Analysis on Phases in the Gross-Neveu Model on the Lattice with Shape-based Clustering Method

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CONTENTS

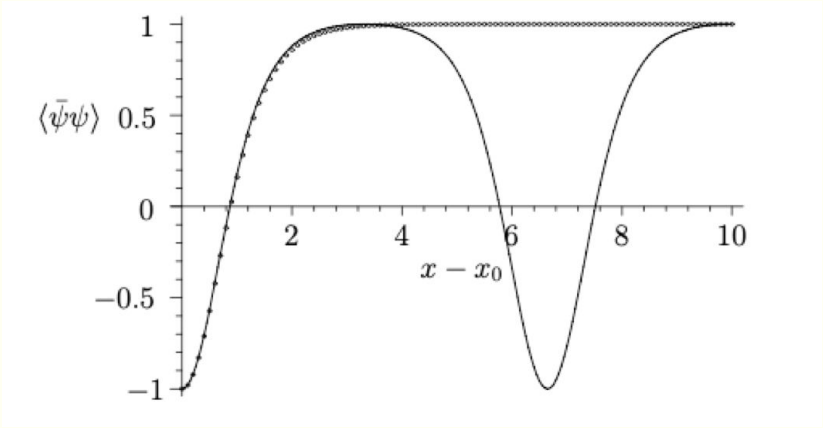
Motivation: QCD Phase Diagram at Finite Density

❖ **Possible interesting phases at high density**

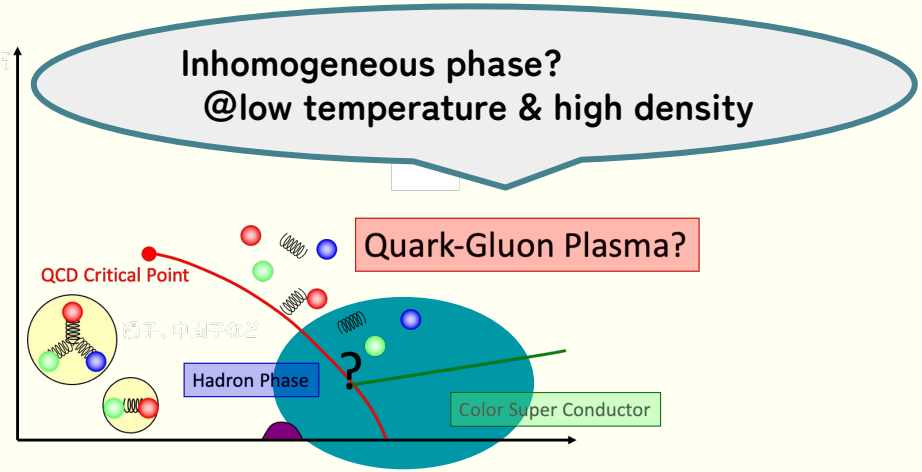
Effective field theory

Inhomogeneous chiral condensation

※spatial dependence of chiral condensate σ(x)



V. Schon and M. Thies, Phys. Rev. D 62, 096002 (2000)



Specific Ansatz @continuous theory

ex) constant, Chiral density wave

A. Heinz et. al., Phys. Rev D 93, no.1 014007 (2016)

Motivation: (1+1)-dimensional Gross-Neveu Model

❖ **Lagrangian density**

$$\mathcal{L} = \bar{\psi} i \gamma^{\nu} \partial_{\nu} \psi + \frac{g^2}{2N} \left(\bar{\psi} \psi \right)^2 \qquad \sigma \sim \langle \bar{\psi} \psi \rangle$$

D. J. Gross and A. Neveu, Phys. Rev. D 10, 3235 (1974)

Important features from comparison with QCD

- ❑ Asymptotic freedom
- ❑ Spontaneous symmetry breaking of discrete chiral symmetry

$$\psi \to \gamma_5 \psi, \quad \bar{\psi} \to -\bar{\psi} \gamma_5$$

- ❑ No sign problem : Monte Carlo simulation
- ❑ Inhomogeneous chiral condensate in large N_f limit

