Use of Inverse Methods for Reconstructing the Hadronic Tensor from Euclidean Correlators

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Introduction

The hadronic tensor $W^{\mu\nu}$ is a field-theoretic quantity that encodes the hadronic response to an external current in inclusive scattering processes. This makes it useful for calculating cross-sections, as the differential cross section goes as

$$d\sigma \propto L_{\mu\nu}W^{\mu\nu},$$

where $L_{\mu\nu}$ is the leptonic tensor, calculable perturbatively in the electroweak theory[1]. The hadronic tensor is defined for a given hadronic state $|H\rangle$ at momentum $p = (p^0, \mathbf{p})$ and external current J^μ via the QCD correlation function:

$$W^{\mu\nu}(p,q) \propto \int d^4x e^{iqx}\langle H,\mathbf{p}|J^\mu(x)J^\nu(0)|H,\mathbf{p}\rangle.$$

For low-energy scattering, it is natural to calculate this quantity non-perturbatively using Lattice QCD. This presents a unique challenge: in order to carry out lattice calculations, it is necessary to perform a Wick rotation from Minkowski to Euclidean spacetime to avoid the numeric instabilities of the sign problem, but after conducting the calculation on the lattice the Wick rotation back to Minkowski spacetime is an ill-posed problem

$$W^{\text{Euc.}}_{\mu\nu}(\mathbf{q}^2,\tau) = \int d\omega e^{-\omega\tau}W_{\mu\nu}(\mathbf{q}^2,\omega).$$

Numerical reconstruction methods such as Bayesian Reconstruction[2][3], the Maximum Entropy Method[3][4], and linear inverse methods like Backus-Gilbert [4][5][6] have been proposed as possible solutions to the inverse problem.

Our Work

Here we present our **preliminary investigation of the electromagnetic hadronic tensor of the pion** ($|H\rangle = |\pi\rangle$, $J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d$) using a modification of Backus-Gilbert developed by Hansen, Lupo, and Tantalo which we will refer to as the **HLT method**[6]. This project is a proof-of-concept and our hope is to refine the reconstruction process to apply to more sophisticated systems, such as neutrino-nucleon scattering.

Correlation functions

Calculation on the lattice requires the correlation functions

$$C^{\text{4-pt}}_{\mu\nu}(\mathbf{q},\tau) = \lim_{t_i\to-\infty}\lim_{t_f\to\infty}\sum_{\mathbf{x}_f}\sum_{\mathbf{x}_1,\mathbf{x}_2}e^{-i\mathbf{q}(\mathbf{x}_2-\mathbf{x}_1)} \langle 0|\chi_\pi(x_f)J_\mu(x_2)J_\nu(x_1)\bar{\chi}_\pi(x_i)|0\rangle,\ \tau = t_2 - t_1, \quad (1)$$

$$C^{\text{2-pt}} = \lim_{t_i\to-\infty}\lim_{t_f\to\infty}\sum_{\mathbf{x}_f}\langle 0|\chi_\pi(x_f)\bar{\chi}_\pi(x_i)|0\rangle, \quad (2)$$

with χ_π being an interpolating operator that couples to the pion. The ratio of (1) and (2) gives us the Euclidean hadronic tensor:

$$W^{\text{Euc.}}_{\mu\nu}(\mathbf{q},\tau) \sim \frac{C^{\text{4-pt}}_{\mu\nu}(\mathbf{q},\tau)}{C^{\text{2-pt}}}. \quad (3)$$

On a finite lattice, we take $t_f \gg t_2$, $t_1 \gg t_i$ to ensure exponentially supressed excited states and that the current couples to the appropriate hadronic states. The distinct topologies of the four-point correlation function can be seen in Fig.1.

Numerical details

- **Ensemble information:** Computations were performed on two MILC ensembles using $2+1+1$ flavors of dynamical quarks with physical masses, one-loop Symanzik-improved gauge action, and HISQ fermion action[7][8]. For the valence quarks we again use the HISQ action. The configurations are as follows:

$L^3\times T$	a (fm)	q	t_{sep}	N_{low}	N_{high}	#conf
$32^3\times 48$	0.15	[0,0,0]	8a	4000	144	30
$48^3\times 64$	0.12	{ [0,0,0], [±1,0,0], [±1,±1,0] [±2,0,0]}	10a	4000	192	47

Table 1. Configuration paramters for the two lattices used. Because we compute on a finite lattice, $t_{sep} = t_f - t_2 = t_1 - t_i$ defines our working asymptotic region where excited states are exponentially supressed and our current couple to the desired hadronic states.

