Investigation of the HLbL contribution to a_μ using staggered fermions

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for the BMW collaboration

Centre de Physique Théorique Marseille

40th International Symposium on Lattice Field Theory

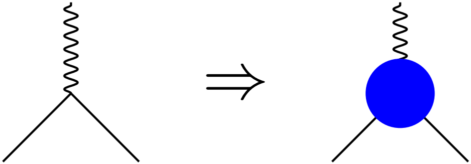
August 1, 2023

Content

Content

The anomalous magnetic muon moment g_μ

- For Dirac fermions $g = 2$ at the tree-level
- Deviation by quantum corrections of the fermion photon vertex:



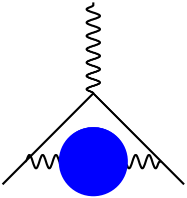
- Corrections quantified by $a = (g-2)/2$
- The magnetic moment of the muon can be determined precisely in the experiment as well as in theory
- Sensitive to new physics
- **Recent experimental values:**
  $a_\mu = 116\;592\;080(54)(33) \times 10^{-11}$ (BNL, 2006)
  [Phys.Rev.D 73 (2006) 072003]
  $a_\mu = 116\;592\;040(54) \times 10^{-11}$ (Fermilab, 2021)
  [Phys.Rev.Lett. 126 (2021) 14, 141801]
- More precise measurements are expected in the near future (error reduction by factor 4) $\Rightarrow$ error on the theory side needs to be reduced as well

Theoretical determination of a_μ

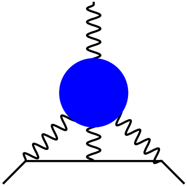
Standard Model contributions and current state results (see "white paper" *[arXiv:2006.04822]*):

contrib	$a_\mu \times 10^{11}$
QED	116 584 718.931(104)
Electroweak	153.6(1.0)
LO-HVP (phenom)	6845(40)
LO-HVP (BMWc'20)	7075(55)
HLbL (phenom & latt)	92(18)
total SM	116 591 810(43)

Have to consider two types of hadronic contributions:



LO-HVP
$\mathcal{O}(\alpha^2)$



HLbL
$\mathcal{O}(\alpha^3)$

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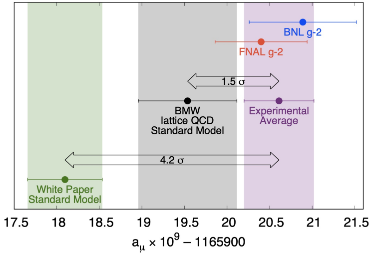
Tension between experiment and theory:

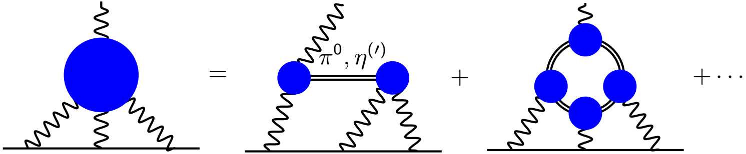


[BMWc'20]

Determination of hadronic contributions

HLbL: Two approaches:

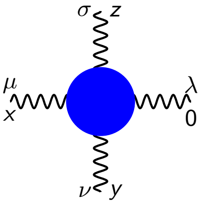
- **Dispersive Method:**



  - Data driven approach, input by lattice calculations (→ talk by Willem)

- **Evaluate 4pt function on the lattice (this work)**

$\widetilde{\Pi}_{\mu\nu\lambda\sigma}(x, y, z) =$

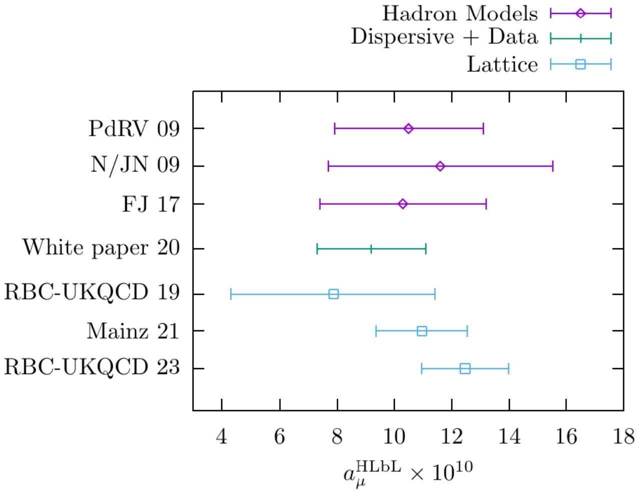


$= \langle j_\mu(x) j_\nu(y) j_\lambda(0) j_\sigma(z) \rangle$

  - Need only precision of 10% to match the future experimental precision
  - BUT: 4pt functions are in general challenging on the lattice

