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# Static Force from the Lattice with Gradient flow

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Julian Mayer-Steudte* (TUM)
Nora Brambilla (TUM)
Viljami Leino (HIM)
Antonio Vairo (TUM)

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# Motivation: Static Energy E(r)

- Interested in the QCD static energy of a quark-antiquark pair $E(r)$
- Given by the Wilson loop

$$E(r) = -\lim_{T\to\infty} \frac{\ln\langle \mathrm{Tr}(W_{r\times T})\rangle}{T} \qquad W_{r\times T} = P\left\{\exp\left(i\oint_{r\times T} dz_\mu g A_\mu\right)\right\}$$

- Can be described by perturbation theory and measured non-perturbatively on the lattice
- For $r\Lambda_{\mathrm{QCD}} \ll 1$ both descriptions should agree

Can be used for precise $\alpha_S$ extraction by comparing PT and lattice

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# Motivation: Issues with E(r)

- Perturbative form of $E(r)$

$$E(r) = \Lambda_S - \frac{C_F \alpha_S}{r} \left(1 + \#\alpha_S + \#\alpha_S^2 + \#\alpha_S^3 + \#\alpha_S^3 \ln \alpha_S + \dots\right)$$

- $E(r)$ is known up to $\mathrm{N}^3$LL
- The perturbative expansion affected by a renormalon ambiguity of order $\Lambda$ in PT side
- On lattice: Linear UV divergence
- All interesting physics is in the slope

$\Rightarrow$ take a derivative of $E(r)$ for the force $F(r) = \partial_r E(r)$

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# Setup: Alternative definition of F(r)

- Direct measurement of $F(r)$

A.Vairo Mod.Phys.Lett. A 31 (2016) & EPJ Web Conf. 126 (2016), Brambilla et.al.PRD63 (2001)

$$F(r) = -\lim_{T\to\infty} \frac{i}{\langle \mathrm{Tr}(W_{r\times T})\rangle} \left\langle \mathrm{Tr}\left( P\left\{ \exp\left( i\oint_{r\times T} dz_\mu g A_\mu \right) \hat{\mathbf{r}} \cdot g\mathbf{E}(\mathbf{r}, t^*) \right\} \right) \right\rangle$$

$$= -\lim_{T\to\infty} \frac{\langle \{ P W_{r\times T} g E_j(r, t^*) \} \rangle}{\langle \mathrm{Tr}\{ P W_{r\times T} \} \rangle}$$

- Chromoelectric field $E$ inserted into Wilson loop
- The insertion location $t^*$ is arbitrary → reduce boundary terms and choose $t^* = T/2$
- Can be used to extract $\alpha_S$ without the usual renormalon issues and for scale setting
- Serves as preparation for observables with field insertions needed in NREFTs

⇒ On the lattice: modifying Wilson loop with a discretized $E$-field insertion

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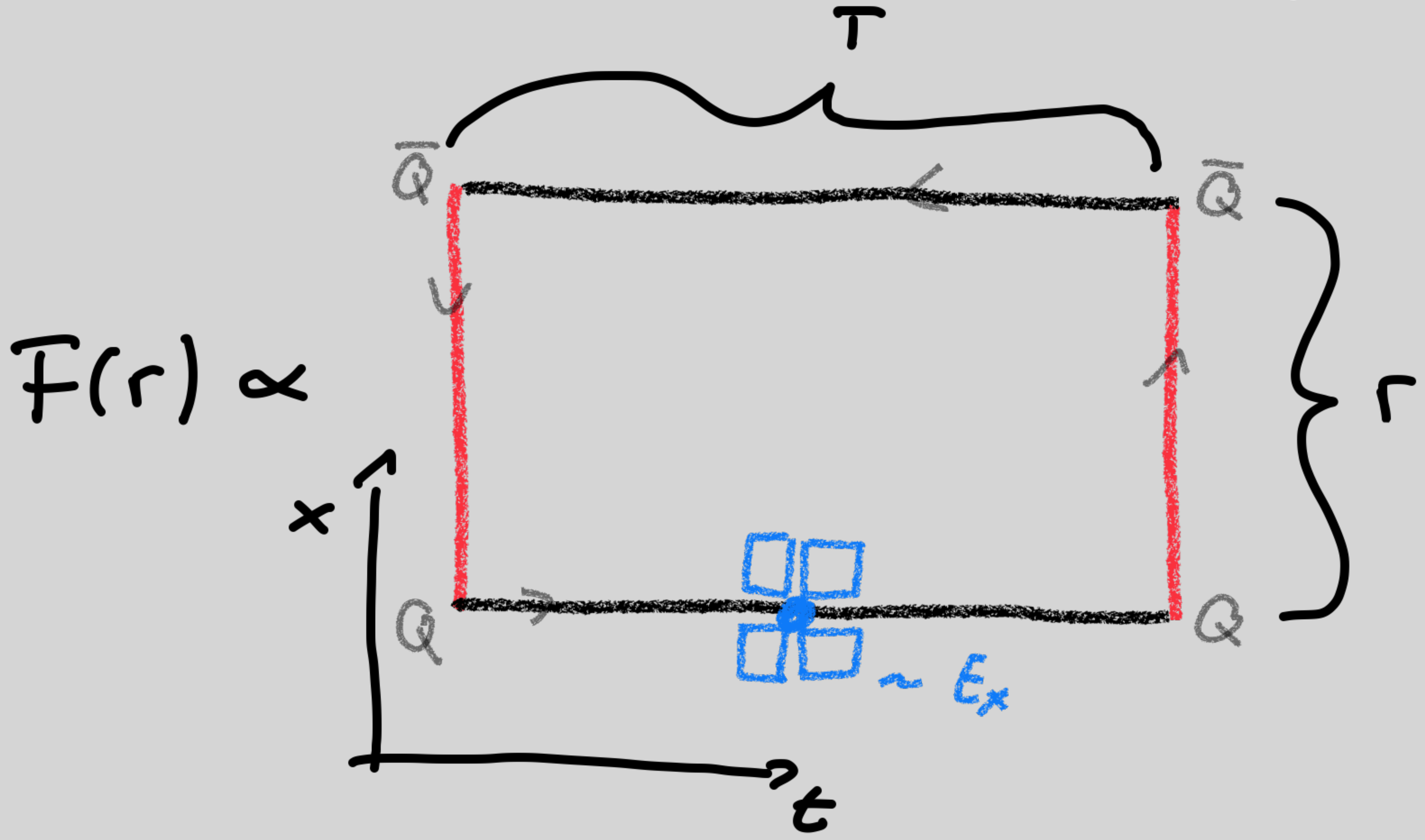
# Setup: Discretization of the $E$-field insertion

- Clover discretization of $E$

$$E_i = \frac{1}{2iga^2}\left(\Pi_{i0} - \Pi_{i0}^\dagger\right)$$

$$\Pi_{\mu\nu} = \frac{1}{4}(P_{\mu,\nu} + P_{\nu,-\mu} + P_{-\mu,-\nu} + P_{-\nu,\mu})$$




- $E$ has finite size on the lattice

- The self energy contribution of $E$ converges slowly to continuum

  See e.g. Lepage et.al.PRD48 (1993), G.Bali Phys.Rept. 343 (2001), and many others…

