THE TWISTED GRADIENT FLOW STRONG COUPLING WITH PARALLEL TEMPERING ON BOUNDARY CONDITIONS

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THE GRADIENT FLOW

Renormalization procedure in which the gauge field $A_\mu(x)$ is replaced by a set of smooth, time-dependent flow fields $B_\mu(x)$ driven by the so-called "flow equations" [arXiv:1101.0963]:

$$\partial_t B_\mu(x,t) = D_\nu G_{\nu\mu}(x,t) \quad B_\mu(x, t=0) = A_\mu(x)$$

SMOOTHING THE GAUGE FIELDS IN A RANGE $\sqrt{8t}$

Gauge-invariant composite observables are automatically renormalized quantities for $t > 0$

Renormalized couplings can be introduced trivially, e.g. with the energy density:

$$E(t) = \frac{1}{2} Tr\left(G_{\mu\nu}(x,t) G_{\mu\nu}(x,t)\right)$$

THE COUPLING:

$$\lambda(\mu) = \mathcal{N} \left\langle t^2 E(t) \right\rangle \Big|_{\sqrt{8t}=\mu^{-1}}$$

IN ADDITION TO AND PECULIAR GEOMETRY

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GEOMETRY AND BOUNDARY CONDITIONS

The Twisted Gradient Flow (TGF) scheme is defined by introducing $SU(N)$ YM theories on an asymmetric hyperbox of size $l^2 \times (Nl)^2$. [arXiv:2107.03747]

SHORT DIRECTIONS: TWISTED BOUNDARY CONDITIONS

$$A_\mu(x + l\hat{\nu}) = \Gamma_\nu A_\mu(x) \Gamma_\nu^\dagger, \text{ for } \nu = 1,2$$

$$\Gamma_1 \Gamma_2 = Z_{12} \Gamma_2 \Gamma_1$$

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LONG DIRECTIONS: PERIODIC BOUNDARY CONDITIONS

$$\tilde{l} \equiv N \times l$$

THE (TWISTED) GRADIENT FLOW

$$\lambda_{TGF}(\mu) = \frac{128\pi^2 t^2}{3N\mathcal{A}(\pi c^2)} < E(t) > \Big|_{\sqrt{8t} = c\tilde{l} = \mu^{-1}}$$

Characterising the scheme ($c = 0.3$)

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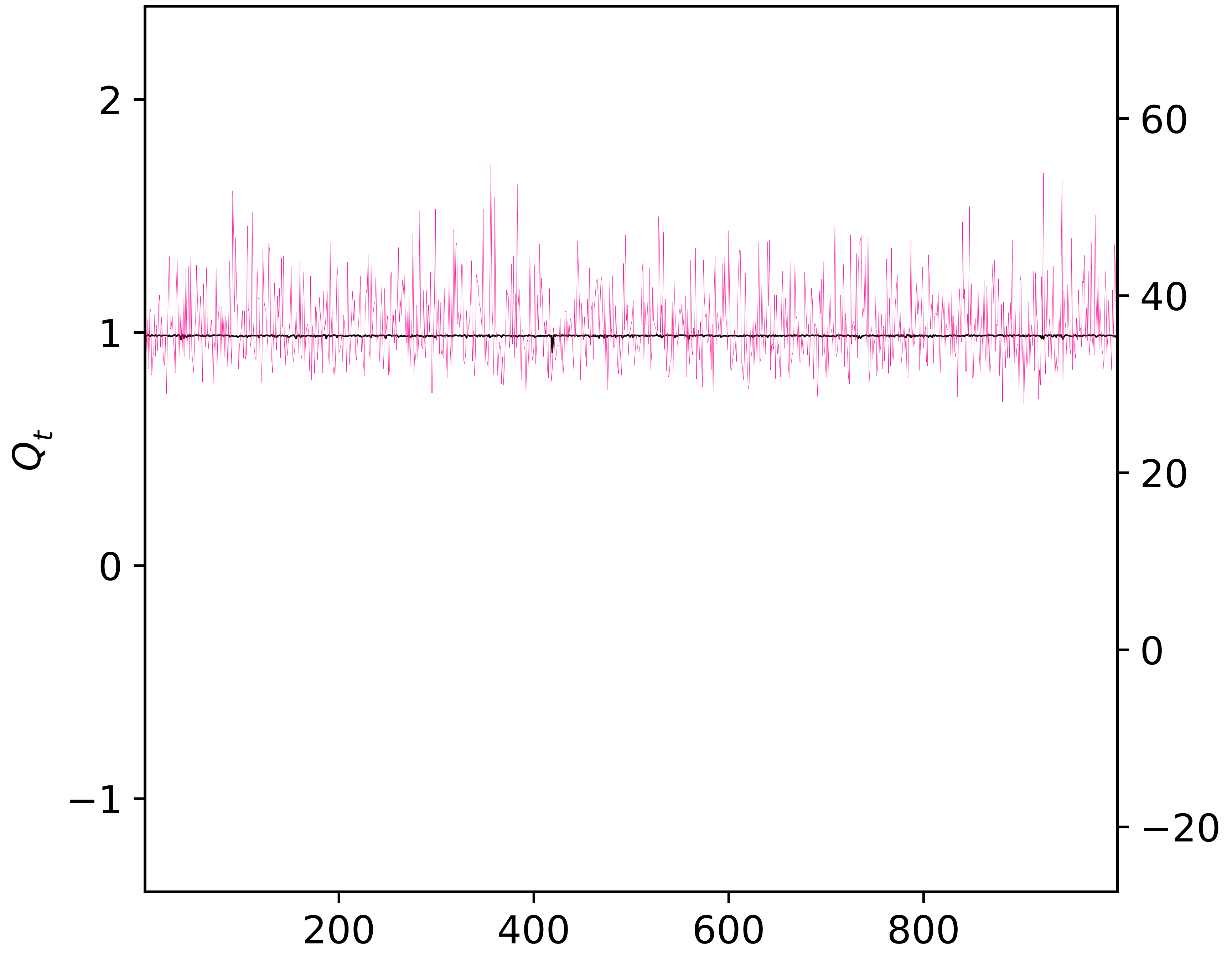

