Status update: $\pi^0 \to \gamma^*\gamma^*$ transition form factor on CLS ensembles

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- The transition form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}$ describes the interaction of an on-shell pion with two off-shell photons.
- $\mathcal{F}_{\pi^0\gamma^*\gamma^*}$ is the main ingredient in the calculation of the pion-pole contribution to hadronic light-by-light scattering in the muon $g-2$, $a^{\mathrm{HLbL};\pi^0}$.
- There is also a direct relation between $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0)$ and the partial decay width $\Gamma(\pi^0 \to \gamma\gamma)$.

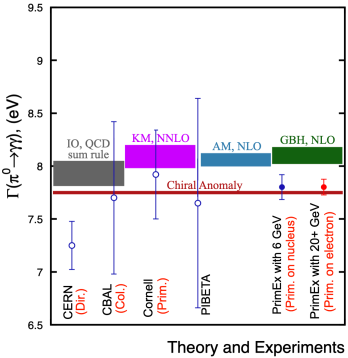
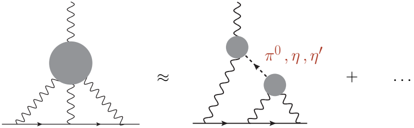


Figure from JLab whitepaper arXiv:2306.09360.

- $\Gamma(\pi^0 \to \gamma\gamma) = \dfrac{m_{\pi^0}^3 \alpha_e^2 N_c^2}{576\pi^3 F_{\pi^0}^2}$

  from LO χPT, tension between theory and experiments when NLO corrections are added.



The transition form factor is extracted from matrix elements

$$M_{\mu\nu}(p,q_1) = i\int \mathrm{d}^4x \mathrm{e}^{iq_1\cdot x}\langle 0|T\{J_\mu(x)J_\nu(0)\}|\pi^0(p)\rangle = \epsilon_{\mu\nu\alpha\beta}q_1^\alpha q_2^\beta \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2),$$

where J_μ is the electromagnetic current. q_1 and q_2 are the four-momenta associated with the two currents, and p is the four-momentum of the pion.

The Euclidean matrix elements read

$$M_{\mu\nu} = (i^{n_0})M^E_{\mu\nu},\ M^E_{\mu\nu} = -\int_{-\infty}^{\infty}\mathrm{d}\tau \mathrm{e}^{\omega_1\tau}\int \mathrm{d}^3z \mathrm{e}^{-i\vec{q}_1\cdot\vec{x}}\langle 0|T\{J_\mu(\vec{x},\tau)J_\nu(\vec{0},0)\}|\pi^0(p)\rangle,$$

and defining $\widetilde{A}_{\mu\nu}(\tau)$, the matrix elements can be obtained by integration

$$M^E_{\mu\nu}(p,q_1) = \frac{2E_\pi}{Z_\pi}\int_{-\infty}^{\infty}\mathrm{d}\tau \mathrm{e}^{\omega_1\tau}\widetilde{A}_{\mu\nu}(\tau),$$

where τ is the time separation between the two EM currents.

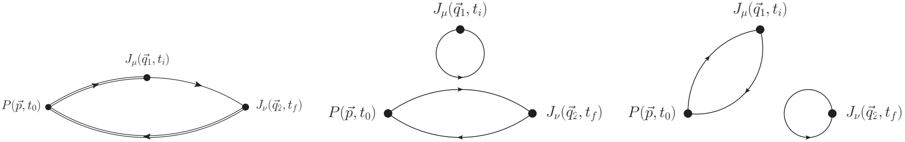
$\widetilde{A}_{\mu\nu}(\tau)$ is connected to a 3-point correlator calculated on the lattice by

$$C^{(3)}_{\mu\nu}(\tau, t_\pi) \equiv a^6 \sum_{\vec{x},\vec{z}} \langle J_\mu(\vec{x}, t_i) J_\nu(\vec{0}, t_f) P^\dagger(\vec{z}, t_0)\rangle \mathrm{e}^{i\vec{p}\cdot\vec{z}} \mathrm{e}^{-i\vec{q}_1\cdot\vec{x}}$$

$$\widetilde{A}_{\mu\nu}(\tau) \equiv \lim_{t_\pi \to +\infty} \mathrm{e}^{E_\pi (t_f - t_0)} C^{(3)}_{\mu\nu}(\tau, t_\pi),$$

where t_π is the time separation between the pion and the closest EM current.

In addition to the quark-line connected diagram, there are contributions from two quark-line disconnected diagrams that have to be calculated.