- **All-to-All:** As seen in Fig. 1 the four-point function consists of both connected and disconnected diagrams and three time separations. In order to extract as much information from the hadronic spectrum as we can, we use the all-to-all method for quark propagators, as implemented in `Grid` and its management workflow system `Hadrons`[9][10][11].

$$D^{-1}_{A2A}(x,y) = \sum_{l=1}^{N_{\text{low}}} v_l(x)w_l^\dagger(y) + \sum_{h=N_{\text{low}}+1}^{N_{\text{total}}} v_h(x)w_h^\dagger(y), \quad (4)$$

$$v_l(x) = \phi_l(x),\ w_l(x) = \phi_l(x)/\lambda_l.$$

From these low- and high-modes we can construct spatially summed meson fields

$$\Pi_{ij}(t_x;\Gamma) \equiv \sum_{\mathbf{x}} w_i^\dagger\Gamma v_j(x), \quad (5)$$

such that correlation functions are matrix multiplications of the appropriate meson fields. For example, the pseudoscalar two-point function is computed as

$$C^{\text{2-pt}}(t_f - t_i) = \sum_{jk}\Pi_{jk}(t_f;\gamma_5\otimes\gamma_5)\Pi_{kj}(t_f;\gamma_5\otimes\gamma_5).$$

We have developed a **contraction code in Python** to maximize efficiency on Fermilab LQ cluster.

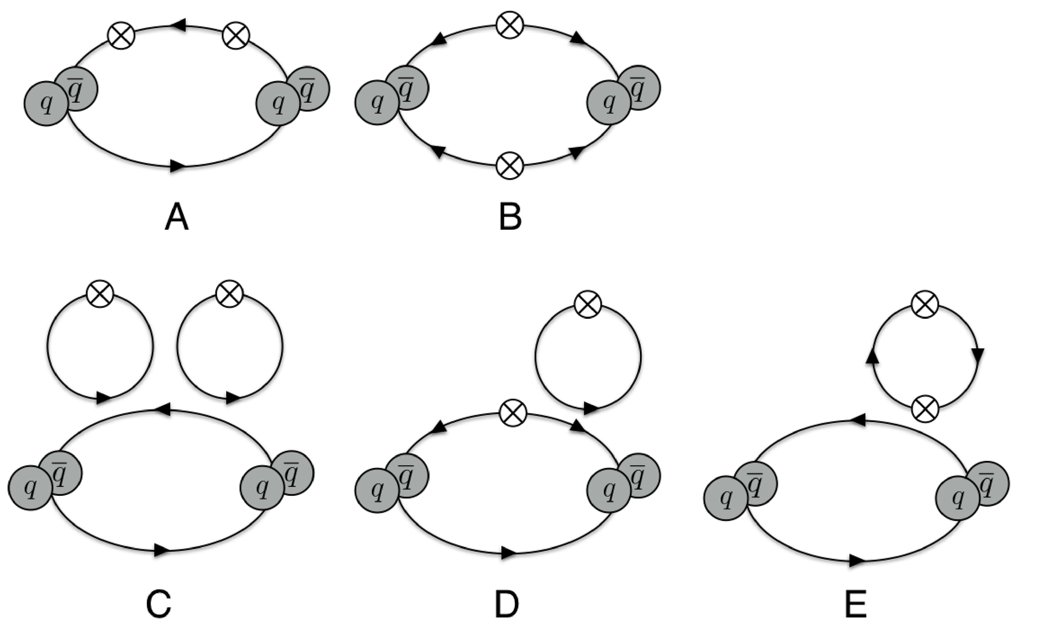

Figure 1. Topologically distinct quark-flow diagrams contributing to the hadron tensor. The ⊗ represents an insertion of the electromagnetic current, and the shaded circles the pion interpolating fields.