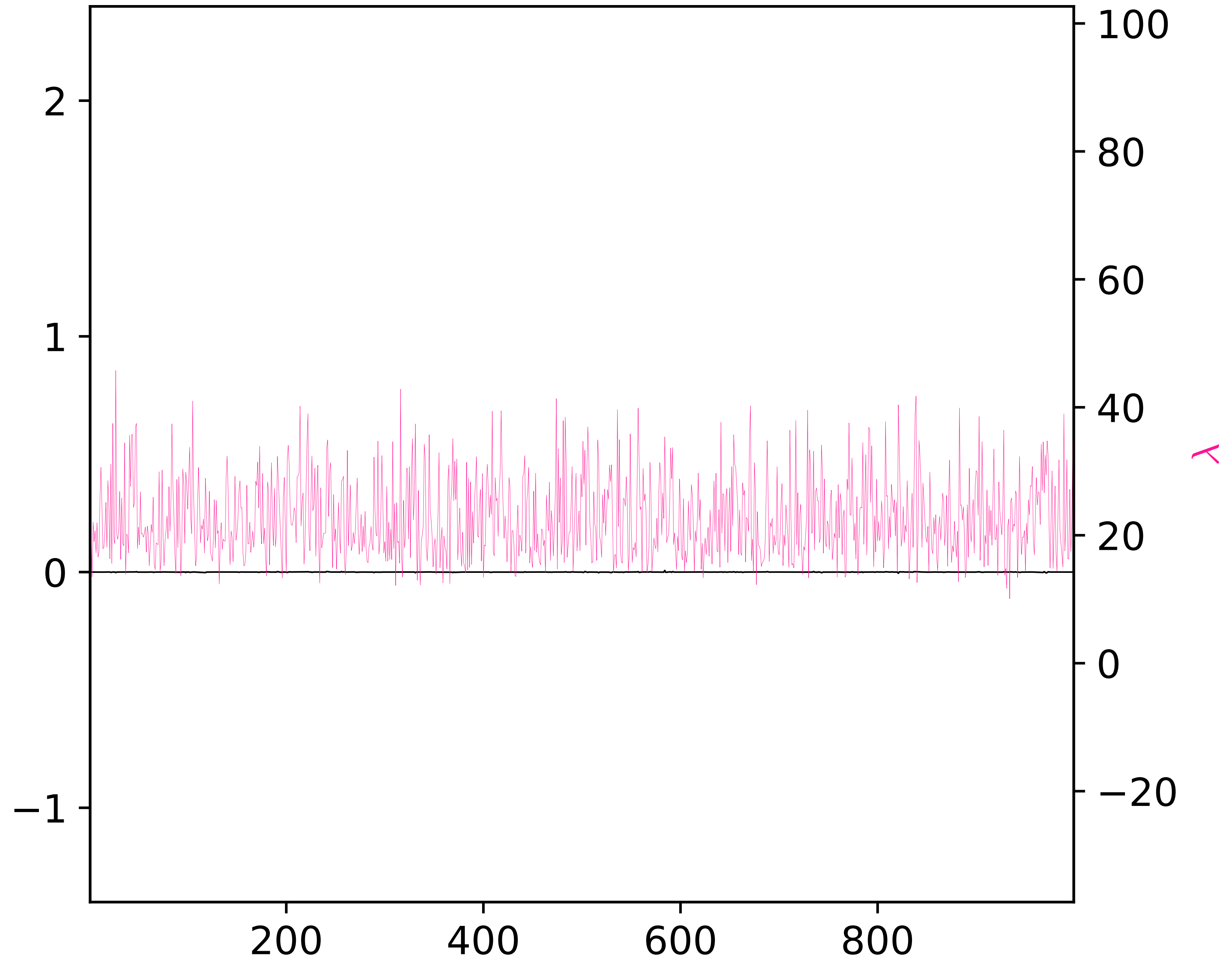

$$\lambda_{TGF}(\mu) = \frac{128\pi^2 t^2}{3N\mathscr{A}(\pi c^2)} \frac{< E(t)\,\delta_Q >}{< \delta_Q >}\Bigg|_{\sqrt{8t}=c\tilde{l}=\mu^{-1}}$$

Projection into the zero charge sector

[arXiv:1311.7304,1905.05147]

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POSSIBLE SOLUTION: PARALLEL TEMPERING ON BOUNDARY CONDITIONS

Proposed for 2d CP^{N-1} models [arXiv:1706.04443,1911.03384], and recently implemented for 4d SU(N) pure-gauge theories [arXiv:2205.06190], it alienates open and periodic boundary conditions in a parallel tempered manner.