Determination of hadronic contributions

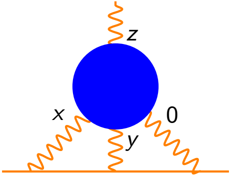
HLbL: Recent results:



[RBC-UKQCD '23]

Content

HLbL contributions to a_μ with staggered fermions



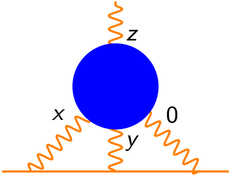
HLbL contributions to a_μ with staggered fermions

Master formula for hadronic light-by-light contribution to a_μ *[Mainz'21]*:

$$a_\mu^{\mathrm{HLbL}} = \frac{m_\mu e^6}{3} \int_{x,y,z} \mathcal{L}_{[\rho\sigma]\mu\nu\lambda}(x,y)\ (-z_\rho)\ \widetilde{\Pi}_{\mu\nu\sigma\lambda}(x,y,z)$$

- $\widetilde{\Pi}_{\mu\nu\sigma\lambda}(x,y,z) := \langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0)\rangle$: with vector currents $j_\mu(x)$
- $\mathcal{L}_{[\rho\sigma]\mu\nu\lambda}(x,y)$: QED-kernel (not unique) *[Mainz'20]*
- 12-dimensional integral, 8 can be evaluated directly in the simulation
- Remaining object is rotational invariant $\Rightarrow$ Integrate over $|y|$

$$a_\mu^{\mathrm{HLbL}} = \frac{m_\mu e^6}{3} 2\pi^2 \int_{|y|} |y|^3 f(y)$$



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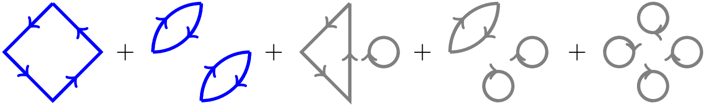
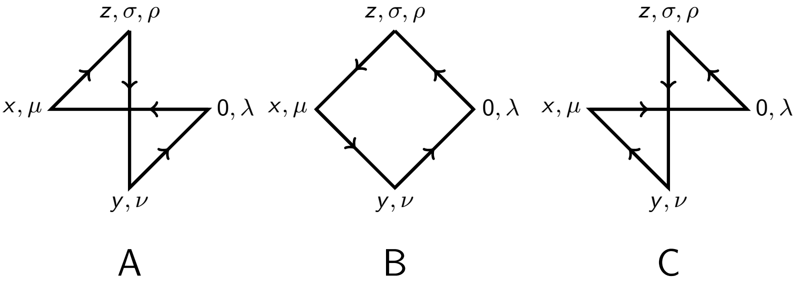
Decomposition of the HLbL tensor $\widetilde{\Pi}$ in terms of Wick contractions:

$\widetilde{\Pi} =$



Connected graphs



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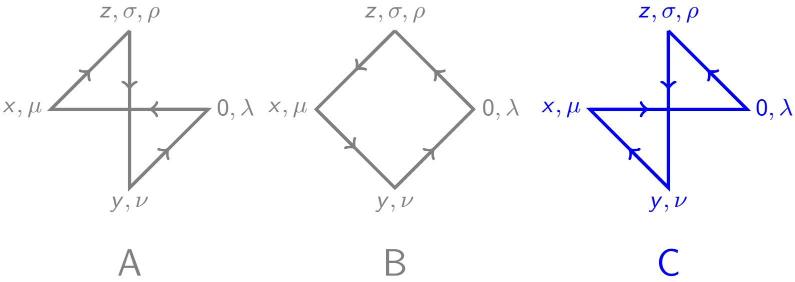

$$a_\mu^{\mathrm{HLbL,c}} = \frac{m_\mu e^6}{3} \sum_{x,y,z} \mathcal{L}^{(\mathrm{sym})}_{\rho\sigma\mu\nu\lambda}(x,y)\,(x_\rho - 3z_\rho)\,\mathrm{Re}\left\{\widetilde{\Pi}^{(\mathrm{C})}_{\mu\nu\sigma\lambda}(x,y,z)\right\}$$