  → need lattice-only renormalization $Z_E$

- Correlator has a bad signal-to-noise ratio

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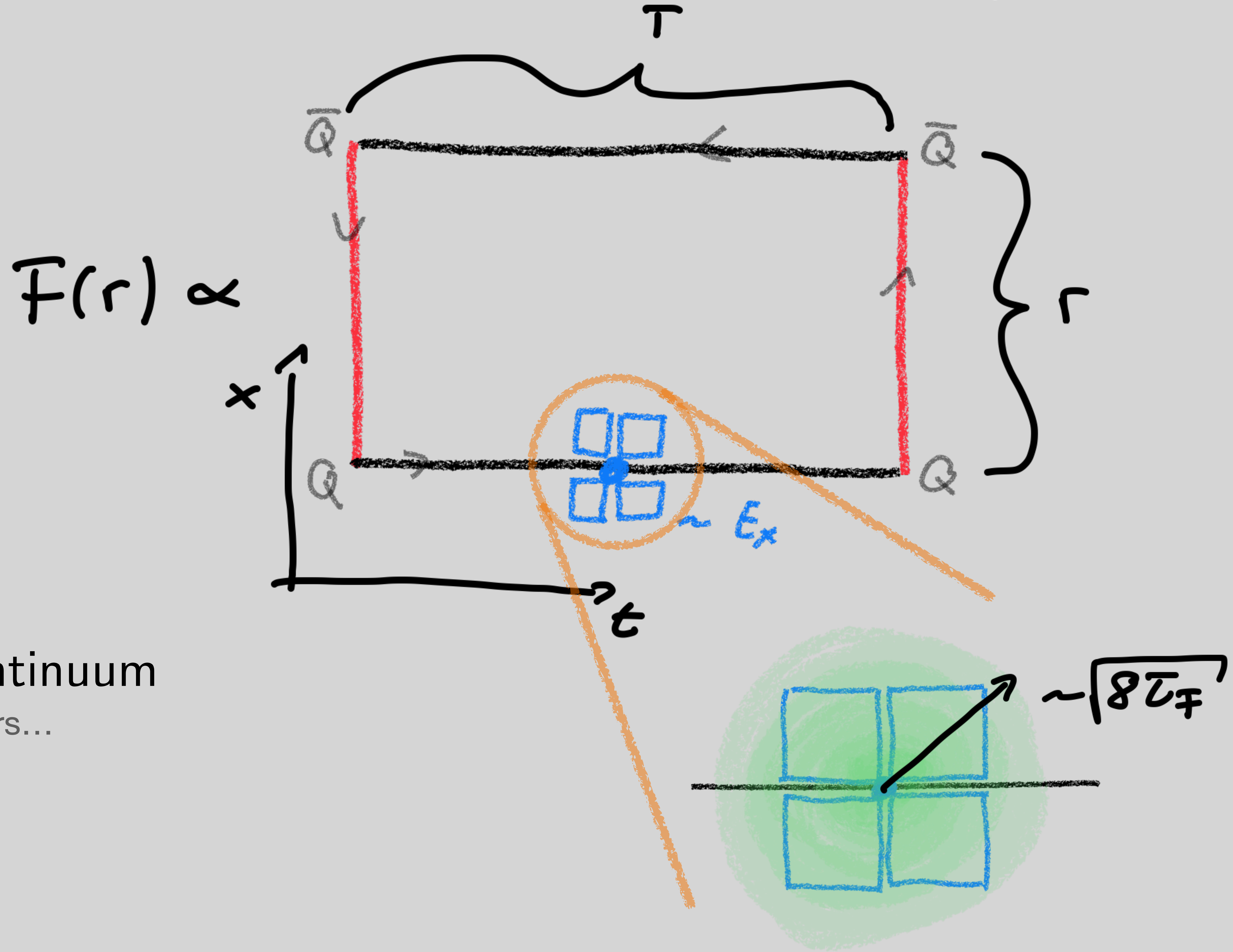


We use Gradient flow to target the renormalization and the Signal-to-noise ratio problems, new scale: flow time $\tau_F$

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# Setup: Continuum results

- One-loop calculation of the flowed force is known
  N. Brambilla, HS Chung, A. Vairo, JP Wang JHEP01 (2022)184

- Relevant scale:

$$\mu_b = \frac{1}{\sqrt{r^2 + 8b\tau_F}} \qquad \mu_0 = \frac{1}{r} \qquad \mu_1 = \frac{1}{\sqrt{r^2 + 8\tau_F}} \qquad \left( \lim_{\tau_F \to 0} \mu_b \to \mu_0 \right)$$

- We focus on the $n_f = 0$ result
- Small $\tau_F$ expansion:

$$r^2 F(r, \tau_F) \approx r^2 F(r, \tau_F = 0) + \frac{\alpha_S^2 C_F}{4\pi} \underbrace{\left[-12\beta_0 - 6C_A c_L\right]}_{8n_f} \frac{\tau_F}{r^2} \qquad c_L = -\frac{22}{3}$$

At small flow time the force is constant in pure gauge ($n_f = 0$)

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# Lattice results: setup and parameters

- Parameters:

| $N_S$ | $N_T$ | $\beta$ | $a$[fm] | $t_0/a^2$ | $N_{\mathrm{conf}}$ | Label |
|---|---|---|---|---|---|---|
| 20 | 40 | 6.284 | 0.060 | 7.868(8) | 6000 | L20 |
| 26 | 52 | 6.481 | 0.046 | 13.62(3) | 6000 | L26 |
| 30 | 60 | 6.594 | 0.040 | 18.10(5) | 6000 | L30 |
| 40 | 80 | 6.816 | 0.030 | 32.45(7) | 3300 | L40 |

- Pure gauge ensembles produced with overrelaxation and heatbath
- Scale setting with $(r_0 = 0.5\ \mathrm{fm})$

$$\ln(a/r_0) = -1.6804 - 1.7331(\beta - 6) + 0.7849(\beta - 6)^2 - 0.4428(\beta - 6)^3$$

S.Necco & R.Sommer Nucl.Phys. B622 (2002)

- Gradient flow with **fixed** and **adaptive** solver, with **Symanzik** action

Bazavov and Chuna 2101.05320 (2021)

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# Lattice results: Impact of gradient flow

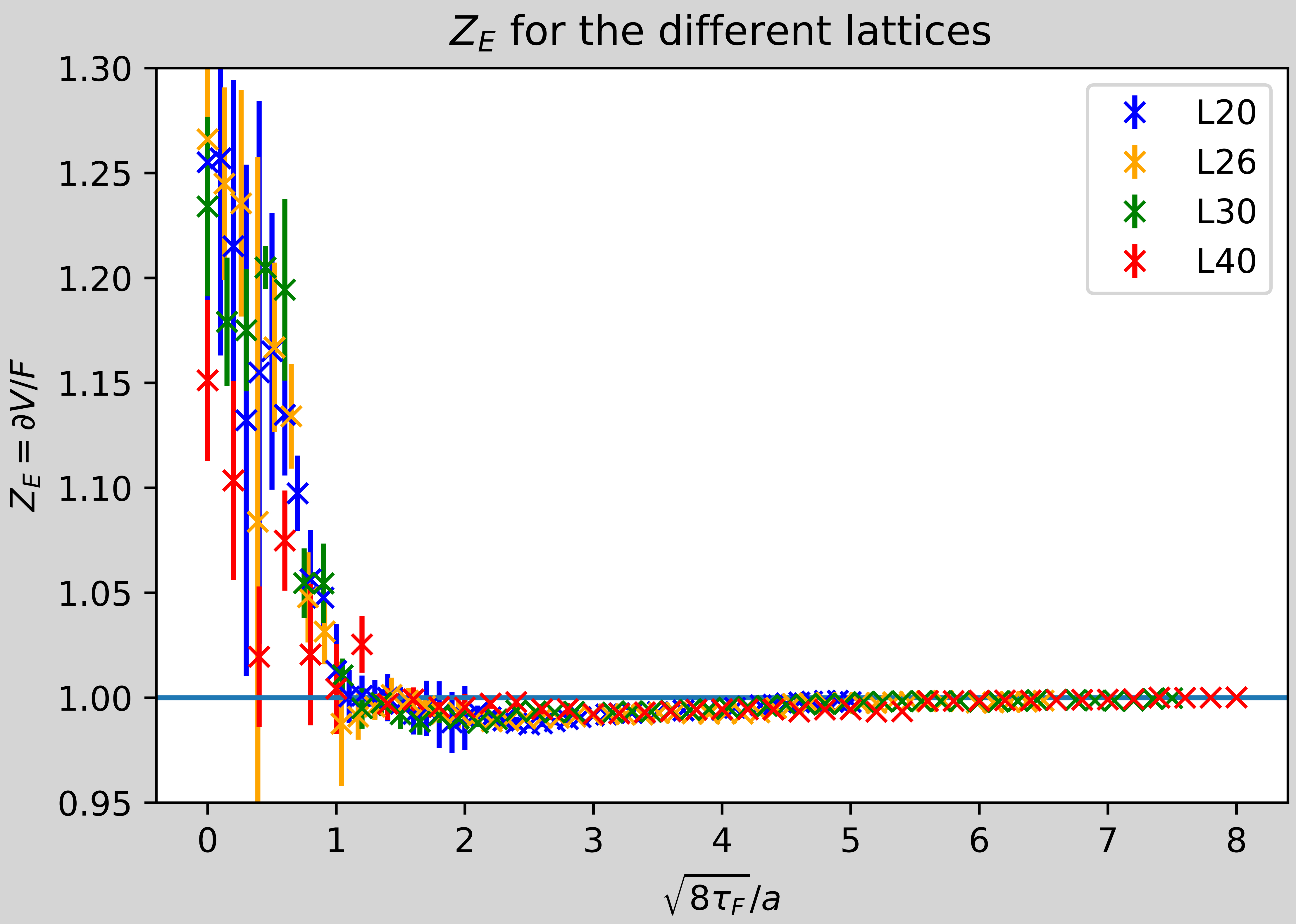
- Non-perturbative determination of $Z_E$:

$$Z_E(r) = \frac{\partial E(r)}{F(r)}$$

- $Z_E$ has small dependence on $r$
  Brambilla et.al. Phys.Rev.D 105 (2022)

- Examine the flowed $Z_E$

- $Z_E \to 1$ for flow radius $\sqrt{8\tau_F} > a$

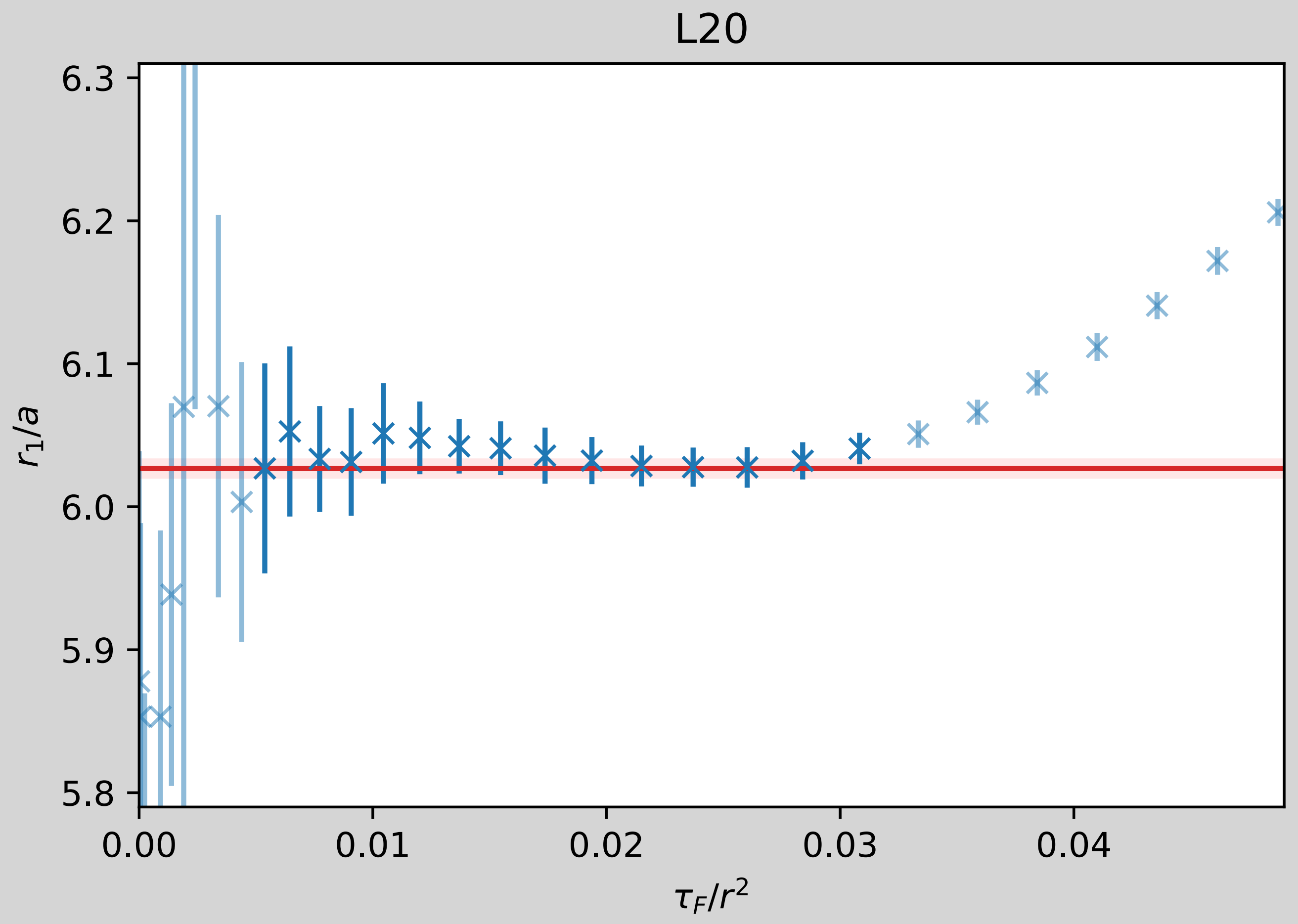


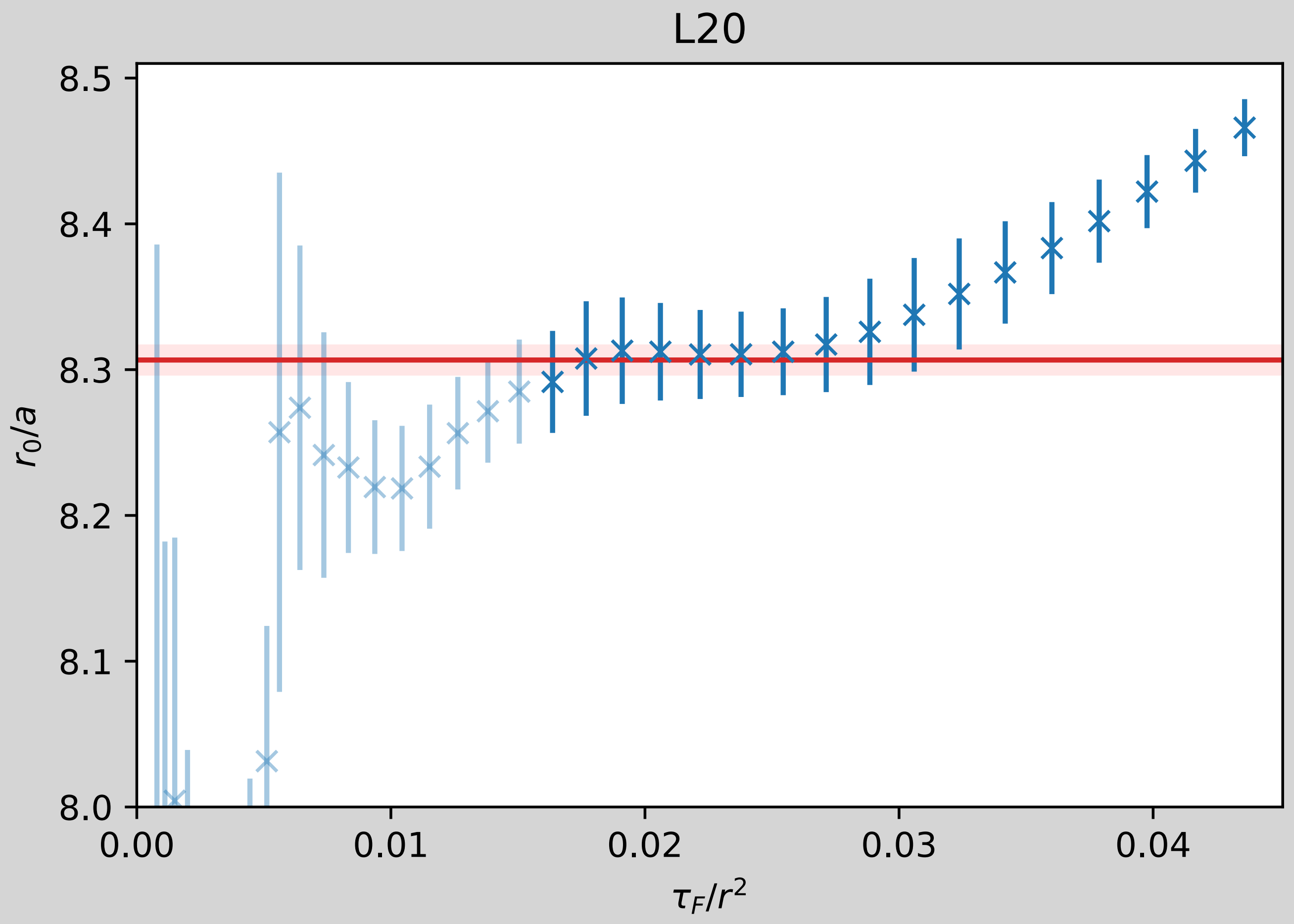

Gradient flow renormalizes non-perturbatively field insertions for $\sqrt{8\tau_F} > a$