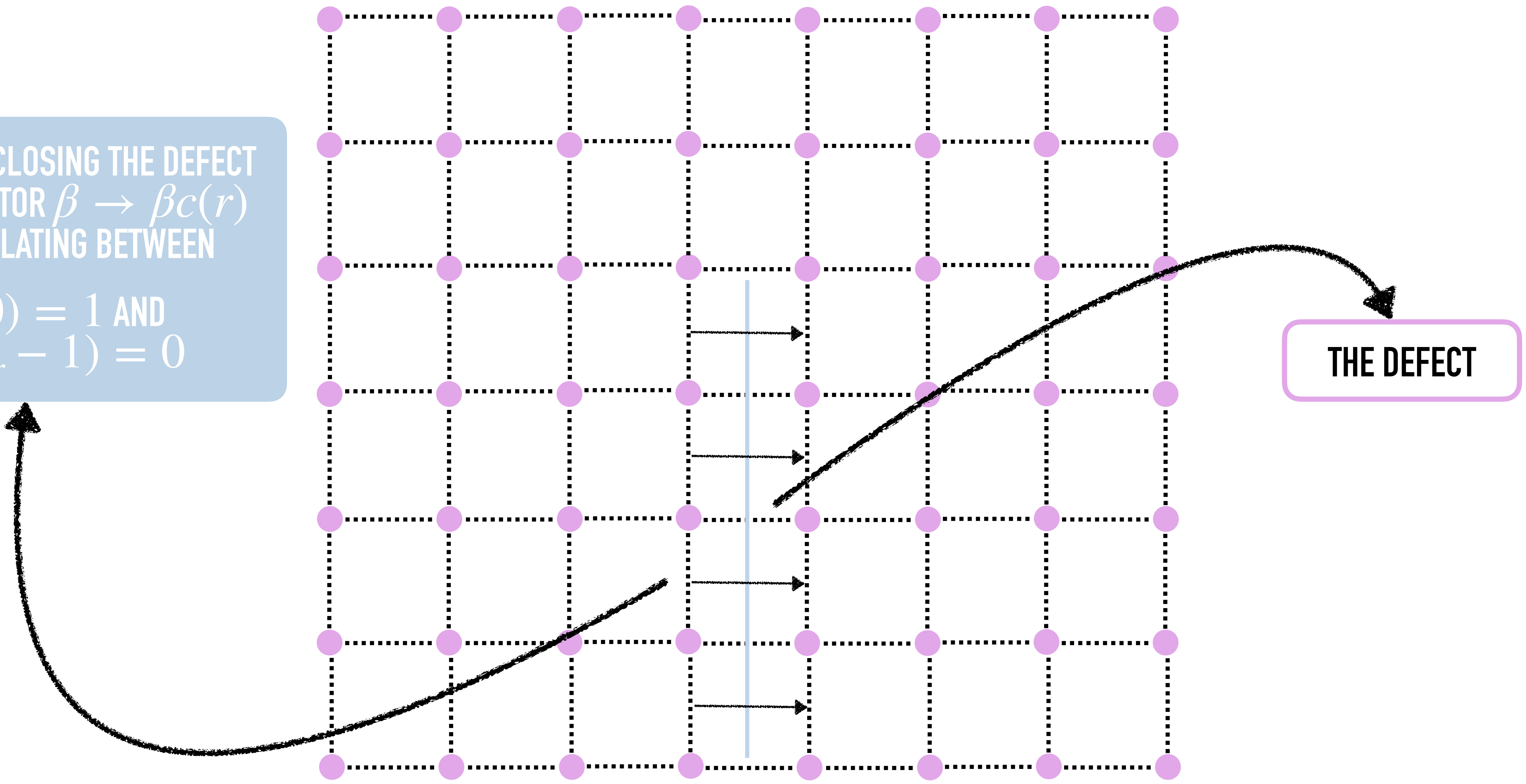
Implementation

- Consider N_r replicas of the target lattice.
- The boundary conditions in each replica change ONLY along a hypercube: **THE DEFECT**
- Each replica is updated using a standard algorithm (a combination of heatbatch and overelaxation steps).
- After updates, propose swaps between configurations via Metropolis test, introducing new plausible topology fluctuations.
- After swaps, update more frequently those links close to the defect (Hierarchical Updates) and move the defect randomly to improve performance.

POSSIBLE SOLUTION: PARALLEL TEMPERING ON BOUNDARY CONDITIONS

THE LINKS CLOSING THE DEFECT PICK A FACTOR $\beta \to \beta c(r)$ INTERPOLATING BETWEEN

$c(0) = 1$ AND
$c(N_r - 1) = 0$



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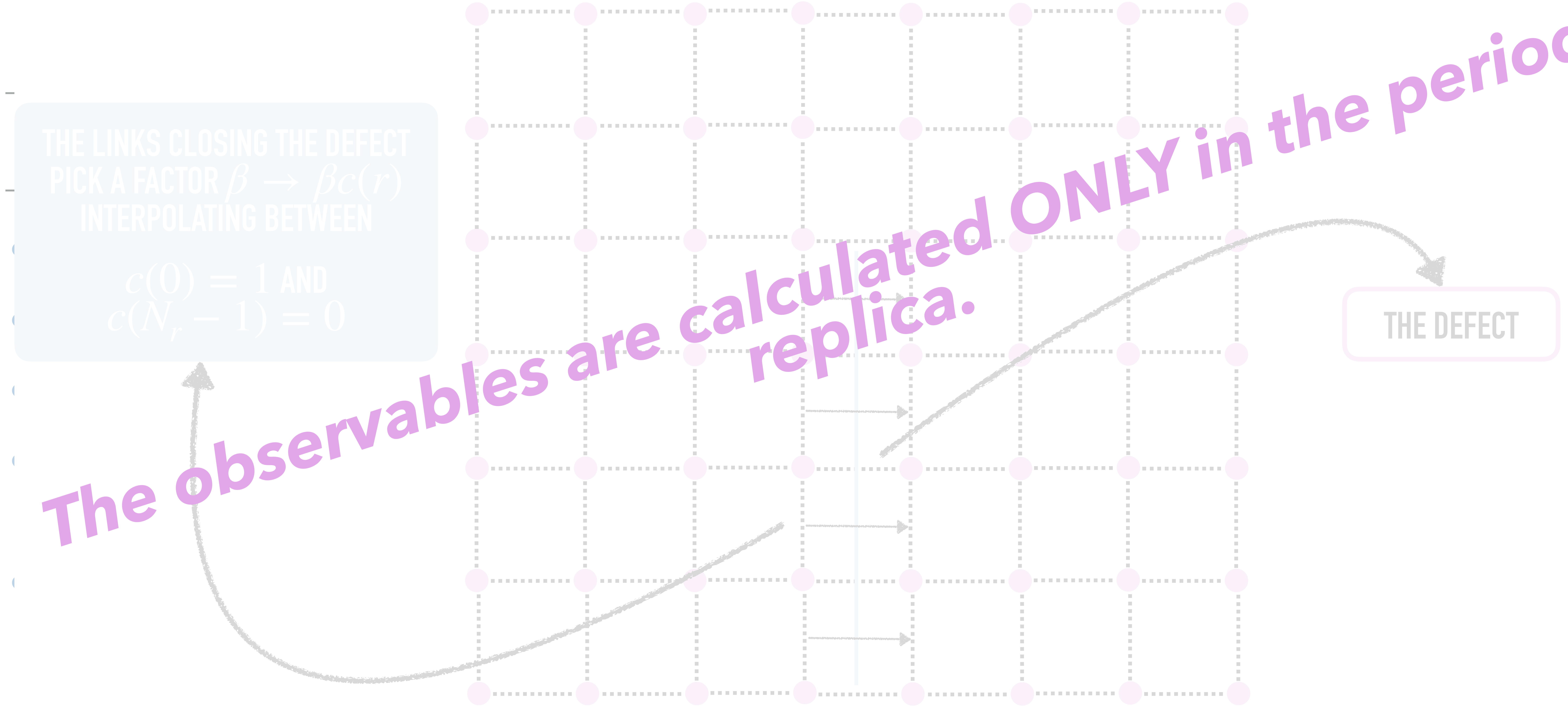

The observables are calculated ONLY in the periodic replica.

A FIRST EXPLORATION: FROZEN VS NON-FROZEN

We will test the parallel tempering in our setup by analyzing two cases, one well-sampled in the standard algorithm, and one suffering from topology freezing, corresponding to the same physical volume.

Our simulations

- $N_r = 18$ replicas.
- Defect of size $d = 4$ in lattice units.
- Acceptance probability for the Metropolis swapping steps of 20 %. [arXiv:2012.14000]

$\tilde{L} = 24$ and $\beta = 6.4881$

- $N_r = 32$ replicas.
- Defect of size $d = 6$ in lattice units.
- Acceptance probability for the Metropolis swapping steps of 20 %.