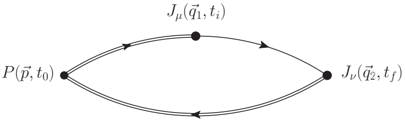


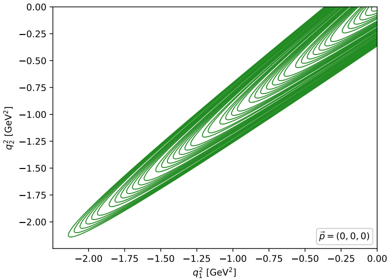
Photon virtualities

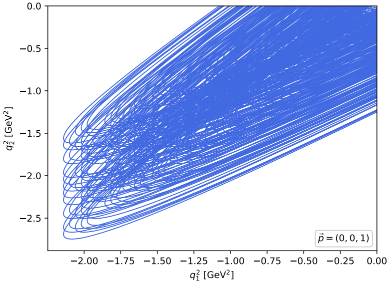
- Use both the rest frame of the pion, $\vec{p} = (0,0,0)$, and a moving frame $\vec{p} = (0,0,1)$ (in units of $2\pi/L$)
- The four-momenta associated with the EM currents are

$$q_1 = (\omega_1, \vec{q}_1)$$
$$q_2 = (E_\pi - \omega_1, \vec{p} - \vec{q}_1)$$

- Each curve in the plot represents a fixed value of $\vec{q}_1$ and $\vec{p}$
- ω_1 is a free parameter (this tracks the curve from one end to another)







- non-perturbatively $\mathcal{O}(a)$-improved Wilson fermions
- tree-level improved Lüscher-Weisz gauge action
- four lattice spacings, multiple pion masses, large volumes ($M_\pi L \geq 4$)

ID	β	$L^3 \times T$	a/fm	κ_l	κ_s	M_π/MeV	$M_\pi L$	N_{conf}
H101	3.40	$32^2 \times 96$	0.08636	0.136760	0.13675962	416	5.8	1000
H102		$32^2 \times 96$		0.136865	0.13654934	354	5.0	1900
H105		$32^2 \times 96$		0.136970	0.13634079	281	3.9	2800
N101		$48^2 \times 128$		0.136970	0.13634079	280	5.9	1600
C101		$48^2 \times 96$		0.137030	0.13622204	224	4.7	2200
S400	3.46	$32^2 \times 128$	0.07634	0.136984	0.13670239	349	4.3	1700
N401		$48^2 \times 128$		0.137062	0.13654808	286	5.3	950
H200	3.55	$48^2 \times 96$	0.06426	0.137000	0.137000	419	4.4	2000
N202		$48^2 \times 128$		0.137000	0.137000	411	6.4	900
N203		$48^2 \times 128$		0.137080	0.13684028	346	5.4	1500
N200		$48^2 \times 128$		0.137140	0.13672086	284	4.4	1700
D200		$64^2 \times 128$		0.137200	0.13660175	200	4.2	1100
E250		$96^2 \times 192$		0.137232867	0.136536633	129	4.0	800
N300	3.70	$48^2 \times 128$	0.04981	0.137000	0.137000	342	5.1	1200
N302		$48^2 \times 128$		0.137064	0.13687218	343	4.2	1100
J303		$64^2 \times 192$		0.137123	0.13675466	258	4.2	650

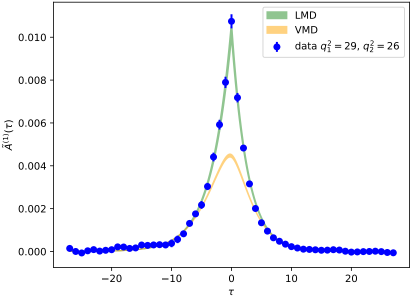
Correlators: E250

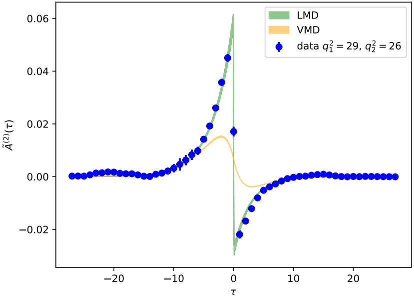
Recall that $\widetilde{A}_{\mu\nu}(\tau)$ is directly related to the 3-point correlators $C^{(3)}_{\mu\nu}$.
For convenience we define two scalar funtions:

$$\widetilde{A}_{0k}(\tau) = (\vec{q}_1 \times \vec{p})^k \widetilde{A}^{(1)}(\tau)$$

$$\epsilon'^k \widetilde{A}_{kl}(\tau)\epsilon^l = -i(\vec{\epsilon}' \times \vec{\epsilon}) \cdot \left(\vec{q}_1 E_\pi \widetilde{A}^{(1)}(\tau) + \vec{p}\frac{\mathrm{d}\widetilde{A}^{(1)}(\tau)}{\mathrm{d}\tau} \right)$$

In the moving frame we define for simplicity $\widetilde{A}_{12}(\tau) \equiv -iE_\pi p_z \widetilde{A}^{(2)}(\tau)$.