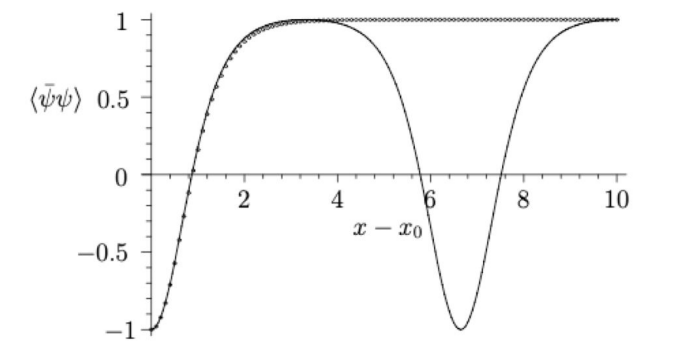
Motivation: (1+1)-dimensional GN Model @ Continuous Theory

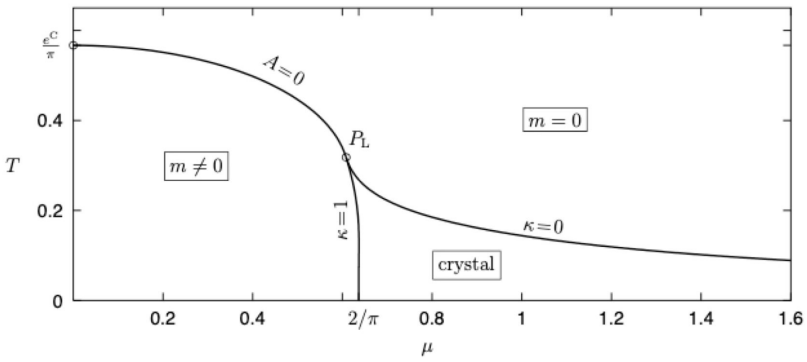
- **Lagrangian density in the continuous theory**

$$\mathcal{L} = \bar{\psi} i \gamma^{\nu} \partial_{\nu} \psi + \frac{g^2}{2N} \left(\bar{\psi} \psi \right)^2 \qquad \sigma \sim \langle \bar{\psi} \psi \rangle$$

in the large N_f limit

- **Specific ansatz & phase diagram**





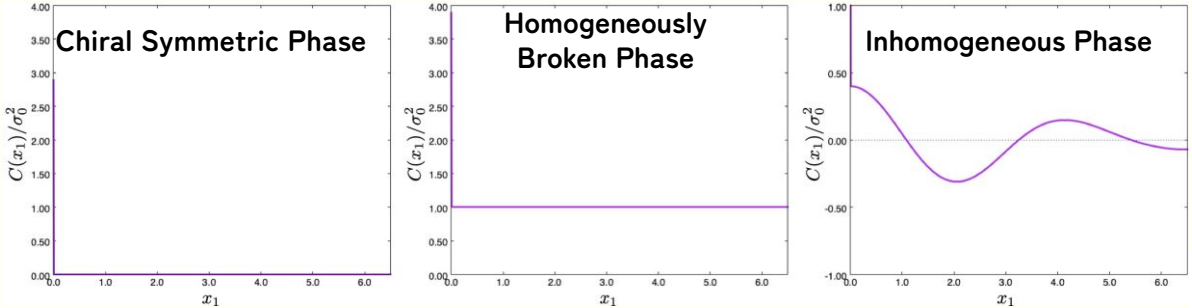
O. Schnetz, M. Thies and K. Urlichs, Annals Phys. 314, 425-447 (2004)