$\tilde{L} = 36$ and $\beta = 6.7790$

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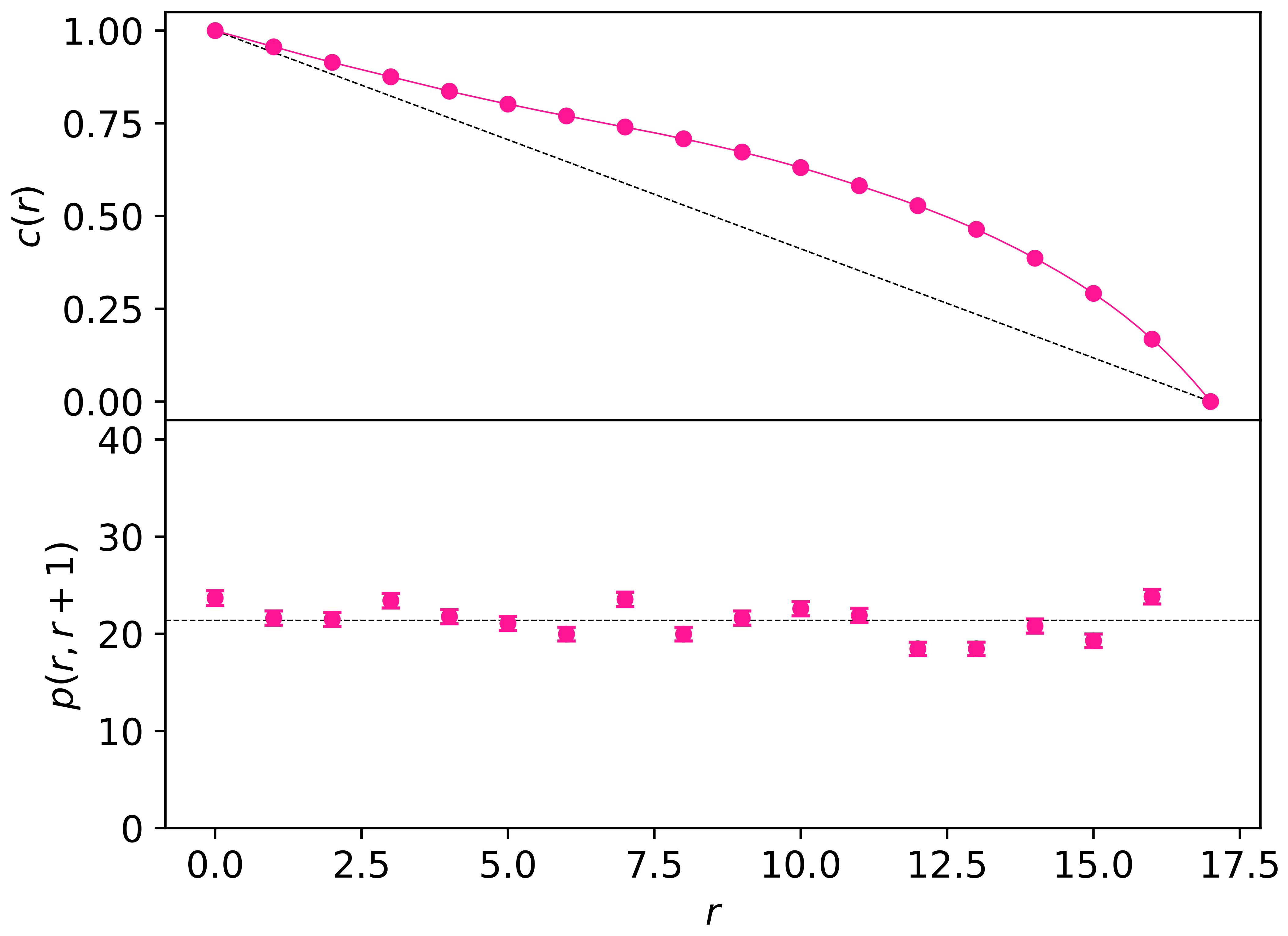

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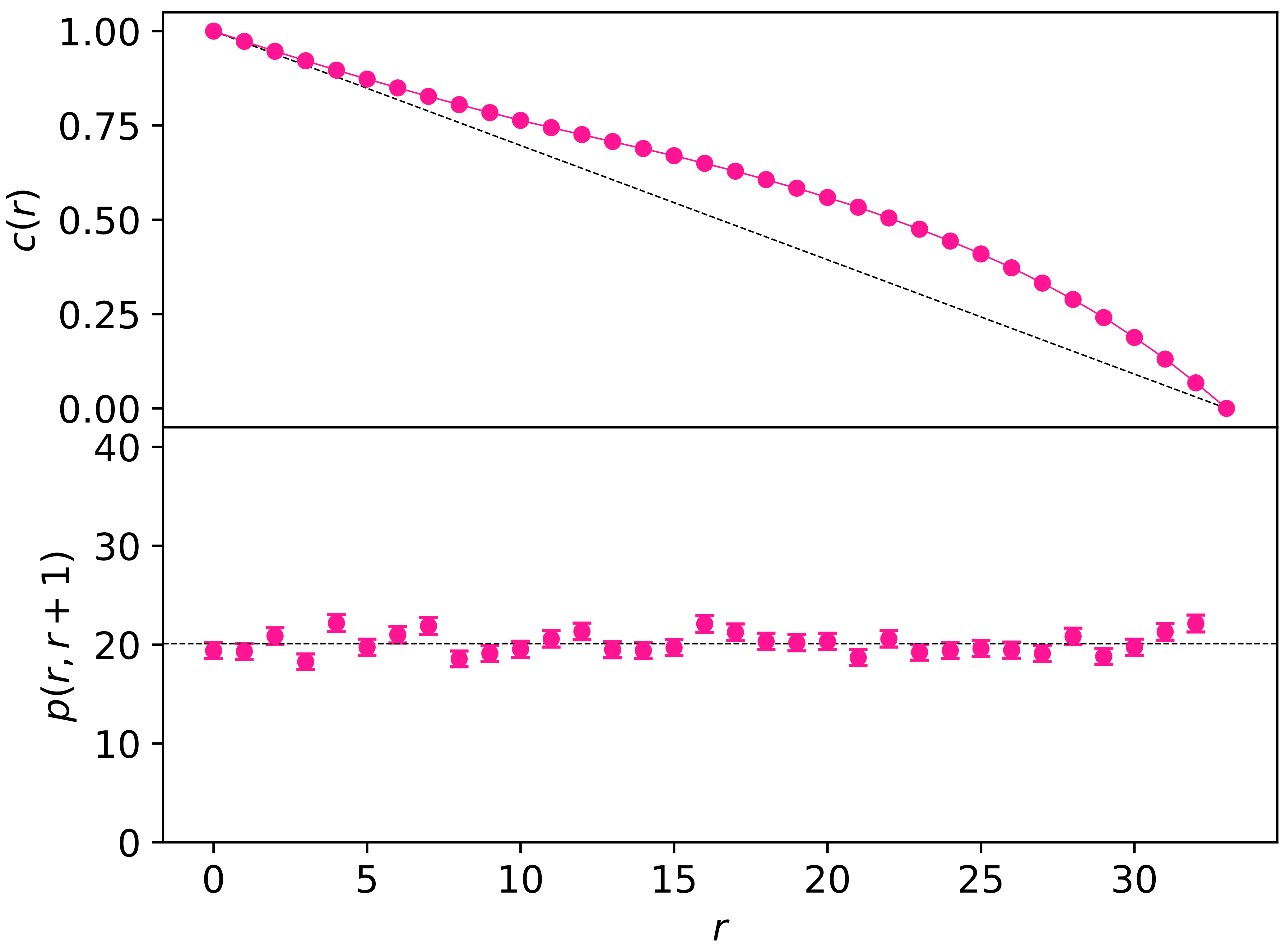