The HLT Method

The HLT method is a modification of the Backus-Gilbert method and as such works much the same way, producing coefficients $\mathbf{g} = (g_0, ..., g_t, ...g_{t_{\max}})$ to a regularized smearing function constructed from a set of basis functions $b(\omega, t)$ that approaches a delta function in the limit of zero smearing width. Hansen, Lupo, and Tantalo's modification consists of **altering the minimization functional in such a way to include a smearing function** $\Delta_\sigma(E_*, E)$ **as an input** so that the minimization occurs on the L_2-norm of the input and output smearing functions

$$A[g] = \int_{E_0}^{\infty} dE e^{\alpha E}\left|\sum_{t=1}^{t_{\max}} g_t(E_*)b(E, t+1) - \Delta_\sigma(E_*, E)\right|^2. \quad (6)$$

In reality, however, since our correlator data contains uncertainties, **the actual minimization occurs on a weighted sum of the** $A[g]$ **functional and a covariance term**, with the weighting parameter chosen in such a way that balances the systematic errors and the statistical errors

$$W[\lambda, g] = (1-\lambda)A[g] + \lambda\frac{B[g]}{C(0)^2}, \quad B[g] = \mathbf{g}^T\ \text{Cov}\ \mathbf{g}. \quad (7)$$

Our implementation of the HLT method was done in C++ in order to take advantage of the Eigen library for the extended-precision matrix computations necessary to generate the $\mathbf{g}$ coefficients. We use a Gaussian as the input smearing function. As a check of our implementation, we have replicated the three-peak "toy model" as well as the two-particle term of their exact "benchmark model" below.

