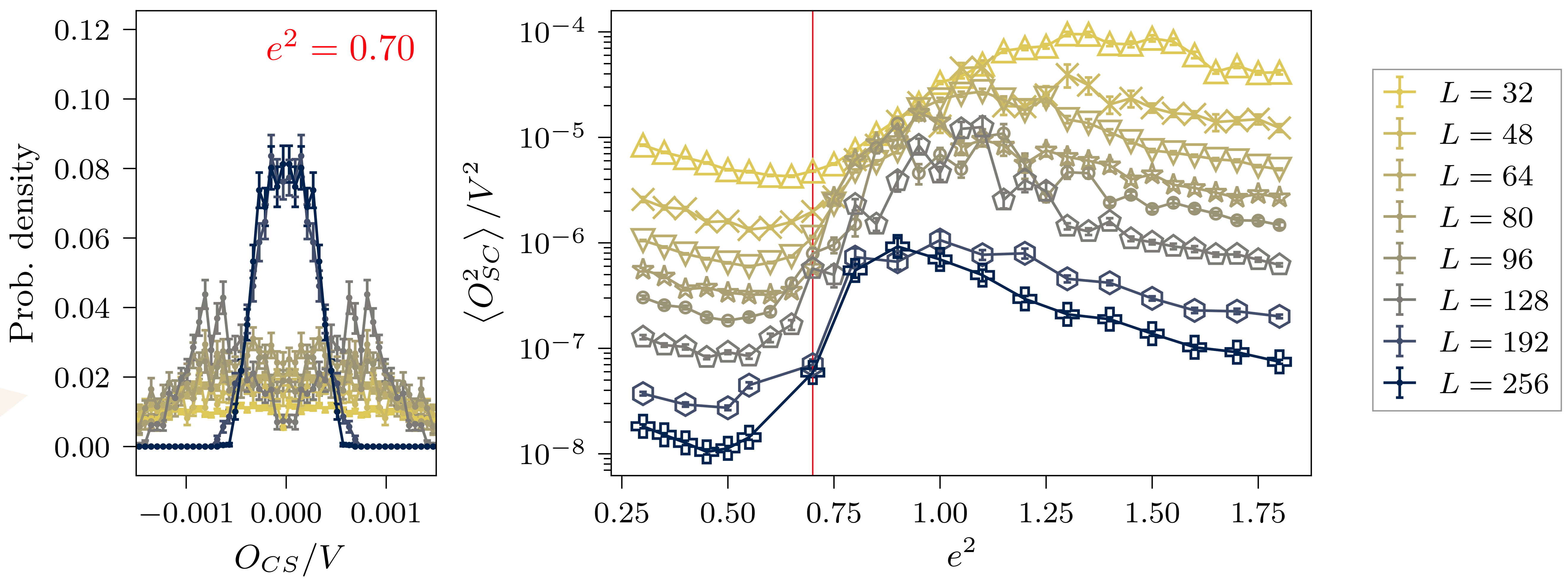
(where $\bar{h}_c = \frac{1}{8} \sum_{x \in c} h_x$)