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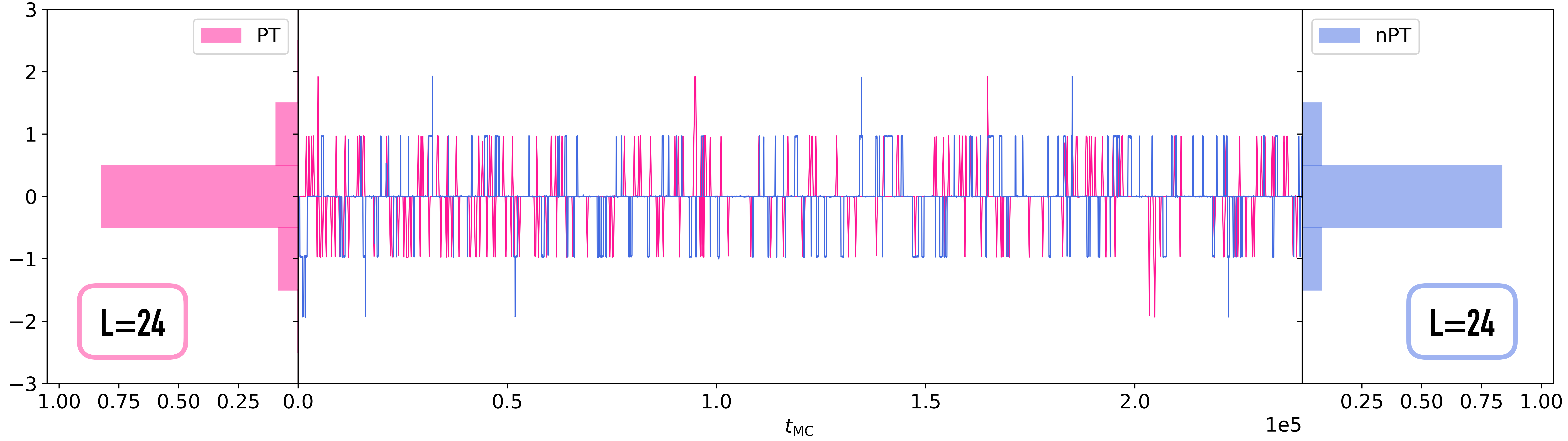
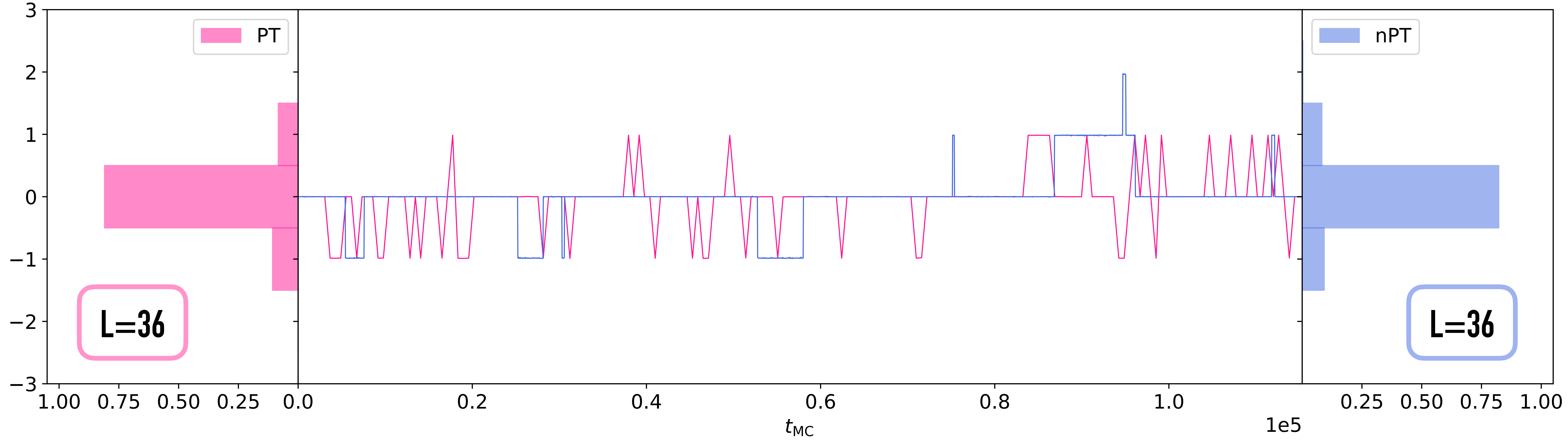
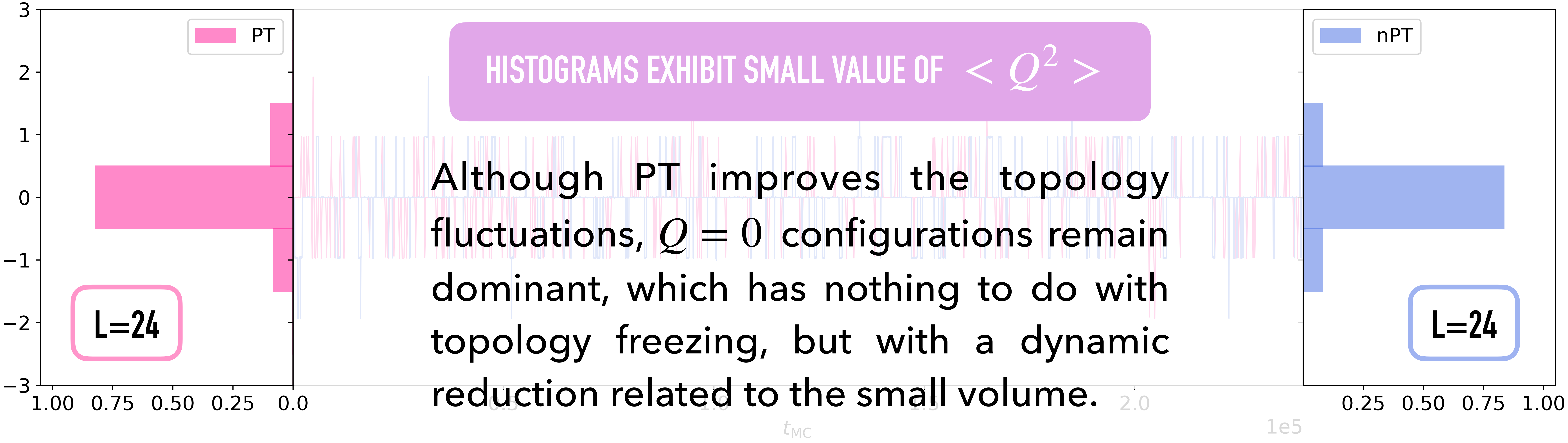
FIRST EXPLORATION: DO WE GAIN SOMETHING USING PT?