Modeling the tail

Recall that $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) \propto \int_{-\infty}^{\infty} \mathrm{d}\tau \mathrm{e}^{\omega_1 \tau} \widetilde{A}_{\mu\nu}(\tau)$.

We want to model $\widetilde{A}_{\mu\nu}(\tau)$ at large $|\tau|$ to get the tail contribution.

- Lowest Meson Dominance (LMD)

$$\widetilde{A}_{\mu\nu}^{\mathrm{LMD}} = \frac{Z_\pi}{4\pi E_\pi} \int_{-\infty}^{\infty} \mathrm{d}\tilde{\omega} \frac{\left(P_{\mu\nu E}\tilde{\omega} + Q_{\mu\nu}^E\right)\left(\alpha M_V^4 + \beta(q_1^2 + q_2^2)\right)}{\left(\tilde{\omega} - \tilde{\omega}_1^{(+)}\right)\left(\tilde{\omega} - \tilde{\omega}_1^{(-)}\right)\left(\tilde{\omega} - \tilde{\omega}_2^{(+)}\right)\left(\tilde{\omega} - \tilde{\omega}_2^{(-)}\right)}$$

$$\text{with} \quad P_{\mu\nu}^E = i\epsilon_{\mu\nu 0 i} p^i, \qquad \tilde{\omega}_1^{(\pm)} = \pm i\sqrt{M_V^2 + |\vec{q}_1|^2}$$

$$Q_{\mu\nu}^E = \epsilon_{\mu\nu i 0} E_\pi q_1^i - i\epsilon_{\mu\nu i j} q_1^i p^j, \qquad \tilde{\omega}_2^{(\pm)} = -i\left(E_\pi \mp \sqrt{M_V^2 + |\vec{q}_2|^2}\right)$$

This gives an explicit expression for $\widetilde{A}_{\mu\nu}^{\mathrm{LMD}}$, which we use to fit our data using α, β and M_V as fit parameters.

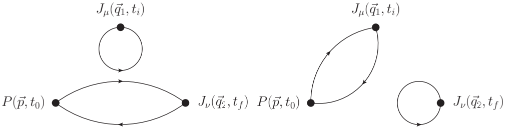
- Vector Meson Dominance (VMD): Set $\beta = 0$ in the LMD model

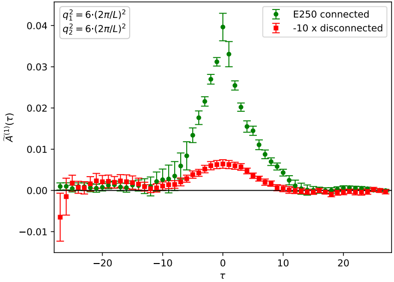
Disconnected contribution

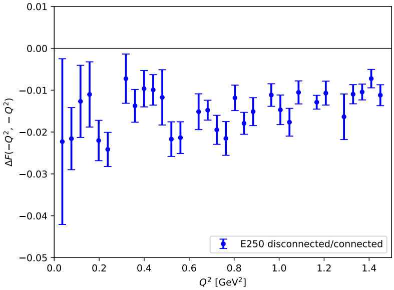
- In addition to the quark-line connected piece, we need two quark-line disconnected diagrams.
- The quark loops are computed using stochastic all-to-all methods, while the two-point functions are computed using point sources.
- We find the disc. contribution

$$\Delta F\left(-Q_1^2, -Q_2^2\right) = \frac{\mathcal{F}^{\mathrm{disc}}_{\pi^0\gamma^*\gamma^*}\left(-Q_1^2, -Q_2^2\right)}{\mathcal{F}^{\mathrm{conn}}_{\pi^0\gamma^*\gamma^*}\left(-Q_1^2, -Q_2^2\right)}$$

is at a few percent level.







Parameterizing the form factor: z-expansion

After obtaining the transition form factor at several virtualities $(q_1^2, q_2^2) \equiv (-Q_1^2, -Q_2^2)$, we parameterize it using a conformal mapping

$$z_k = \frac{\sqrt{t_{\mathrm{cut}} + Q_k^2} - \sqrt{t_{\mathrm{cut}} - t_0}}{\sqrt{t_{\mathrm{cut}} + Q_k^2} + \sqrt{t_{\mathrm{cut}} - t_0}}, \text{ with } t_{\mathrm{cut}} = 4m_\pi^2,\ t_0 = t_{\mathrm{cut}}\left(1 - \sqrt{1 + \frac{Q_{\mathrm{max}}^2}{t_{\mathrm{cut}}}}\right).$$