- Re-express contractions A and B in terms of C by exploiting translational invariance
- Fully symmetric kernel ($x \leftrightarrow y$, $y \leftrightarrow x - y$)

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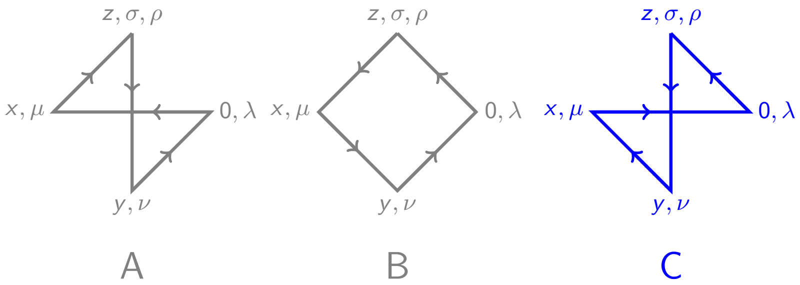

$$a_\mu^{\mathrm{HLbL,c}} = \frac{m_\mu e^6}{3} 2\pi^2 \sum_{|y|} |y|^3 \underbrace{\sum_{x,z} \mathcal{L}^{(\mathrm{sym})}_{\rho\sigma\mu\nu\lambda}(x,y)\,(x_\rho - 3z_\rho)\,\mathrm{Re}\left\{\widetilde{\Pi}^{(\mathrm{C})}_{\mu\nu\sigma\lambda}(x,y,z)\right\}}_{=f(|y|)}$$

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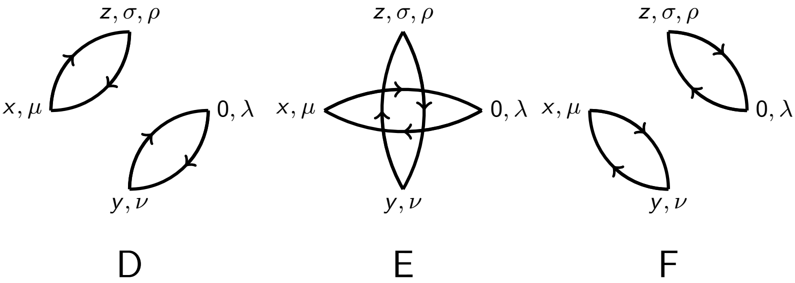
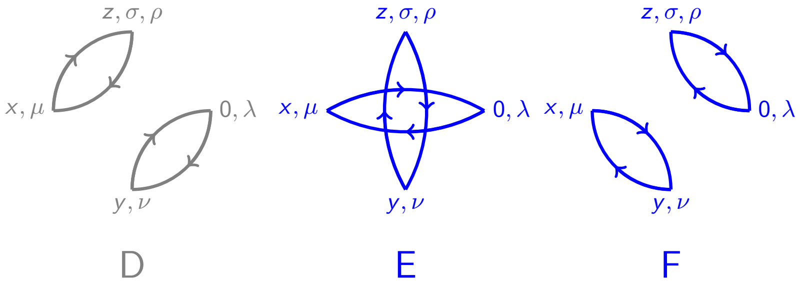


Disconnected graphs



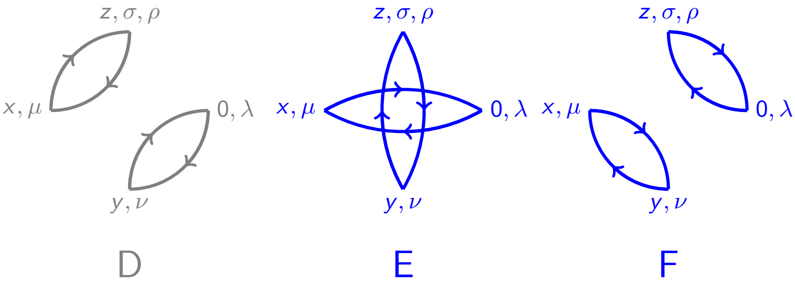
Two versions:

$$a_\mu^{\mathrm{HLbL,d1}} = \frac{m_\mu e^6}{3} 2\pi^2 \sum_{|y|} |y|^3 \sum_{x,z} \mathcal{L}^{(\mathrm{sym})}_{\rho\sigma\mu\nu\lambda}(x,y)\,(y_\rho - 3z_\rho)\,\widetilde{\Pi}^{(\mathrm{E})}_{\mu\nu\sigma\lambda}(x,y,z)$$

$$a_\mu^{\mathrm{HLbL,d2}} = \frac{m_\mu e^6}{3} 2\pi^2 \sum_{|y|} |y|^3 \sum_{x,z} \mathcal{L}^{(\mathrm{sym})}_{\rho\sigma\mu\nu\lambda}(x,y)\,(x_\rho + y_\rho - 3z_\rho)\,\widetilde{\Pi}^{(\mathrm{F})}_{\mu\nu\sigma\lambda}(x,y,z)$$

Same kernel as for connected contributions $\Rightarrow$ re-use

Disconnected graphs



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Take average as final result:

$$a_\mu^{\mathrm{HLbL,d}} = \frac{1}{2}\left(a_\mu^{\mathrm{HLbL,d1}} + a_\mu^{\mathrm{HLbL,d2}}\right)$$

Dealing with staggered tastes

$$a_\mu^{\mathrm{HLbL}} \propto \sum_{|y|} I(|y|)$$

- Have to sample $I(|y|)$ for a given set of points $|y|$
- In contrast to other discretizations (e.g. Wilson fermions), we have contributions for up to 16 tastes for a given y.
- Have to project onto the desired contribution

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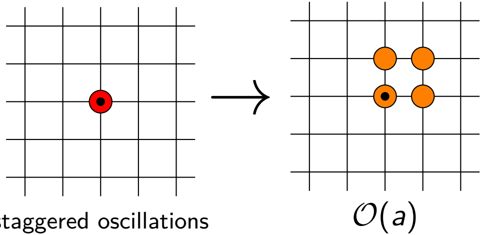
Project by applying "smearing":

$$I(y) \to I^{(1)}(y) = \prod_\mu \mathcal{S}_\mu^{(1)} I(y) \qquad \mathcal{S}_\mu^{(1)} f(y) := \frac{f(y) + f(y+\hat{\mu})}{2}$$



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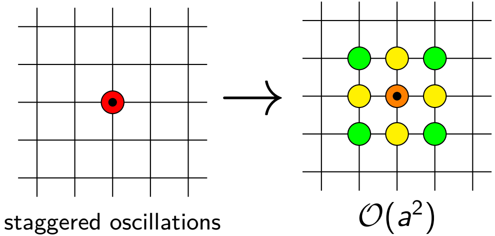
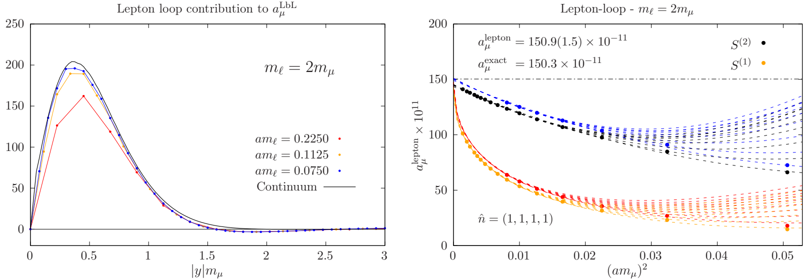
Project by applying "smearing":

$$I(y) \to I^{(2)}(y) = \prod_\mu \mathcal{S}_\mu^{(2)} I(y) \qquad \mathcal{S}_\mu^{(2)} f(y) := \frac{f(y-\hat{\mu}) + 2f(y) + f(y+\hat{\mu})}{4}$$