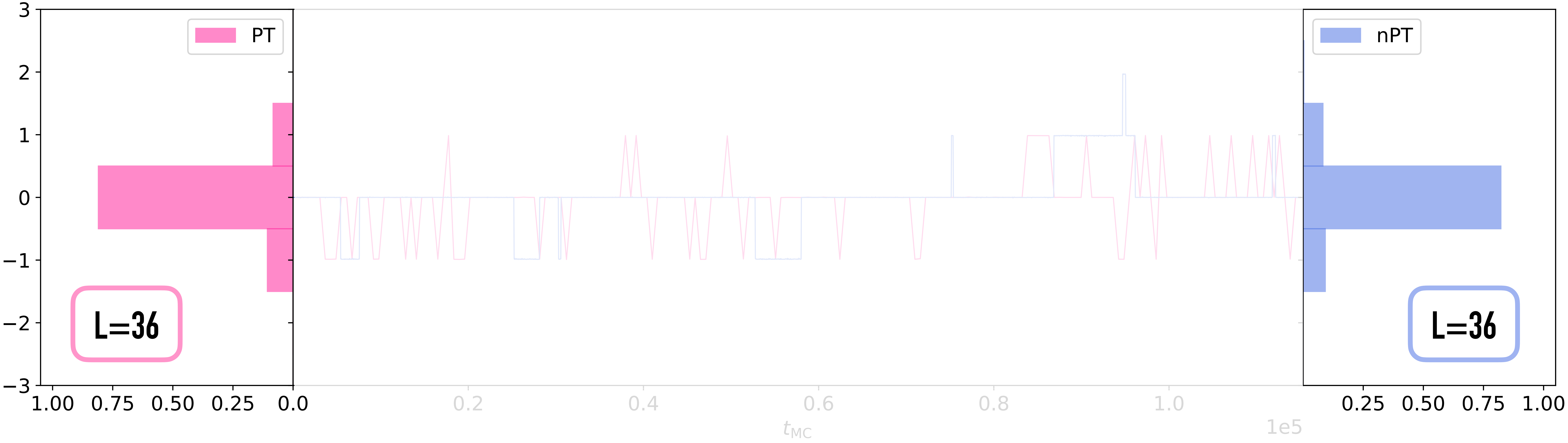




HISTOGRAMS EXHIBIT SMALL VALUE OF $< Q^2 >$

Although PT improves the topology fluctuations, $Q = 0$ configurations remain dominant, which has nothing to do with topology freezing, but with a dynamic reduction related to the small volume.





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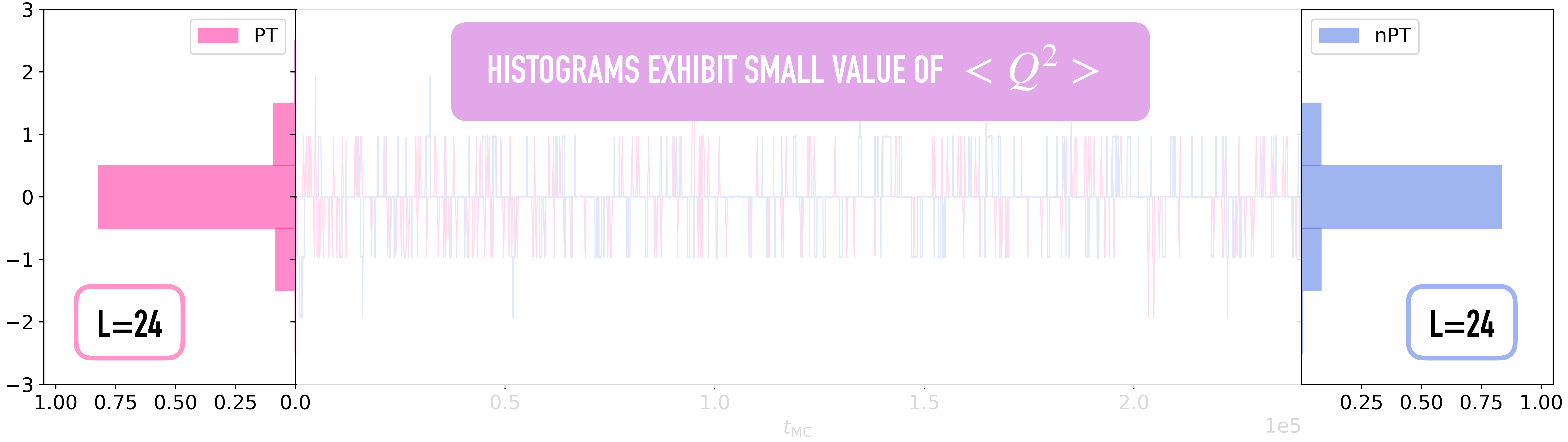
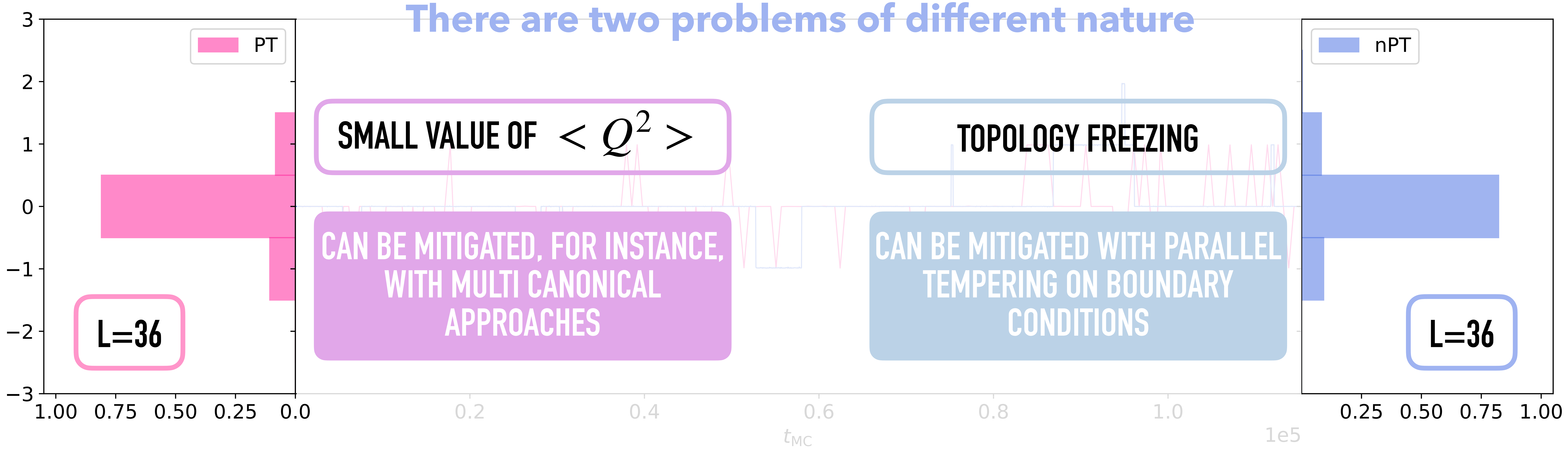