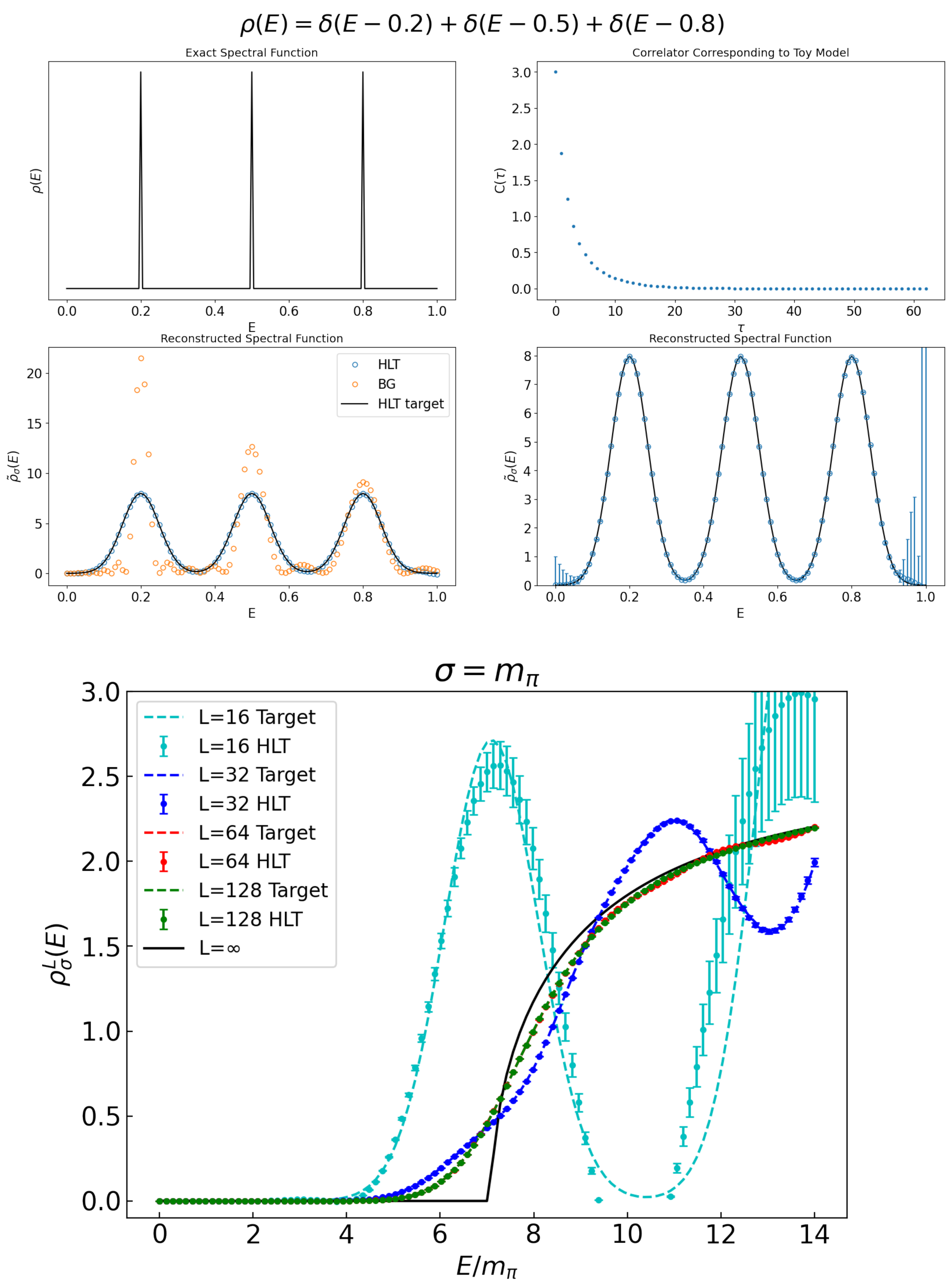

Figure 2. Top: the toy model of three delta-peaks. Bottom: Benchmark system corresponding to [4]. Errors are systematic only.

Preliminary Findings

- Application of the HLT method to the two-point pseudoscalar correlation function is performed as a sanity check. In the reconstruction the higher-statistics lattice shows a strong peak at m_π with a possible feature at the $\pi(1300)$ state.
- A preliminary spectral analysis of the vector two-point functions has been carried out using standard multi-exponential fits. Present work is underway to obtain stable spectral reconstructions using the HLT method. An interesting technical complication arises due to the use of staggered fermions: the even and odds sites correspond to different spectral densities. The connected four-point functions are shown on the bottom in Fig. 4. The region relevant for the spectral reconstruction on the hadronic tensor is on the left-hand side, $a\tau = [0, 44)$.