# Lattice results: $r_i$ scale

- Scale defined as

$$r^2 F(r)\big|_{r=r(c)} = c \qquad r_0 \equiv r(1.65) \qquad r_1 \equiv r(1.0)$$

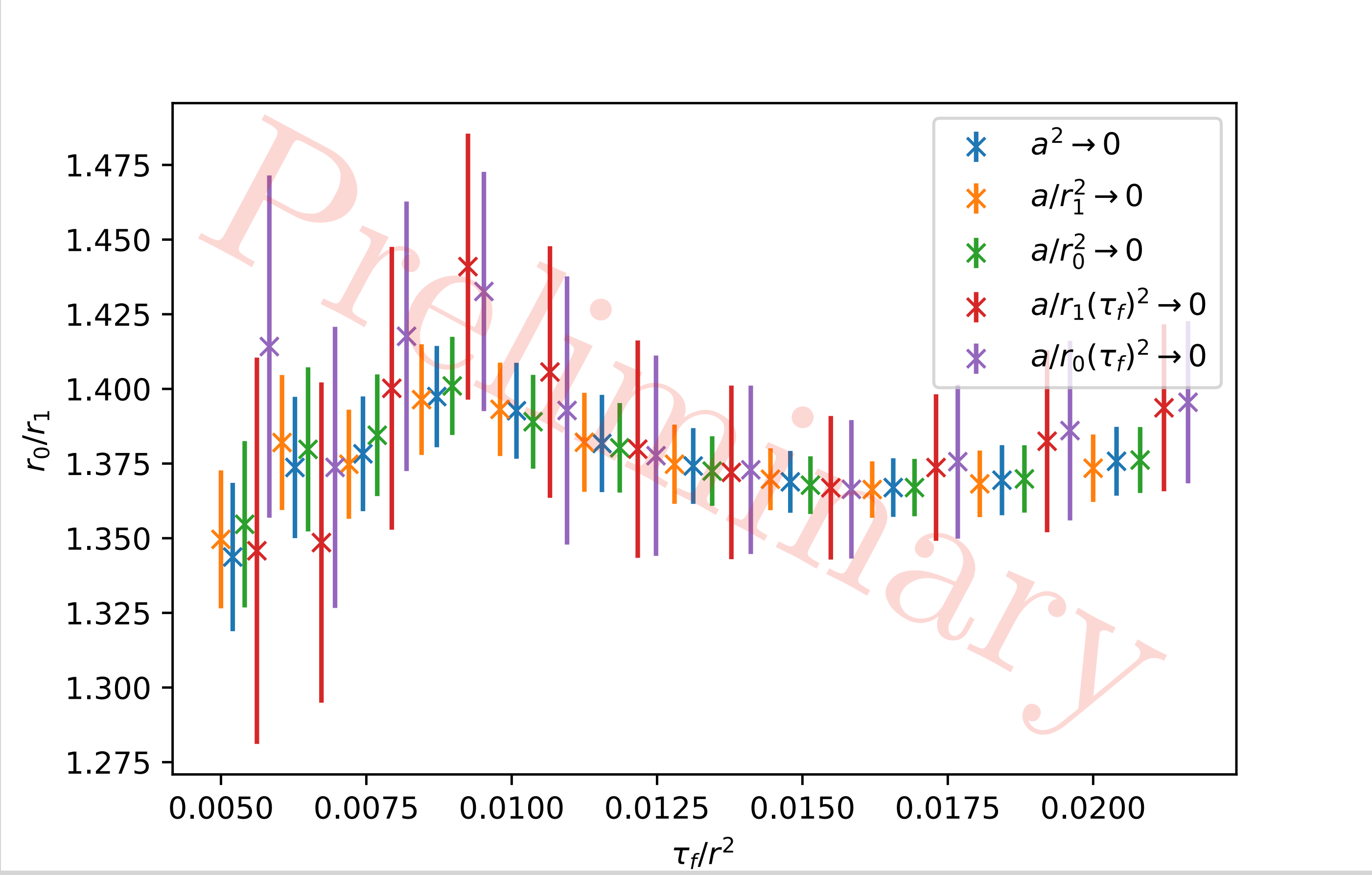



- $r_i(\tau_F)$ approaches a plateau in a certain flow time range

- Continuum limit of the ratio $r_0/r_1$ at finite flow time, constant zero flow time limit:

$$r_0/r_1 = 1.38(3)$$

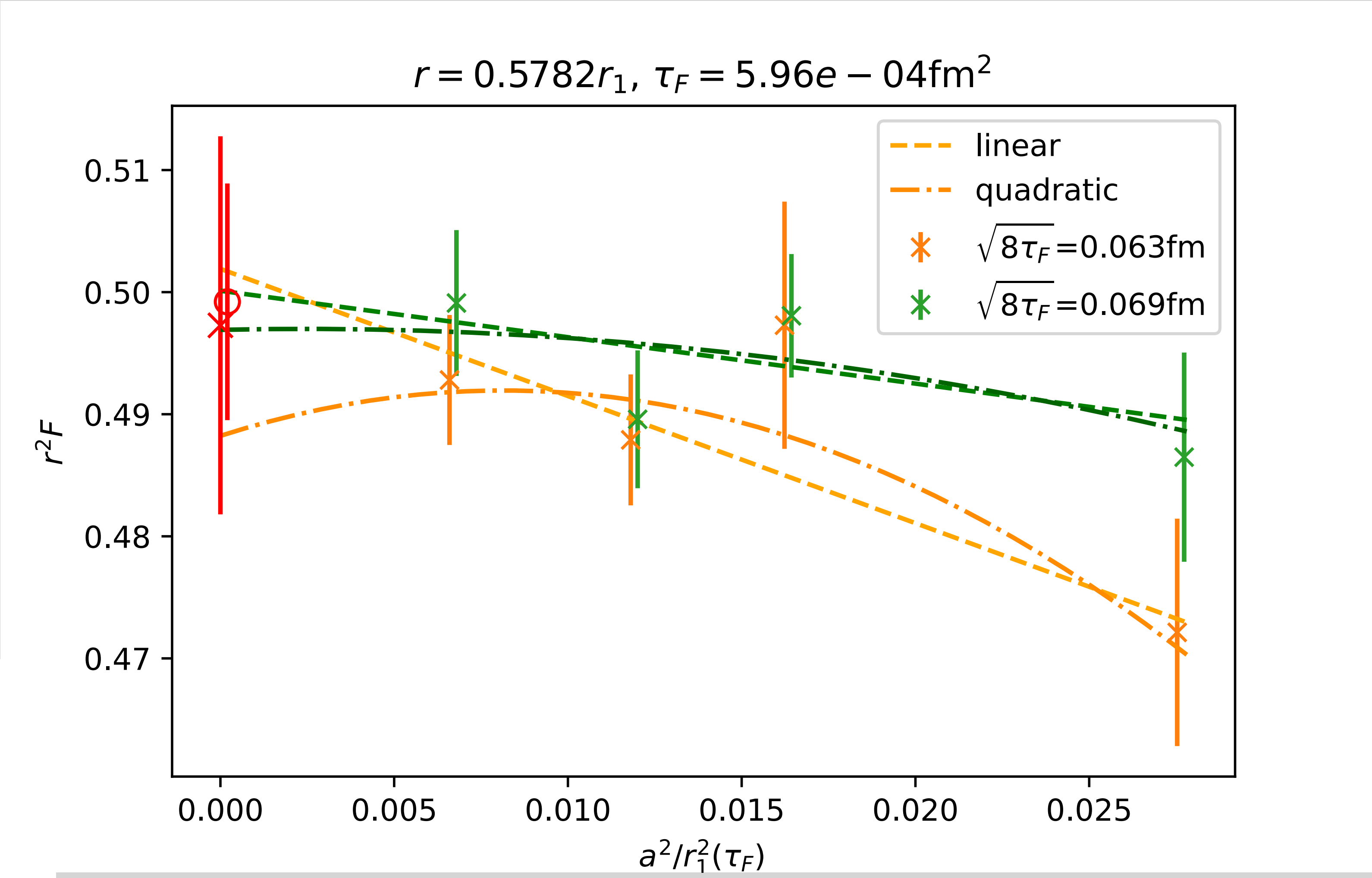


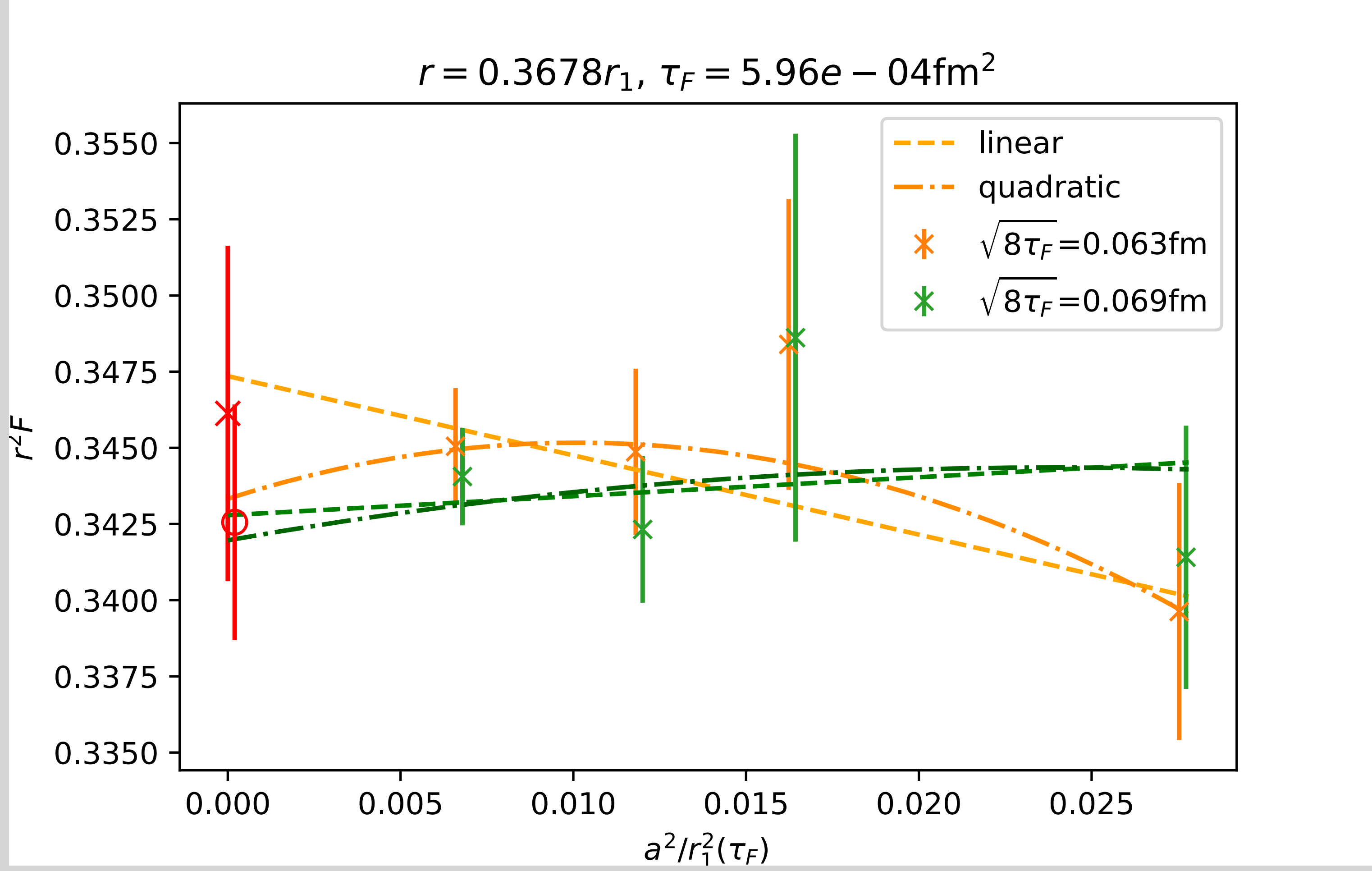
# Lattice results: Continuum limit

- Tree level improvement:

$$F^{\mathrm{Impr}}_{\mathrm{Latt}}(r,\tau_F) = \frac{F^{\mathrm{Tree}}_{\mathrm{Cont}}}{F^{\mathrm{Tree}}_{\mathrm{Latt}}} \cdot F^{\mathrm{Meas}}_{\mathrm{Latt}}$$

- Continuum limit at fixed $r$ and fixed $\tau_F$
- Akaike average of linear and quadratic in $a^2/r_1(\tau_F)^2$ limit
- Restrict to $\chi^2/dof < 3.6$