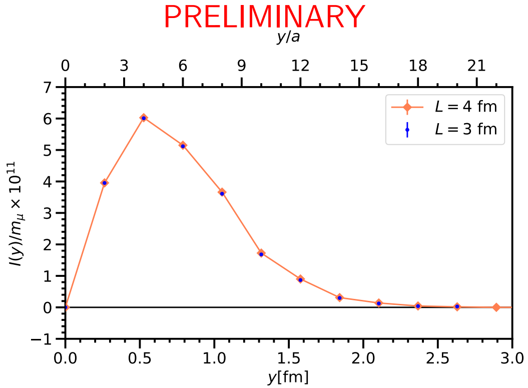
Test: Lepton loop on the lattice



- No staggered oscillations visible at the integrand level
- Ansatz: $a_\mu(a) = c_0 + c_1 a + c_2 a^2 + c_4 a^4$
- $a_\mu^{\rm lepton} = 150.9(1.5) \times 10^{-11}$ (lattice) vs $a_\mu^{\rm lepton} = 150.3 \times 10^{-11}$ (exact)
- **Values consistent** $\Rightarrow$ **First test successful**

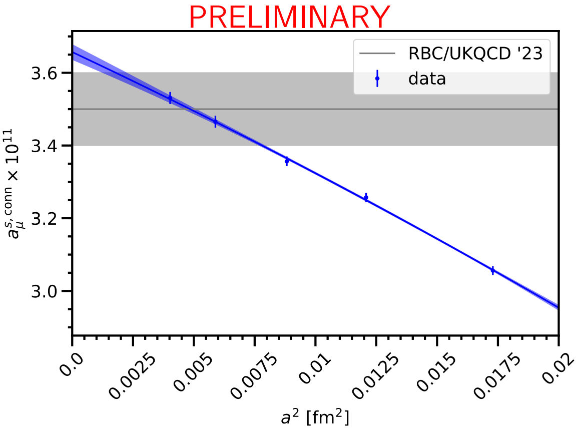
Content

Results: Strange, connected, $a = 0.13$ fm



- Integrated values:
  $L = 3$ fm: $a_\mu^{s,\mathrm{conn}} = 3.056(3)_{\mathrm{stat}}(12)_a$,
  $L = 4$ fm: $a_\mu^{s,\mathrm{conn}} = 3.087(2)_{\mathrm{stat}}(12)_a$
- Statistical error small, uncertainty by the lattice spacing dominates
- Finite volume effects very small

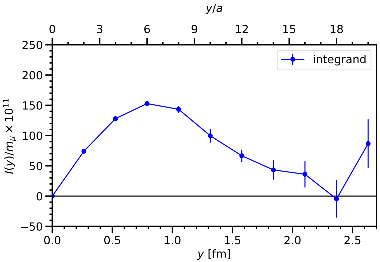
Results: Strange, connected, continuum extrapolation



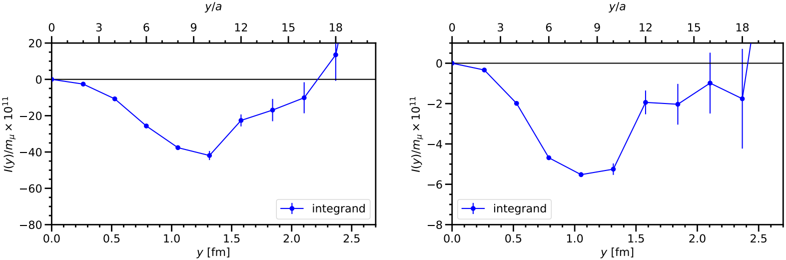
- $\mathcal{O}(a^2)$ discretization effects: Use fit ansatz: $c_0 + c_1 a^2 + c_2 a^4$
- Finite size effects are not yet included
- Continuum value for $a_\mu^{s,\mathrm{conn}} \times 10^{11}$: $3.657(20)_{\mathrm{stat}}(14)_a[\mathrm{sys}]$
- Compatible with RBC-UKQCD value within 1.5σ

First results for physical pion mass: Light, $a = 0.13$ fm

Connected(PRELIMINARY):

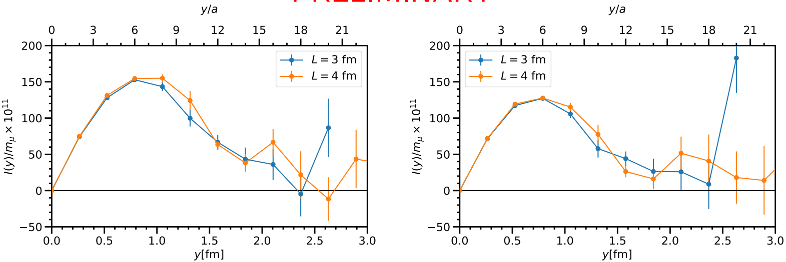


Disconnected (light-light, light-strange)(PRELIMINARY):