To calculation without ansatz, we use lattice field theory

J. Lenz et. al, Phys. Rev. D 101, no.9, 094512 (2020)

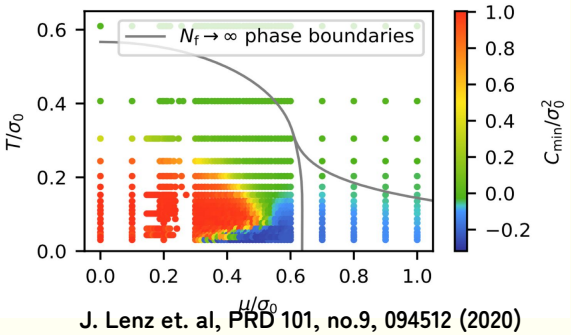
Motivation: (1+1)-dimensional GN Model @ Lattice Theory

- **Spatial correlators in three phases**



- **Classification by minimum of the spatial correlator**

$$C_{\min} := \min_x C(x) \begin{cases} \gg 0, & \text{the Homogeneously Broken Phase} \\ \approx 0, & \text{the Chiral Symmetric Phase} \\ < 0, & \text{the Inhomogeneous Phase} \end{cases}$$



J. Lenz et. al, PRD 101, no.9, 094512 (2020)

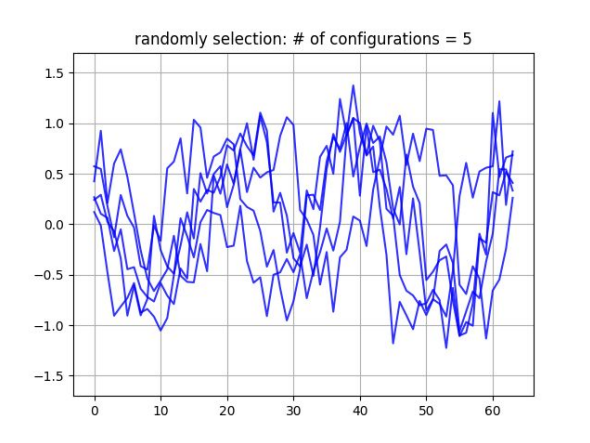
Knowledge of Ansatz is used in the interpretation of the results.

Can we directly extract the spatial dependence of the configurations?

Method: Shape-based Clustering Method

❖ **Why is it difficult directly to extract the spatial dependence?**

The configurations shifted at each Monte Carlo step



When we generate enough number of configurations,
the expectation value at each point becomes zero.

⇨ spatial correlators

Is there a more general and direct method?

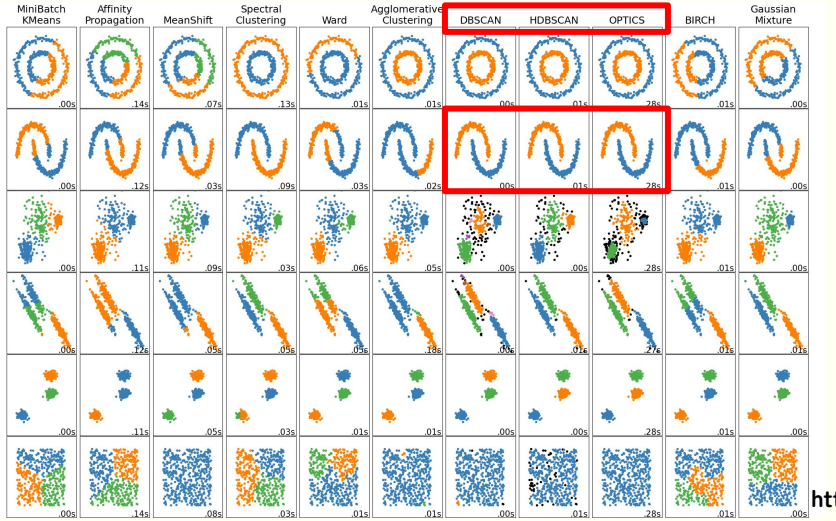
We need shift-invariant clustering method that focuses on the shape of configurations.

Time Series Clustering Method

Method: Shape-based Clustering Method

❖ Clustering Method

- ❑ **Unsupervised Learning**
  A method for grouping data without labeled training data
- ❑ It is important to choose a **"similarity"** that represents how similar two data points are.