Direct force measurement with gradient flow leads to reliable continuum limits

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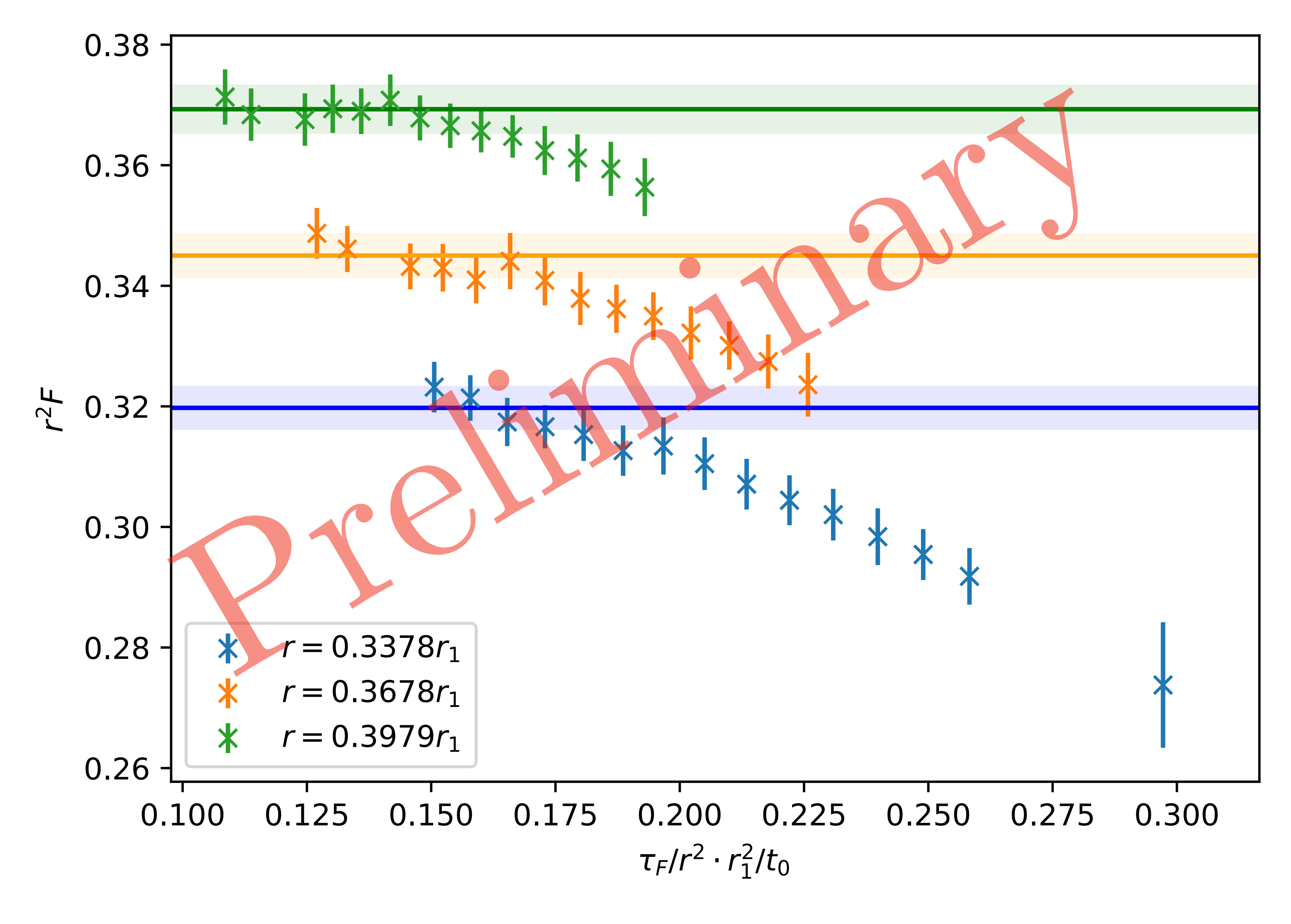
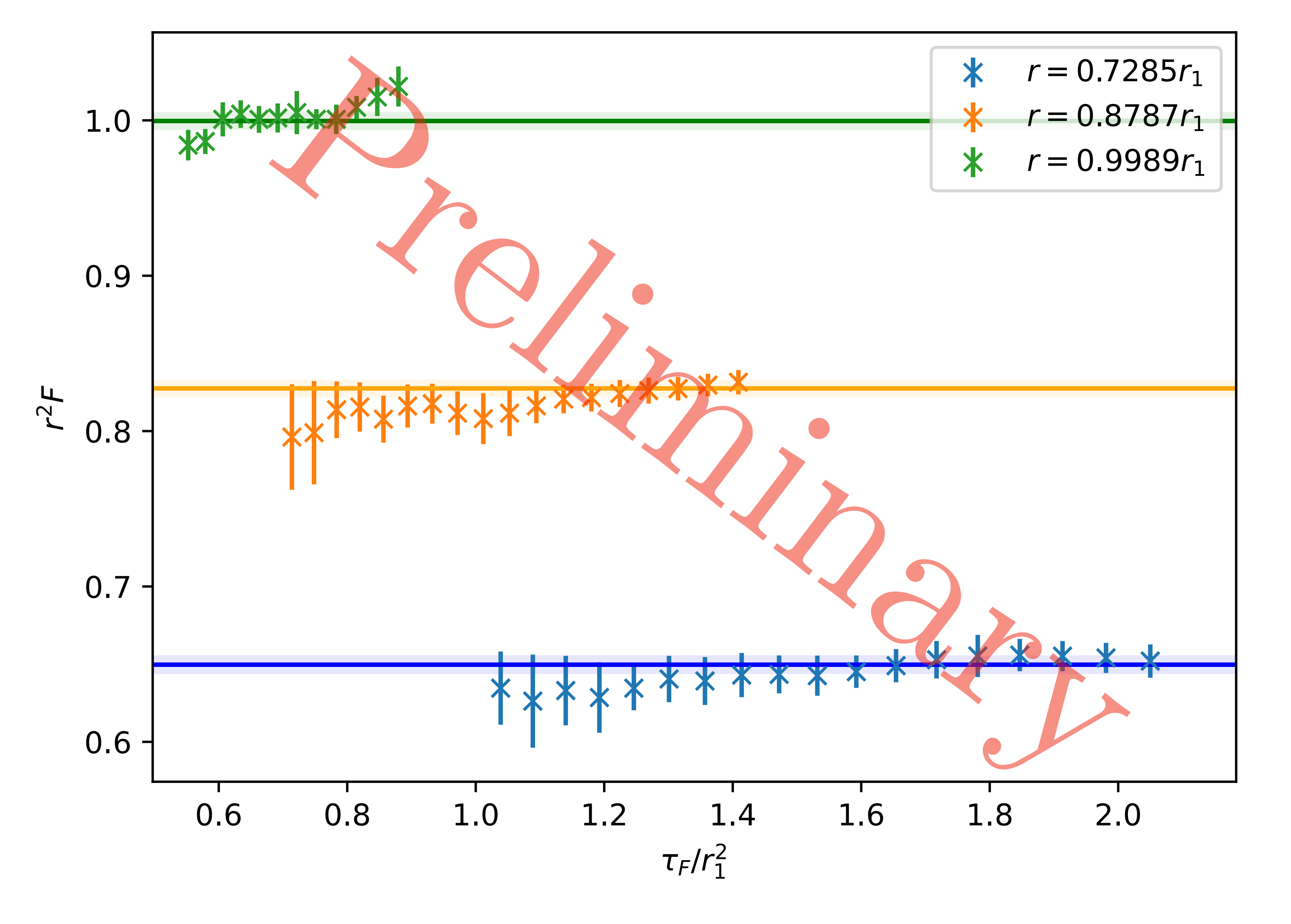
# Lattice results: Continuum results, constant zero flow time limit




- Constant zero flow time limits for $r/r_1 > 0.338$

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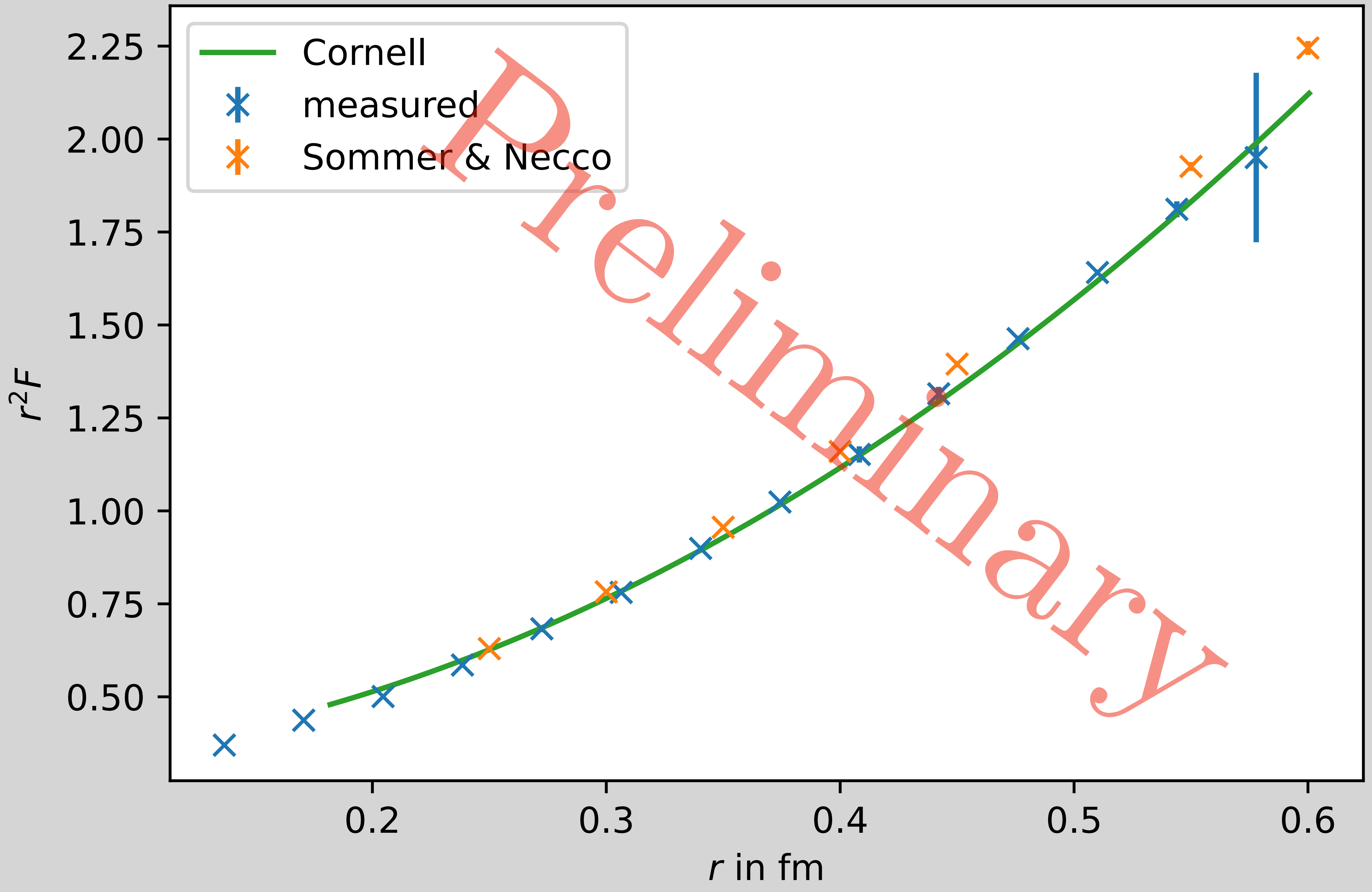
# Lattice results: Continuum results at large $r$, zero $\tau_F$