Results: Light, finite volume effects

PRELIMINARY



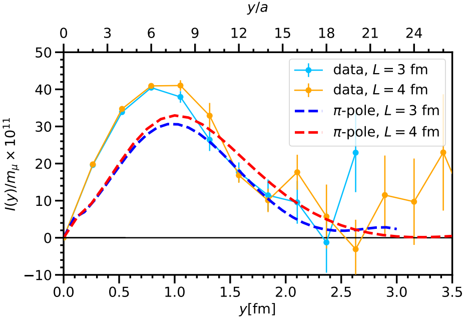
connected

connected + disconnected

- Finite volume effects within statistical error
- Use prediction of the pion-pole to better estimate finite volume effects

Results: Finite volume effect from the π-pole

PRELIMINARY



Evaluate π-pole contribution to the 4pt-function numerically (FFT) using transition form factor (TFF) data *[RBC-UKQCD '23]* (assume LMD parameterization, data from lattice calculation *[→ Talk by Willem]*) :

$$\Pi^{(\mathrm{C})\pi,\mathrm{TFF}}_{\mu\nu\sigma\lambda}(y) = \int_{uv} D_\pi(u-v)\left(M_{\mu\nu}(u,x,y)M_{\sigma\lambda}(v,z,0) + M_{\mu\lambda}(u,x,0)M_{\nu\sigma}(v,y,z)\right)$$

$$M_{\mu\nu}(u,x,y) := i\epsilon_{\mu\nu\rho\sigma}\partial^{(x)}_\rho\partial^{(y)}_\sigma \int_{qk} F(q^2,k^2)\; e^{iq(x-u)}e^{iq(y-u)}$$

Compare with lattice data by calculating $I^{\pi,\mathrm{latt}}(y) = \frac{9}{34} I^{\mathrm{conn},\ell}(y)$

Content

Summary

Achieved:

- We have developed a staggered version of the Mainz method in order to calculate the light-by-light contribution to $g_\mu - 2$ on the lattice
- Reproduce lepon-loop result: Successful cross-check
- Prelimary continuum extrapolation for the connected strange contribution: Compatible with RBC-UKQCD
- First results for light quarks, $a = 0.13$ fm: Good data quality
- Finite size correction: long range π^0 contribution obtained from lattice calculations of transition form factors

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- Charm contributions (work in progress)

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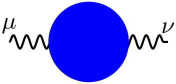
Thanks for your attention!

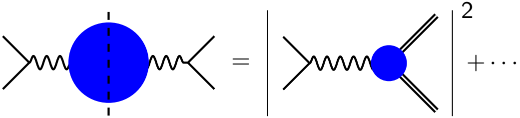
HVP

HVP: Involves hadronic vacuum polarization tensor:

$\Pi^{\mu\nu}(Q) =$ 

$\propto \int \mathrm{d}x^4 e^{-iQx} \langle j^\mu(x) j^\nu(0) \rangle$

- **Dispersive Method (Analyticity + optical theorem):**



R-ratio:

$$a_\mu = \frac{\alpha^2}{3\pi^2} \int_{m_\pi^2}^{\infty} \mathrm{d}s \frac{K(s)}{s} R(s) \qquad R(s) = \frac{3s\ \sigma^0(e^+e^- \to \mathrm{hadrons})}{4\pi\alpha^2}$$

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  - Need sub-per-mille-precision
  - Lattice artifacts (discretization, finite-volume, taste-braking,...) have to be under control

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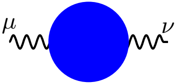
$\Pi^{\mu\nu}(Q) =$  $\propto \int \mathrm{d}x^4 e^{-iQx} \langle j^\mu(x) j^\nu(0) \rangle$

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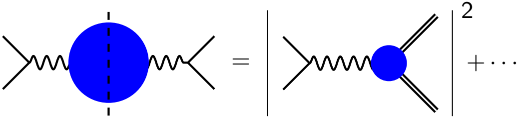