The choice of similarity can lead to differences in the data that can be grouped.

Example:
Moon-shaped data are well-suited for density-based similarity calculations such as DBSCAN.

https://scikit-learn.org/stable/modules/clustering.html

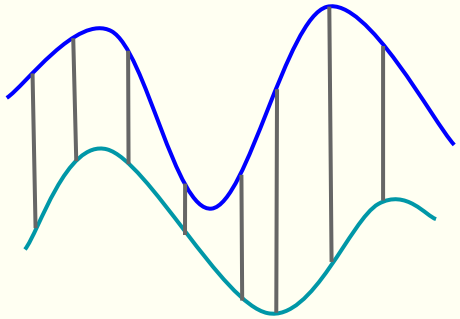
Method: Shape-based Clustering Method

❖ Time Series Clustering Method

- ❑ One of the Clustering methods
- ❑ Clustering methods for time series (1-dimensional data):
- ❑ Major methods:

  Dynamic Time Wraping, K-Shape, k-means, etc...

❖ Simple Example: k-means method with Euclidean distance



"Similarity":

$$d_E(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{(\boldsymbol{x} - \boldsymbol{y})^2}$$

Note：
k-means method is not well suited for time series data because the similarity decreases when the phase is shifted.

Method: Shape-based Clustering Method

❖ K-Shape Method

This method is characterized by shift-invariance and scale-invariance

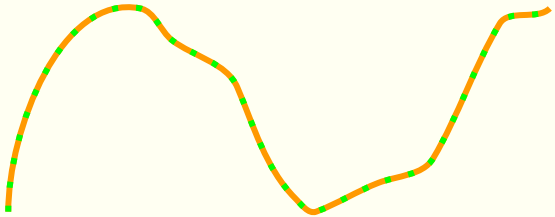
❖ Shape-based Distance (SBD)：the similarity with shift-invariance

$$SBD(\sigma_1(\boldsymbol{x}), \sigma_2(\boldsymbol{x})) = 1 - \max_w \left(\frac{CCF_w(\sigma_1(\boldsymbol{x}), \sigma_2(\boldsymbol{x}))}{\sqrt{ACF(\sigma_1(\boldsymbol{x}))ACF(\sigma_2(\boldsymbol{x}))}} \right)$$

CCF：cross-correlation function
ACF：auto-correlation function

J. Paparrizos, L. Gravano, PROC. ACM SIGMOD Int. Conf. Manage. Data, pp. 1855-1870, 2015

SBD evaluate similarity ignoring the phase shift.



Method: Shape-based Clustering Method

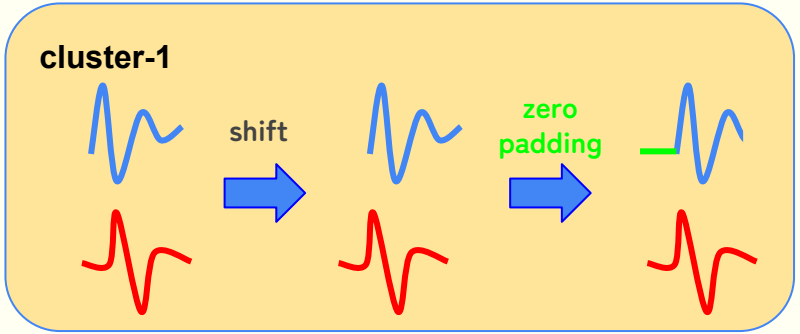
❖ Algorithm – Refinement step

INPUT
X **is an** ***n*****-by-*****m*** **matrix containing** ***n*** **time series of length** ***m*** **that are initially z-normalized.**
k **is the number of clusters to produce.**