There are two problems of different nature



SMALL VALUE OF $< Q^2 >$

CAN BE MITIGATED, FOR INSTANCE, WITH MULTI CANONICAL APPROACHES

TOPOLOGY FREEZING

CAN BE MITIGATED WITH PARALLEL TEMPERING ON BOUNDARY CONDITIONS

A FIRST EXPLORATION: FROZEN VS NON-FROZEN

Let us compare the coupling, computed in different projected sectors, and the $< Q^2 >$ value.

	Algorithm	λ_{TGF}(All Q)	τ_λ(All Q)
$\tilde{L} = 24$	PT	34.09(32)	37.4(79)
$\tilde{L} = 24$	nPT	34.97(20)	43.5(43)
$\tilde{L} = 36$	PT	35.47(26)	149(31)
$\tilde{L} = 36$	nPT	35.65(77)	550(274)

	Algorithm	$\lambda_{\mathrm{TGF}}(Q = 1)$	$\tau_\lambda(Q = 1)$
$\tilde{L} = 24$	PT	42.45(52)	28(12)
$\tilde{L} = 24$	nPT	42.91(25)	31(10)
$\tilde{L} = 36$	PT	43.43(43)	348(165)
$\tilde{L} = 36$	nPT	43.65(22)	250(118)

	Algorithm	$\lambda_{\mathrm{TGF}}(Q = 0)$	τ_λ(All Q)
$\tilde{L} = 24$	PT	31.87(30)	39.1(50)
$\tilde{L} = 24$	nPT	32.17(11)	40.1(43)
$\tilde{L} = 36$	PT	33.37(22)	100(19)
$\tilde{L} = 36$	nPT	33.23(22)	88.7(84)

	Algorithm	$< Q^2 >$	τ_{Q^2}
$\tilde{L} = 24$	PT	0.192(17)	100(30)
$\tilde{L} = 24$	nPT	0.163(17)	207(29)
$\tilde{L} = 36$	PT	0.195(14)	401(91)
$\tilde{L} = 36$	nPT	0.177(88)	2445(966)

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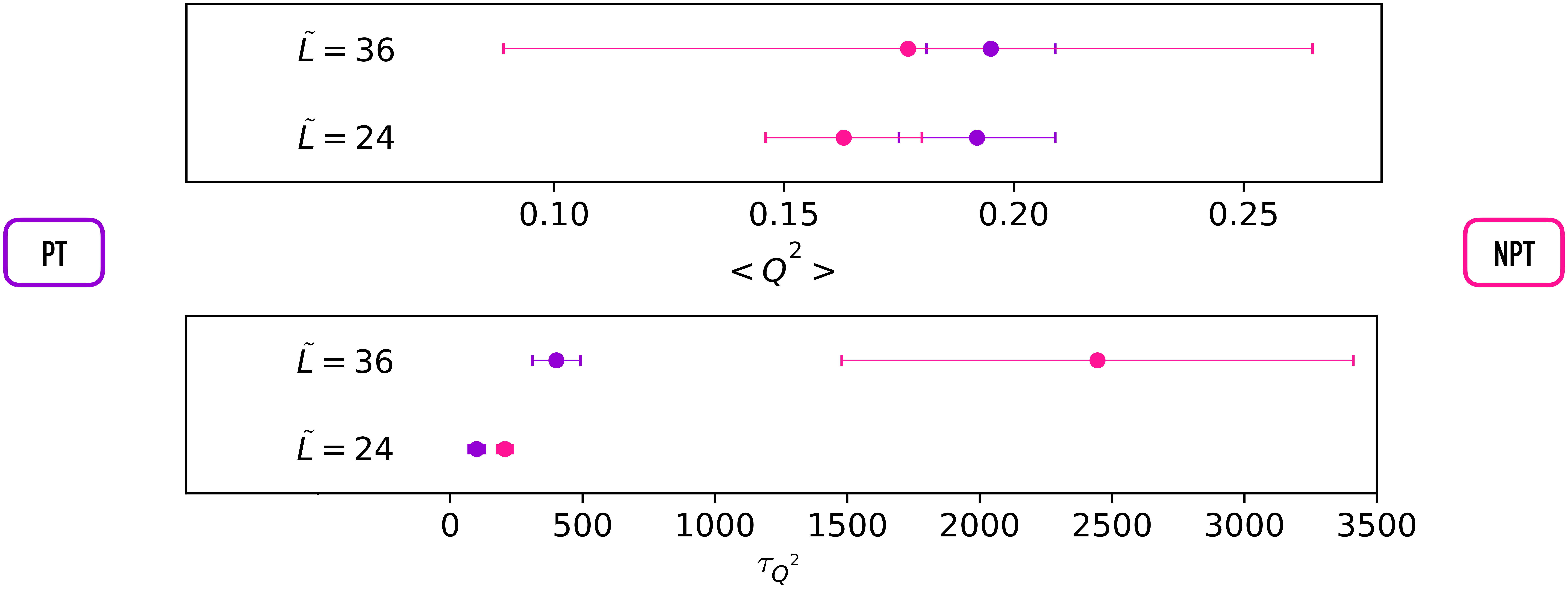