- Close to force from Necco & Sommer
- Cornell fit $r^2F(r) = A + \sigma r^2$
- $\sigma = 5.25(3)\ \mathrm{fm}^{-2}$, literature: $5.5\ \mathrm{fm}^{-2}$
- $A = 0.277(3)$
  Koma & Koma: $A = 0.2808(5)$




Direct force measurement with gradient flow gives reliable large $r$ results

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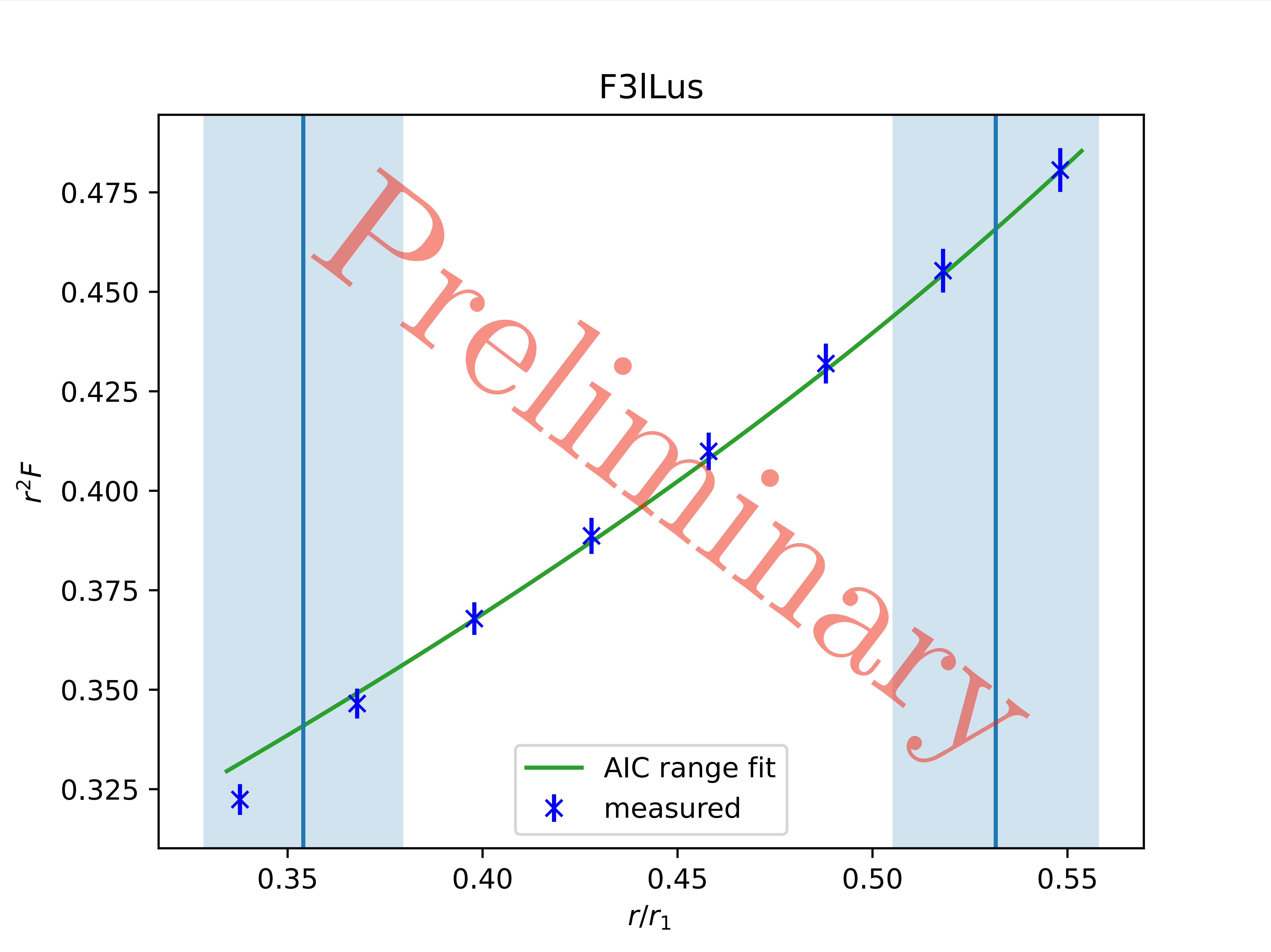
# Lattice results: Continuum results at small $r$, zero $\tau_F$

- Fit PT equation at different orders, fit parameter: $\Lambda_0 \equiv \Lambda_{\overline{\mathrm{MS}}}^{n_f=0}$
- Perform Akaike averaging for different $r$ fit windows
- Vertical lines correspond to the effective lower and upper fit bounds
- Fit result:

| | | | |
|---|---|---|---|
| F1l: | $r_1\Lambda_0 = 0.588(8)$ | | $r_0\Lambda_0 = 0.813(21)$ |
| F2l: | $r_1\Lambda_0 = 0.480(7)$ | $\frac{r_0}{r_1} = 1.38(3)$ | $r_0\Lambda_0 = 0.664(17)$ |
| F2lLus: | $r_1\Lambda_0 = 0.505(8)$ | $\Rightarrow$ | $r_0\Lambda_0 = 0.697(19)$ |
| F3l: | $r_1\Lambda_0 = 0.443(8)$ | | $r_0\Lambda_0 = 0.612(18)$ |
| F3lLus: | $r_1\Lambda_0 = 0.453(8)$ | | $r_0\Lambda_0 = 0.626(17)$ |