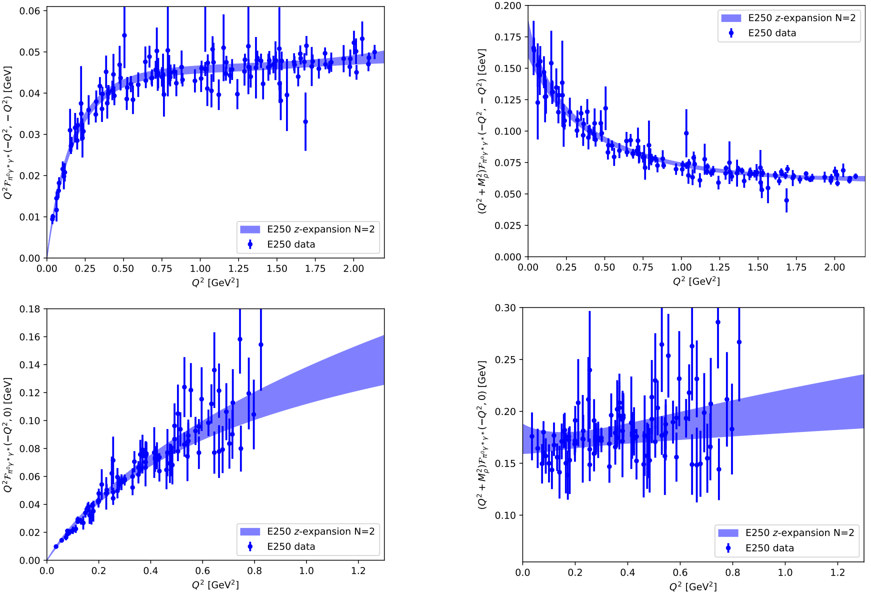
The form factor is then written as an expansion in z_1 and z_2:

$$P(Q_1^2, Q_2^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) = \sum_{n,m=0}^{N} c_{nm}\left(z_1^n + (-1)^{N+n}\frac{n}{N+1}z_1^{N+1}\right)\left(z_2^m + (-1)^{N+m}\frac{m}{N+1}z_2^{N+1}\right),$$

where the coefficients $c_{nm} = c_{mn}$, the fit parameters, are symmetric.

$P(Q_1^2, Q_2^2) = 1 + \dfrac{Q_1^2 + Q_2^2}{M_V^2}$ is the vector meson pole with $M_V = 775$ MeV.

Results: Transition form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}$



Partial decay width $\Gamma(\pi^0 \to \gamma\gamma)$

Recall the relation between the partial decay width $\Gamma(\pi^0 \to \gamma\gamma)$ and the transition form factor:

$$\Gamma(\pi^0 \to \gamma\gamma) = \frac{\pi \alpha_e^2 m_{\pi^0}^3}{4} \mathcal{F}^2_{\pi^0\gamma^*\gamma^*}(0,0)$$

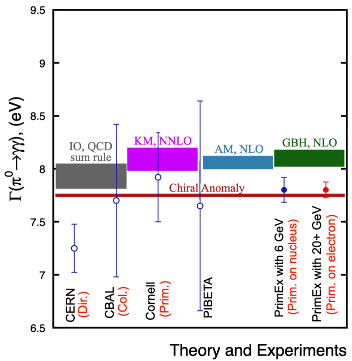
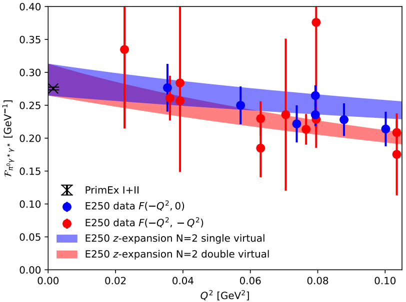
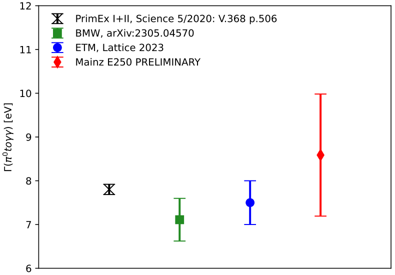


Figure from JLab whitepaper arXiv:2306.09360.





Summary and outlook

- We have calculated the transition form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}$ on a physical pion mass ensemble.
- We compute the disconnected diagrams needed in addition to the quark-line connected piece to construct the full form factor.
- At the final stage of the analysis, the result on E250 will be combined with the previous work published in 2019 to extrapolate the form factor to the continuum and to physical quark masses.
- $\mathcal{F}_{\pi^0\gamma^*\gamma^*}$ is the main ingredient in the estimation of the pion-pole contribution to hadronic light-by-light scattering in the muon $g-2$. The goal is to improve this estimate by including E250 in the analysis.
- If lattice QCD calculations want to address the tension between the partial decay width $\Gamma(\pi^0 \to \gamma\gamma)$ from PrimEx experiment and NLO theory predictions, there is still a long way to go.

Thank you!
Any questions?