spontaneously broken

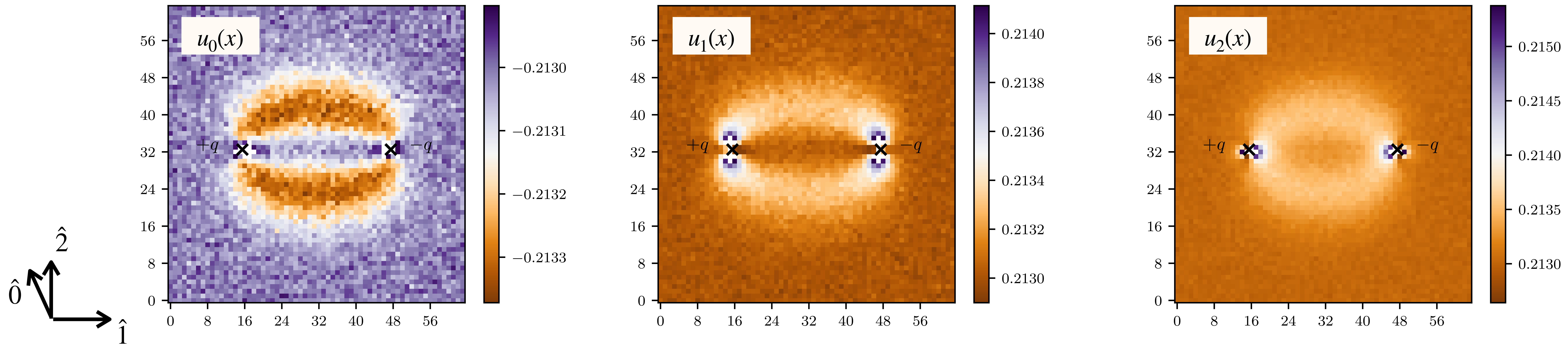


$$O_{CS} = \sum_x (-1)^x h_x$$

preserved



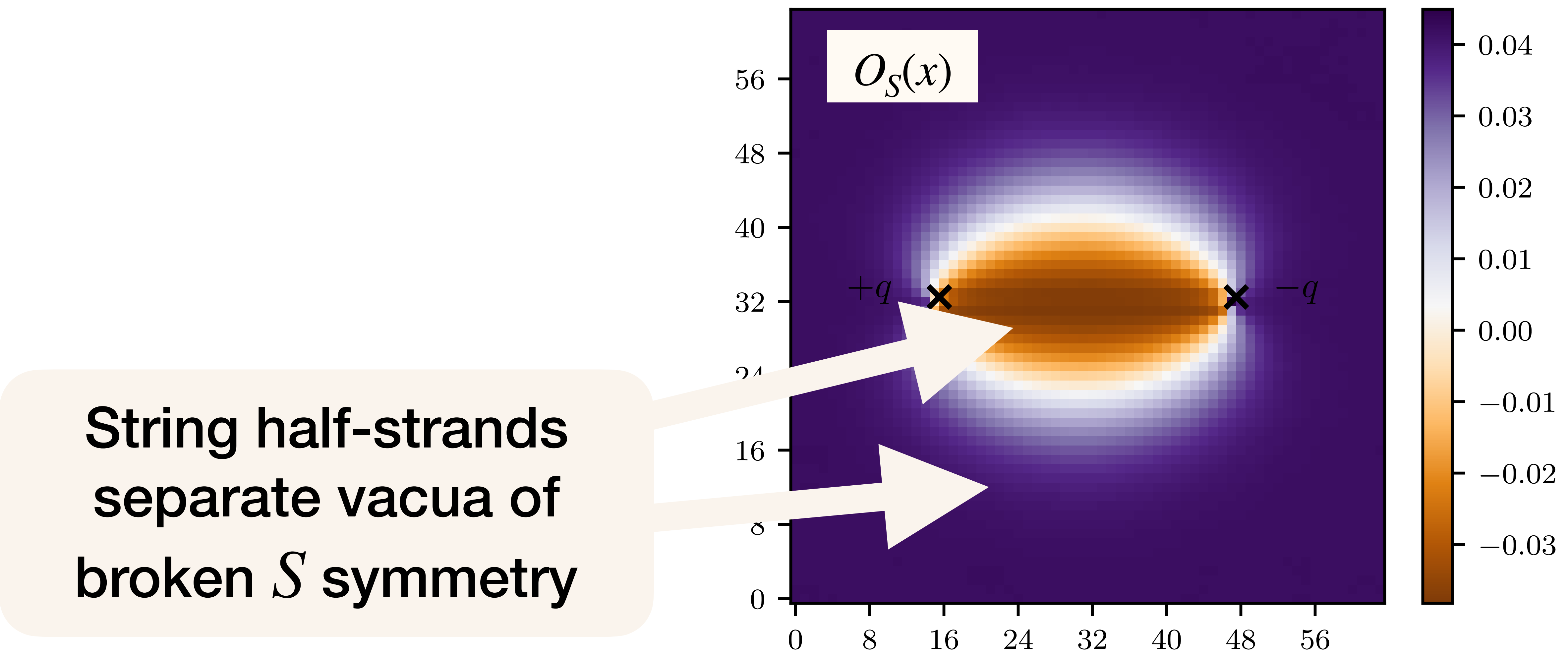
Fractionalized strings



Local energy measure

$$u_\mu(x) = \pm\,(h_x - h_{x+\hat{\mu}} + s_\mu(x))^2$$

- $s_\mu(x) = 1$ where links cross Dirac string connecting charges
- negative sign for timelike, positive for spacelike



Conclusions & Outlook

1. Formulated a "staggered" $U(1)$ gauge theory satisfying all usual symmetries

2. Dual variables $\Longrightarrow$ sign-problem-free lattice theory

3. Verified expected properties

   - Continuum limit as $e^2 \to 0$, confinement, emergence of relativistic $O(3)$ symmetry

New universality class?
Known universality class in
a strange disguise?

4. More exotic results

   - Spontaneous breaking of $\mathbb{Z}_2$ one-site shift symmetry (persisting as $e^2 \to 0$)
   - String fractionalization

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Thanks! Questions?

Backup slides

Staggered theory action

Transfer matrix for electric term Trotter step can be explicitly evaluated:

$$T(U', U) = \left\langle U' \,|\, e^{-\frac{e^2}{2}E^2} \,|\, U \right\rangle = \sum_m \langle U' \,|\, m \rangle \langle m \,|\, U \rangle \, e^{-\frac{e^2}{2}E_m^2}$$

$$= \sum_m e^{-i\varphi_p m} e^{-\frac{e^2}{2}(m+\Theta/2\pi)^2}$$

$$= \sqrt{\frac{2\pi}{e^2}} \exp\left[-\frac{1}{2e^2}\varphi_p^2 + \frac{i\Theta}{2\pi}\varphi_p\right] \vartheta\left(\frac{i}{e^2}\varphi_p + \frac{\Theta}{2\pi}; \frac{2\pi i}{e^2}\right)$$