FLAG: $r_0\Lambda_0 = 0.624(36)$

Preliminary



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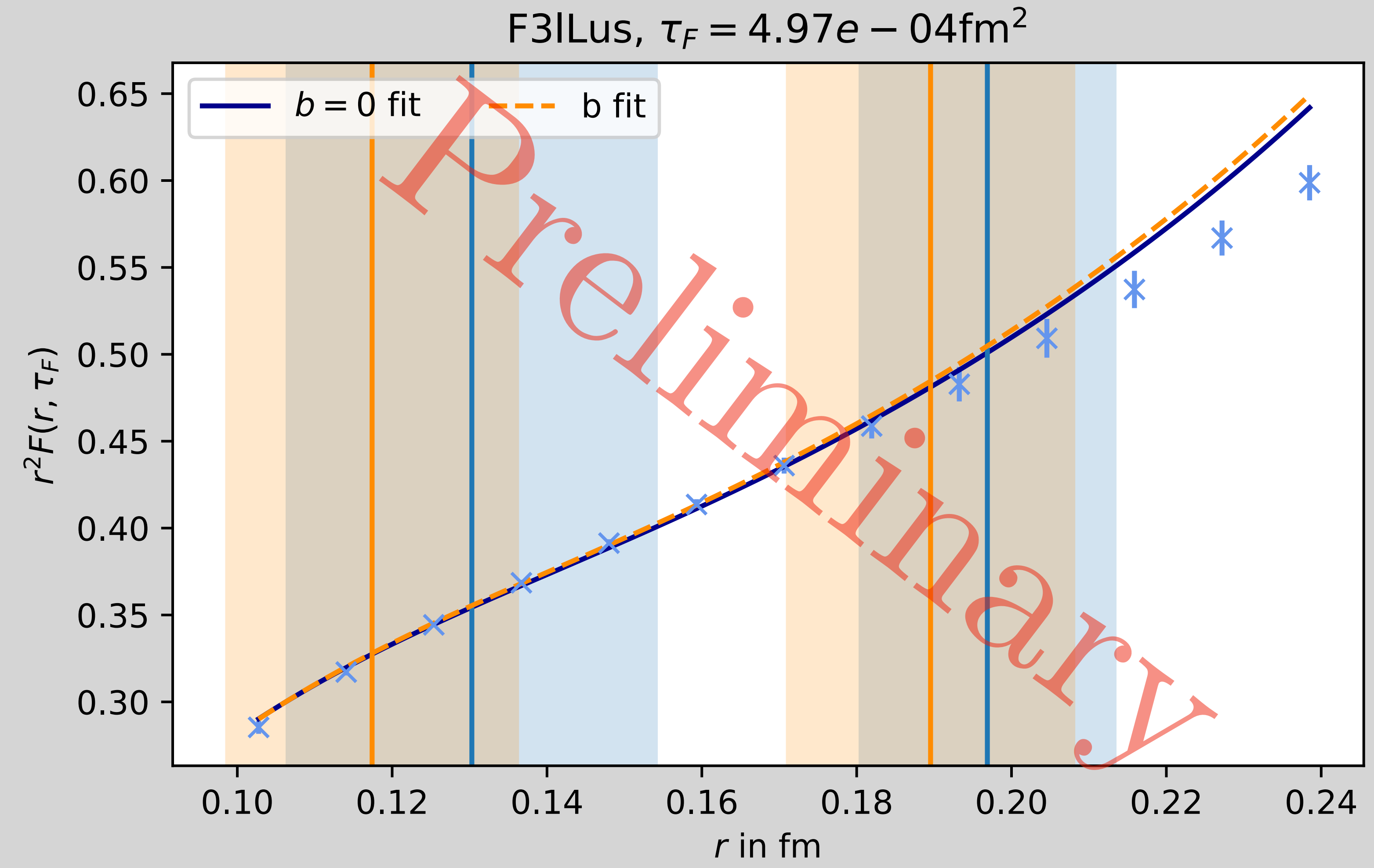
# Lattice results: Continuum results at small $r$, finite $\tau_F$

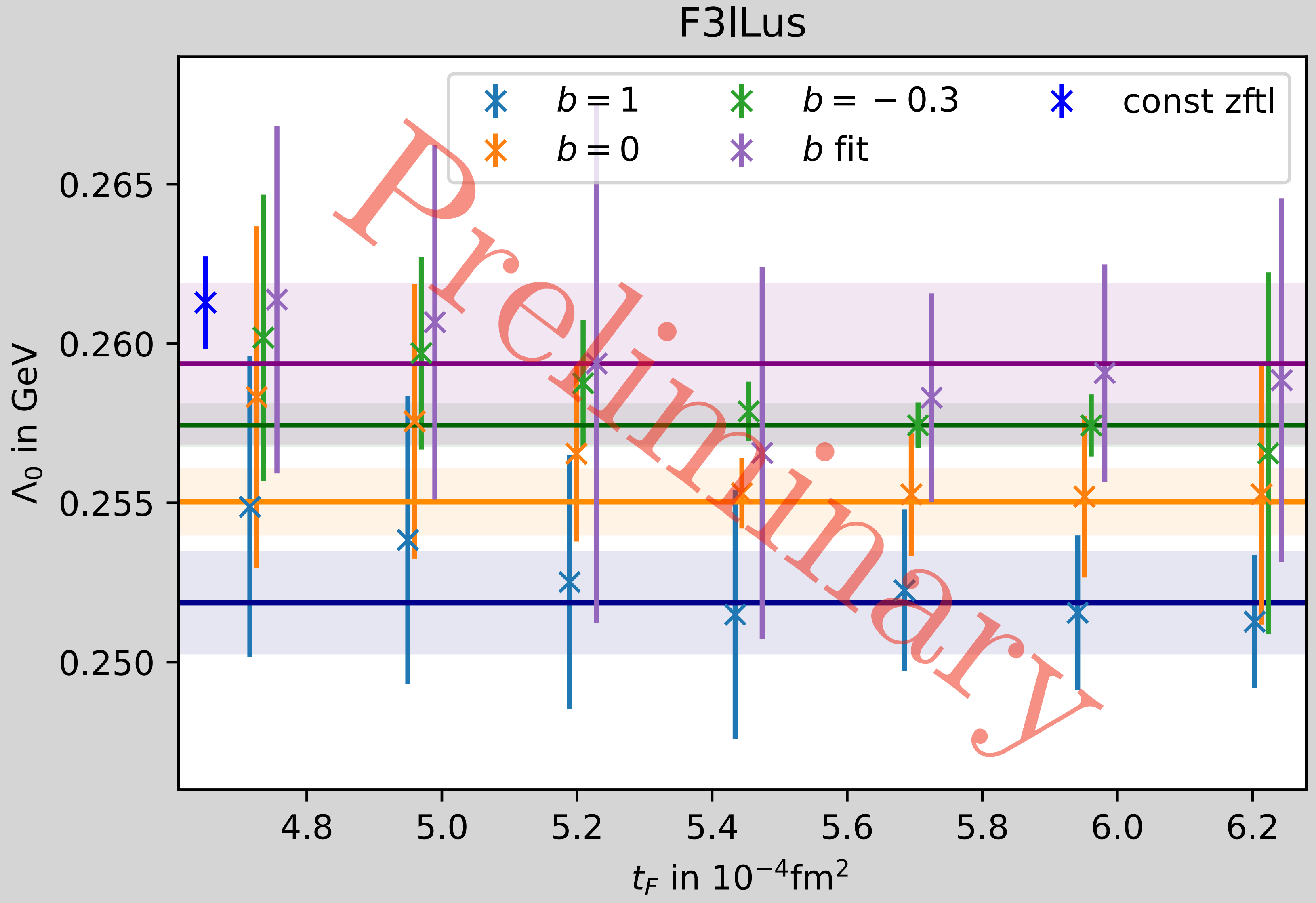
- Combine PT at zero flow time with 1-loop flow time behavior:

$$r^2F^{\text{order}}(r,\tau_F) \equiv r^2F^{\text{order}}(r,\tau_F=0) + f^{1-\text{loop}}(r,\tau_F)$$

$$f^{1-\text{loop}}(r,\tau_F) \equiv r^2F^{1-loop}(r,\tau_F) - r^2F^{1-\text{loop}}(r,\tau_F=0)$$

- Select valid points ($Z_E \approx 1$)
- Multiple values for $\Lambda_0$ at different flow times
- $\Lambda_0(\tau_F)$ is constant within the errors
- The value depends on the scale choice for $\mu_b$



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# Conclusion

- Summary and observations:
  - Gradient flow renormalizes field insertions
  - Gradient flow improves qualitatively the signal-to-noise ratio
  - Direct force measurement with gradient flow can be used for scale setting
  - Good preparation for future applications in NREFTs

- For the future:
  - Go to finer lattices
  - Other operators with field insertions
  - Extend to dynamical fermions

**Thank you for your attention!**