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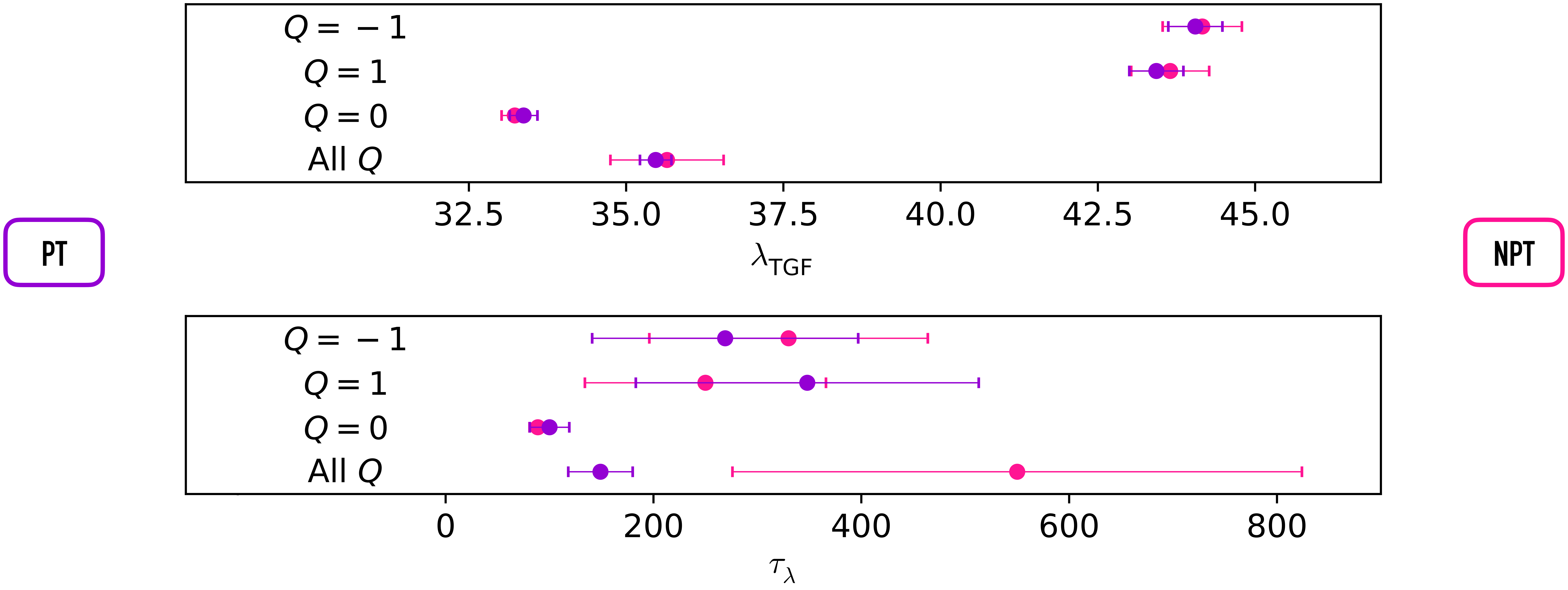

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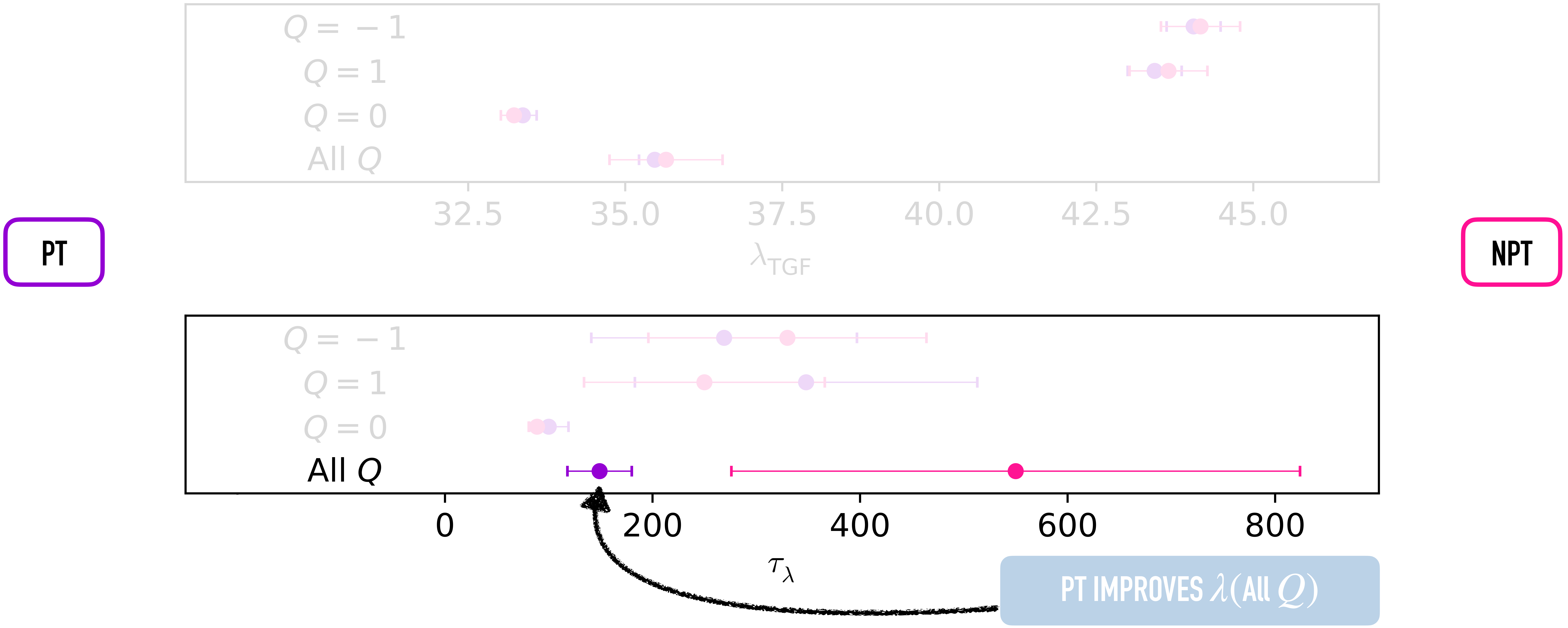

- Parallel Tempering improves the autocorrelation time of the topological charge of our previous TGF calculations.
- At this stage, it appears that topological fluctuations are well sampled once the calculation of the coupling is projected into a specific charge sector.
- Although PT efficiently mitigates topology freezing, $Q = 0$ configurations remain dominant because the mean $< Q^2 >$ is small. This can be solved with other types of algorithms.

FURTHER STUDIES ARE NEEDED TO REPRODUCE THE RUNNING OF THE COUPLING AND TO EXTRACT THE PURE $SU(3)$ Λ PARAMETER.

THANK YOU FOR YOUR ATTENTION

QUESTIONS?

A FIRST EXPLORATION: TOPOLOGICAL SUSCEPTIBILITY