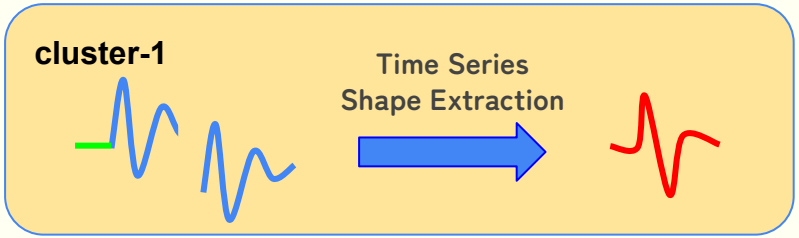
while cluster labels don't change or ***iter*** **< max**
|
| Refinement step
|
| Assignment step
L

OUTPUT
IDX **is as** ***n*****-by-1 vector containing the assignment of** ***n*** **time series to** ***k*** **clusters (initialized randomly).**
C **is a** ***k*****-by-*****m*** **matrix containing** ***k*** **centroids of length** ***m*** **(initialized as vectors with all zeros).**

1. **Shift each data to overlap with the centroid with minimized SBD**



2. **Calculate the optimal centroid for each data in the cluster (Time Series Shape Extraction)**



Method: Shape-based Clustering Method

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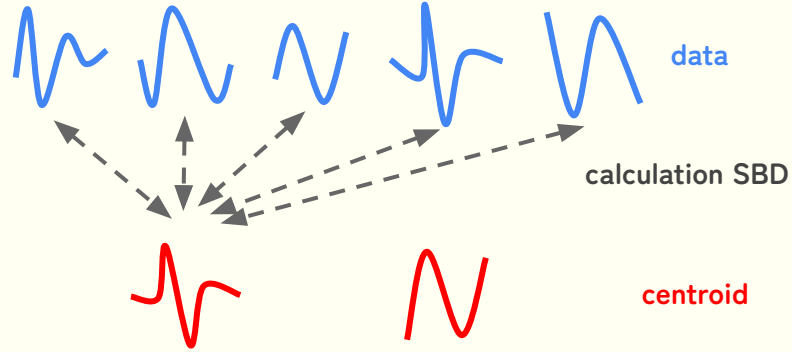
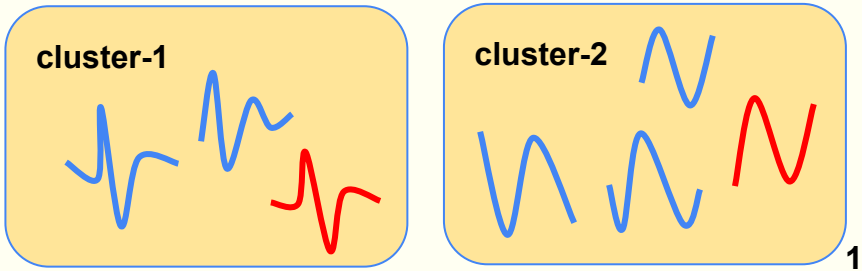
1. Calculate the similarity (SBD) between each data and the centroid



2. Assign the centroid with the minimum similarity to each data



Method: Shape-based Clustering Method

❖ Time Series Shape Extraction

Maximization problem from SBD

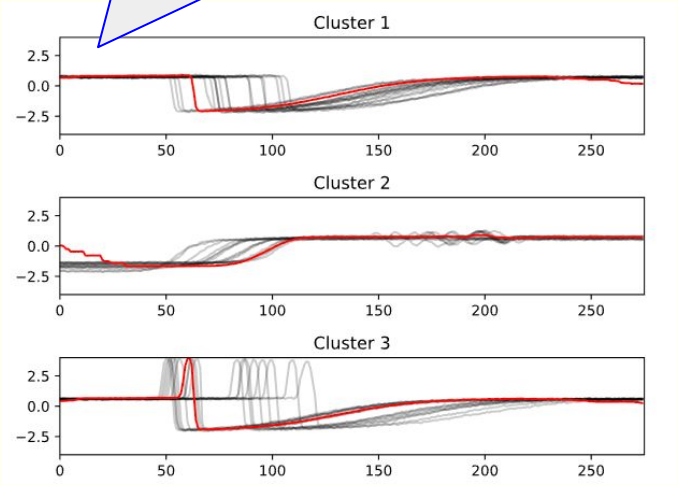
$$\mu_k^* = \underset{\mu_k}{\operatorname{argmax}} \sum_{\sigma_i \in \Sigma_k} \left(\frac{\max_w CCF_w(\sigma_i, \mu_k)}{\sqrt{ACF(\sigma)ACF(\mu_k)}} \right)^2$$