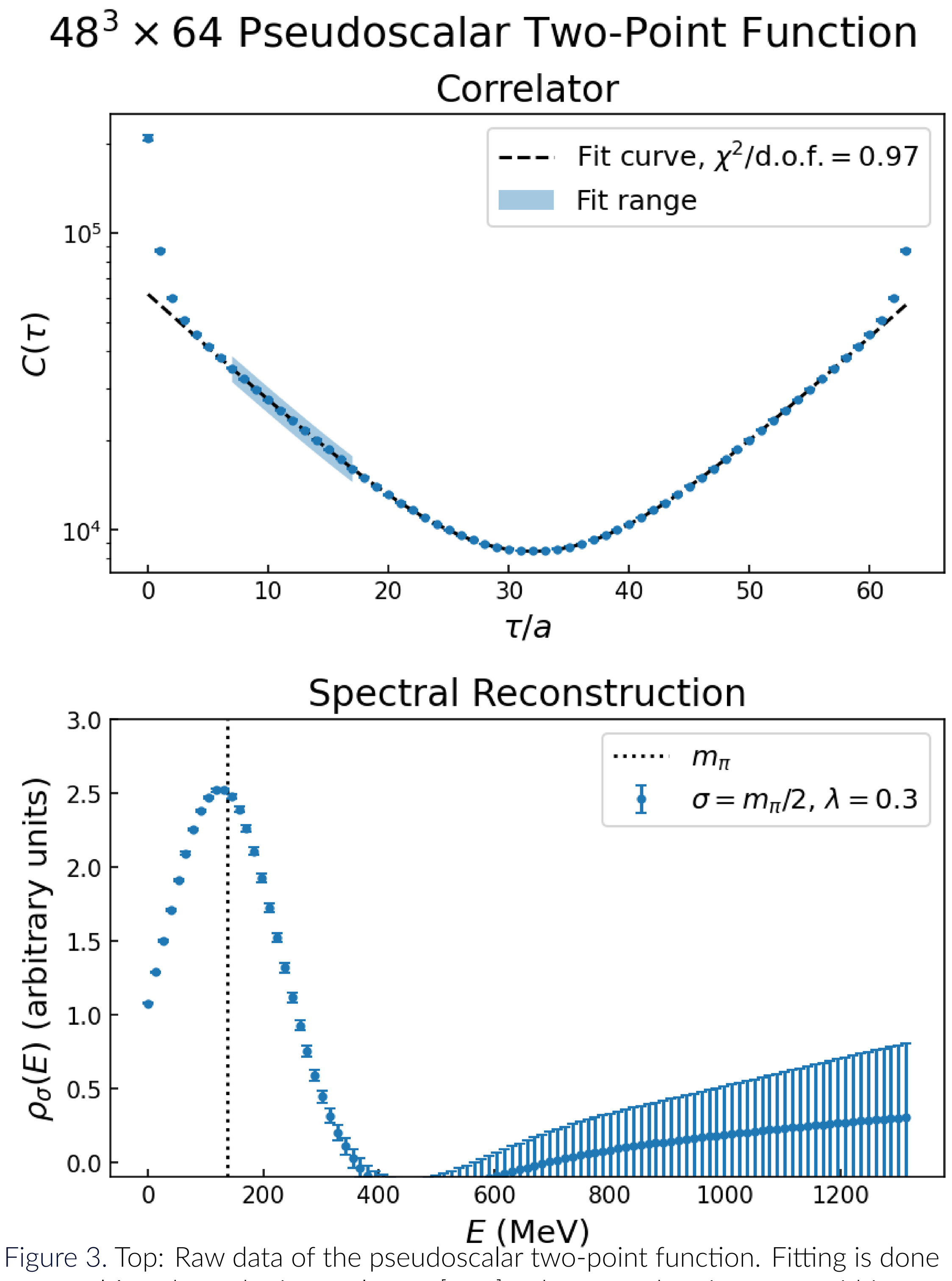

Figure 3. Top: Raw data of the pseudoscalar two-point function. Fitting is done on a multi-cosh on the interval $a\tau = [7, 18]$and returns the pion mass within errors. Bottom: HLT reconstruction using $C^{\text{2-pt}}(\tau)$ as input; $a\tau = [0, 20]$ $\sigma = m_\pi/2, \lambda = 0.3$; errors follow the convention of [6]