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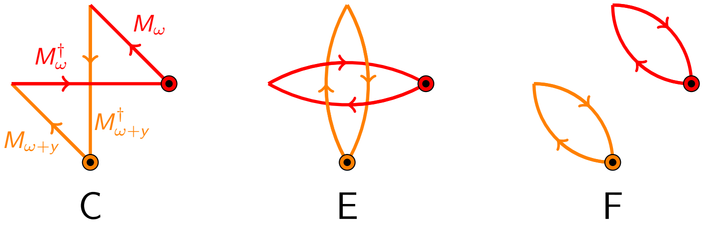
Technical approach

Conserved vector currents for staggered fermions $\chi(x)$, $\bar{\chi}(x)$:

$$j_\mu(x) = -\frac{1}{2}\eta_\mu(x)\left[\bar{\chi}(x+\hat{\mu})\ U^\dagger_\mu(x)\ \chi(x) + \bar{\chi}(x)\ U_\mu(x)\ \chi(x+\hat{\mu})\right]$$



- For given direction $\hat{n}$, consider all combinations of (y, ω) with $y = |y|\hat{n}$, $\omega = |\omega|\hat{n}$, exploit (anti-)periodicity.
- Repeat calculation for all $y' = y + h$, where $h_\mu = 0, \pm 1$ ($\mathcal{O}(a^2)$-smearing)
- Use local stochastic sources in color space $\Rightarrow$ Reduce number of matrix-matrix multiplications

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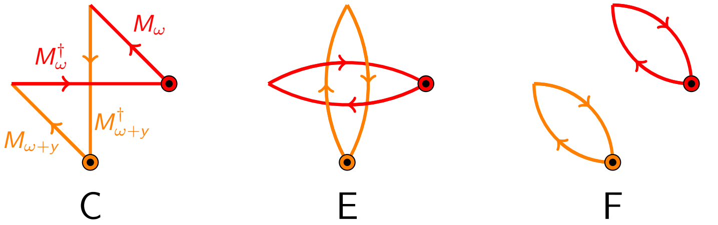

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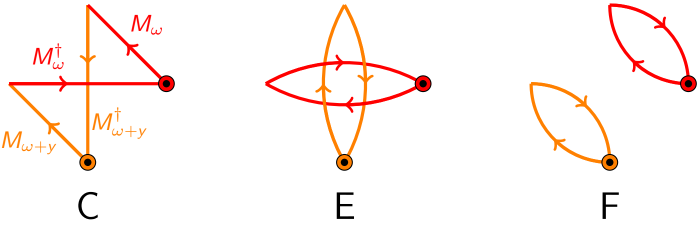
Conserved vector currents for staggered fermions $\chi(x)$, $\bar{\chi}(x)$:

$$j_\mu(x) = -\frac{1}{2}\eta_\mu(x)\left[\bar{\chi}(x+\hat{\mu})\ U_\mu^\dagger(x)\ \chi(x) + \bar{\chi}(x)\ U_\mu(x)\ \chi(x+\hat{\mu})\right]$$



- For given direction $\hat{n}$, consider all combinations of (y, ω) with $y = |y|\hat{n}$, $\omega = |\omega|\hat{n}$, exploit (anti-)periodicity.
- Repeat calculation for all $y' = y + h$, where $h_\mu = 0, \pm 1$ ($\mathcal{O}(a^2)$-smearing)
- Use local stochastic sources in color space $\Rightarrow$ Reduce number of matrix-matrix multiplications

Ensembles and Strategy

- Set of tree-level improved 4stout gauge ensembles employing $2+1+1$ staggered fermions
- 3 physical volumes: $3~\mathrm{fm}$, $4~\mathrm{fm}$, $6~\mathrm{fm}$
- Trajectories: $\hat{n} = (1,1,1,1)/2$ if $T = 2L$ ($L = 3~\mathrm{fm}$), else ($2T = 3L$) $\hat{n} = (1,1,1,3)/\sqrt{6}$

Runs:

$a[\mathrm{fm}]$	$L = 3~\mathrm{fm}$	$L = 4~\mathrm{fm}$	$L = 6~\mathrm{fm}$
0.132	$\ell + s$	$\ell + s$	
0.110	$\ell + s$		ℓ
0.096	$\ell + s$		ℓ
0.077	s		
0.063	s		