μ_k : old centroid in the cluster-k

μ_k^* : updated centroid in the cluster-k

To solve this problem,
we use Rayleigh Quotient.

Red solid lines are centroids μ_k^*



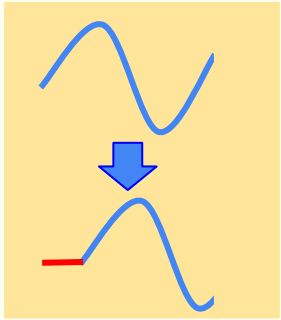
R. Tavenard et. al., JMLR, 21, no.118 (2020) http://jmlr.org/papers/v21/20-091.html

Method: Shape-based Clustering Method

❖ **Modify K-Shape for lattice simulation**

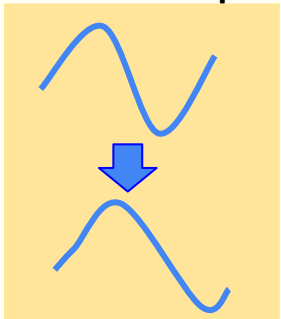
The Original K-Shape：we peform zero padding in the refinement step.

$$\boldsymbol{x} = \begin{cases} \overbrace{0,\ldots,0}^{s}, x_1, x_2, \ldots, x_{m-s}, & s \geq 0 \\ x_{1-s}, \ldots, x_{m-1}, x_m, \underbrace{0,\ldots,0}_{|s|} & s < 0 \end{cases}$$



The Modified K-Shape：we impose periodic boundary condition in the refinement step.

$$\boldsymbol{x} = \begin{cases} x_{m-s+1}, \ldots, x_m, x_1, x_2, \ldots, x_{m-s}, & s \geq 0 \\ x_{1-s}, \ldots, x_{m-1}, x_m, x_{m+1}, \ldots, x_{-s} & s < 0 \end{cases}$$



Results

❖ Lattice Simulation Setup

- ➢ We use a standard hybrid Monte Carlo algorithm.
- ➢ Lattice discretization of fermions is a naive fermion.
- ➢ To set the scale, we use the expectation value at zero temperature and zero chemical potential.
- ➢ We use the same coupling constant as the previous study [J. Lenz et. al, PRD 101, no.9, 094512 (2020)].
- ➢ The other simulation parameters are described in the table below:

fermion	N_f	$N_s = L/a$	$N_t = 1/Ta$	g^2	$a\sigma_0$	μ/σ_0
naive	8	64	14, 24, 64	1.8132	0.42 ± 0.01	0.0, 0.5, 0.6

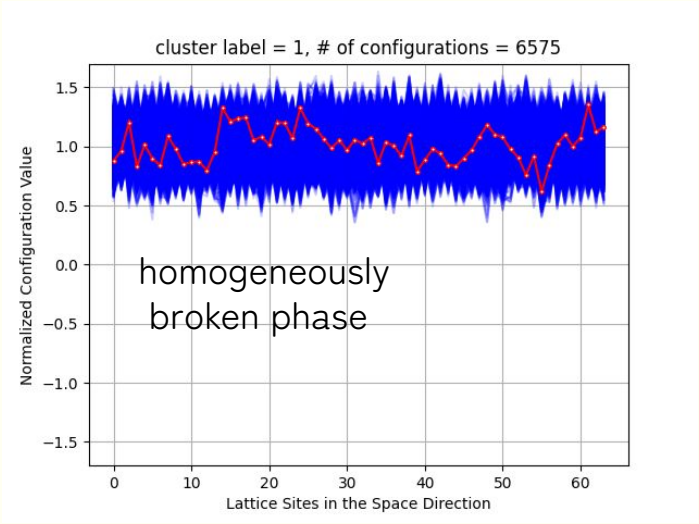
❖ Preprocessing

In the equilibrium, configurations do not have time dependence.