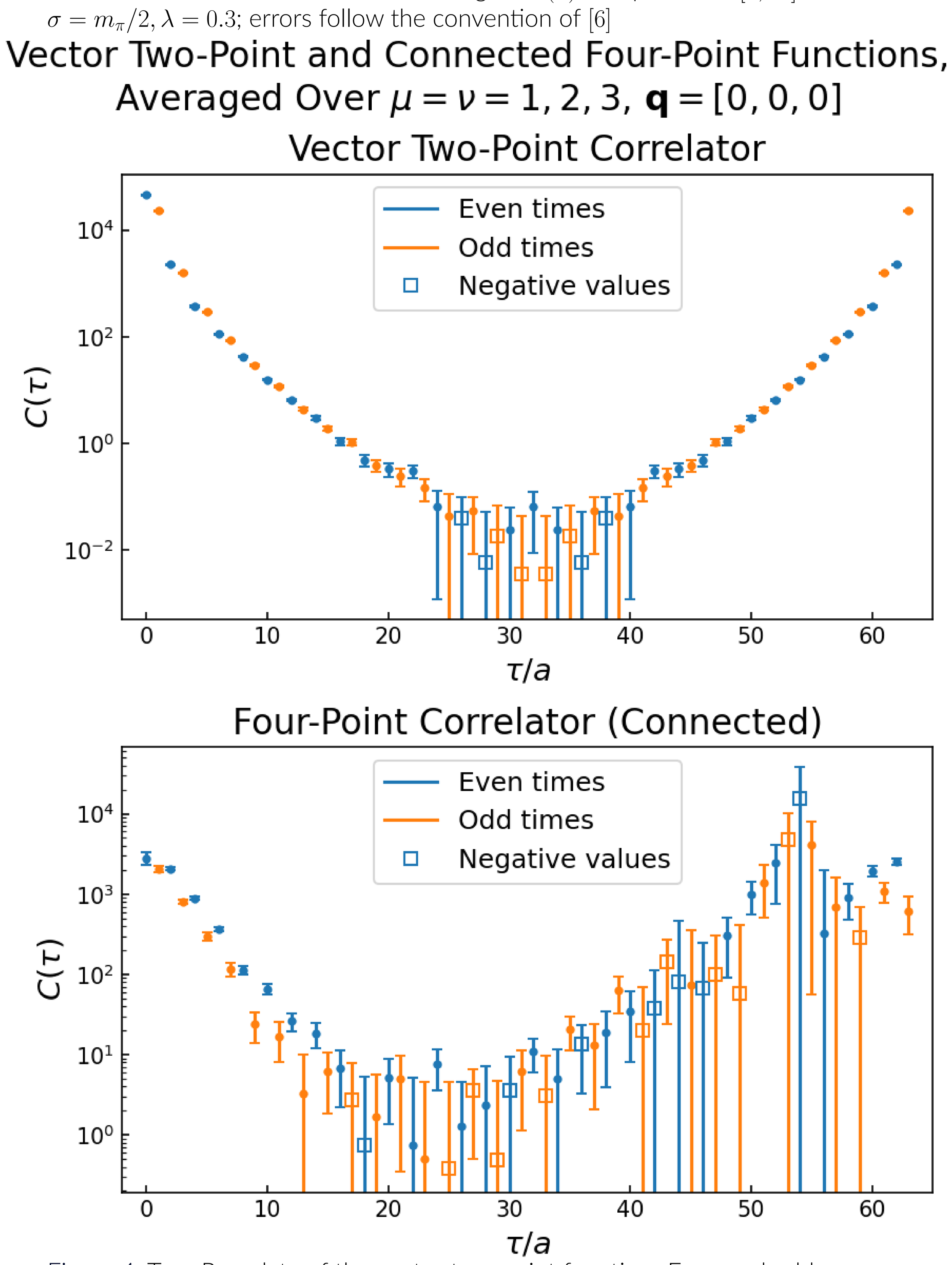

Figure 4. Top: Raw data of the vector two-point function. Even and odd timeslices are differentiated to highlight the result of using staggered fermions. Bottom: The connected diagrams (A and B) of Fig.1. For both an improvement in statistics is needed

Future Work

- We are investigating **an alternate method using conformal mappings and Nevanlinna interpolation**. For more information see [12] and Dr. Jay's Parallel Talk.
- We have started runs to **improve statistics** on our $48^3\times 64$, $a = 0.12$ fm dataset, where we aim to increase the number of configurations from 47 to 175.
- With sufficient compute time we will proceed to a $48^3\times 64$, $a = 0.15$ fm lattice in order to **study the infinite-volume limit**.
- After further analysis, we intend to begin a treatment of the **hadronic tensor of the nucleon**.
- With the hadronic tensor of the nucleon, we would **calculate neutrino-nucleon cross sections**.

References

[1] M. Peskin and D. Schroeder. *An Introduction to Quantum Field Theory*. Westview
[2] Y. Burnier and A Rothkopf. "A novel Bayesian approach to spectral function reconstruction". https://arxiv.org/pdf/1307.6106.pdf, 2013.
[3] J. Liang, K. Liu, Y. Yang. "Towards the nucleon hadronic tensor from lattice QCD". https://www.epj-conferences.org/articles/epjconf/pdf/2018/10/epjconf_lattice2018_14014.pdf, 2018.
[4] M. Hansen, H. Meyer, and D. Robaina. "From deep inelastic scattering to heavy0flavor semi-leptonic decays: Total rates into multi-hadron final states from lattice QCD". https://arxiv.org/pdf/1704.08993.pdf, 2017.
[5] J. Liang, et al. "Lattice calculation of hadronic tensor of the nucleon". https://arxiv.org/pdf/1906.05312.pdf, 2019.
[6] M. Hansen, A. Lupo, and N. Tantalo. "On the extraction of spectral densities from lattice correlators". https://arxiv.org/pdf/1903.06476.pdf, 2019.
[7] A. Bazavov, et al. "Scaling studies of QCD with the dynamical HISQ action". https://arxiv.org/pdf/1004.0342.pdf, 2010.
[8] A. Bazavov, et al. "Lattice QCD ensembles with four flavors of highly improved staggered quarks". https://arxiv.org/pdf/1212.4768.pdf, 2010.
[9] P. Boyle, et al. "Grid: A next generation data parallel C++ QCD library". https://arxiv.org/pdf/1512.03487.pdf, 2013.
[10] A. Portelli, et al. "Hadrons: Hadrons v1.4". https://doi.org/10.5281/zenodo.8023716, 2013.
[11] J. Foley, et al. "Practical all-to-all propagators for lattice QCD". https://arxiv.org/pdf/hep-lat/0505023.pdf, 2005.
[12] T. Bergamaschi, W. Jay, and P. Oare. "Hadronic Structure, Conformal Maps, and Analytic Continuation". https://arxiv.org/pdf/2305.16190.pdf, 2023.

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