Therefore, we calculate the average of configurations along the time axis.

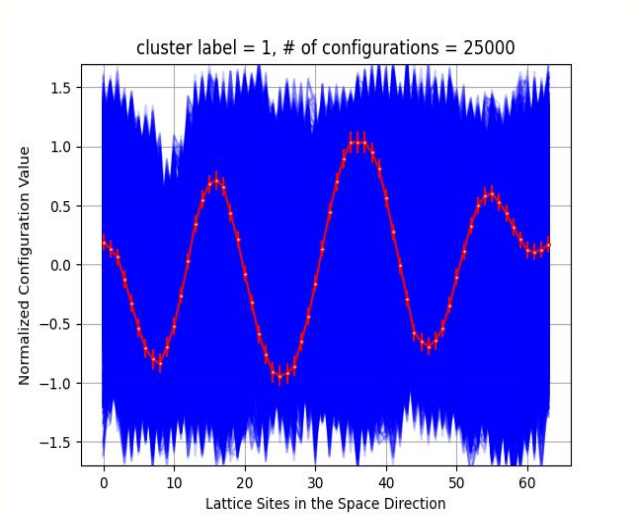
Results: Shape-based Clustering Method

- **Extraction the spatial dependence of the configurations**
  - **Set the cluster number k = 1**



$(T/\sigma_0, \mu/\sigma_0) = (0.037, 0.000)$

inhomogeneous phase



$(T/\sigma_0, \mu/\sigma_0) = (0.098, 0.630)$

We can extract the spatial dependence without the ansatz!

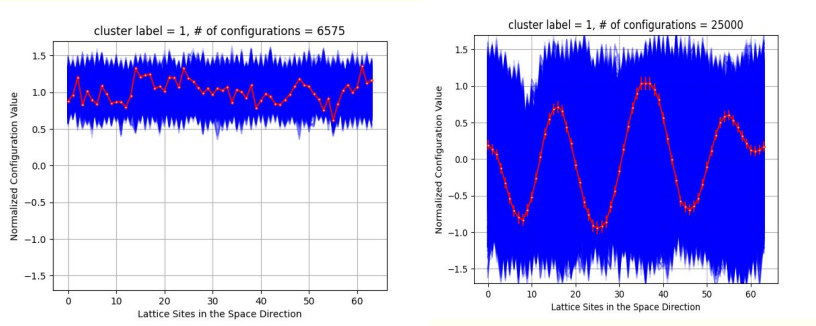
Summary:

Summary

- We applied a shape-based clustering method, a type of unsupervised learning, to the analysis of lattice configurations.
- We modified the method to make it suitable for lattice calculations, including periodic boundary condition.

$$\boldsymbol{x} = \begin{cases} \overbrace{0,\ldots,0}^{s}, x_1, x_2, \ldots, x_{m-s}, & s \geq 0 \\ x_{1-s}, \ldots, x_{m-1}, x_m, \underbrace{0,\ldots,0}_{|s|} & s < 0 \end{cases} \Rightarrow \boldsymbol{x} = \begin{cases} x_{m-s+1}, \ldots, x_m, x_1, x_2, \ldots, x_{m-s}, & s \geq 0 \\ x_{1-s}, \ldots, x_{m-1}, x_m, x_{m+1}, \ldots, x_{-s} & s < 0 \end{cases}$$

- We succeed in extracting the spatial dependence without the knowledge of ansatz.



Future Work

- We will apply this method to the configurations to classify the phases.
- We will use this method to another models with spatial dependent phases.