Constructing approximate semi-analytic and machine-learned trivializing maps for lattice gauge theory

Julian M. Urban  lettucefield.org

MIT / IAIIF
Talks by collaborators

Thursday, 15:10

Practical applications of machine-learned flows on gauge fields

Denis Boyda

Daniel Hackett

Ryan Abbott

Enhancing expressivity in machine learning: application of normalizing flows in LQCD

Michael Albergo

Flow-based sampling for lattice field theories

Gurtej Kanwar

Monday, 10:00

Multiscale normalizing flows for gauge theories

Monday, 16:20
Basic concepts

- Change of variables \( Z = \int dU \ e^{-S} = \int dV \ e^{-S} + \log |J| \)
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- Trivializing map condition $- S + \log |J| = 0$ minimizes relative entropy (Kullback-Leibler divergence) w.r.t. Haar measure $\rightarrow$ full thermodynamic integration
Basic concepts

- Change of variables \( Z = \int dU e^{-S} = \int dV e^{-S + \log |J|} \)

- Trivializing map condition \(- S + \log |J| = 0\) minimizes relative entropy (Kullback-Leibler divergence) w.r.t. Haar measure \(\rightarrow\) full thermodynamic integration

- Less ambitious: \(- S + \log |J| = -S' \rightarrow\) partial thermodynamic integration
Basic concepts

- Change of variables $Z = \int dU \, e^{-S} = \int dV \, e^{-S + \log |J|}$

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- Less ambitious: $-S + \log |J| = -S' \rightarrow$ partial thermodynamic integration

- $S' \equiv S_{\text{defect}} \rightarrow$ restoration of topological ergodicity (cf. Dan Hackett’s talk)
Basic concepts

- Change of variables $Z = \int dU \, e^{-S} = \int dV \, e^{-S + \log |J|}$

- Trivializing map condition $-S + \log |J| = 0$ minimizes relative entropy (Kullback-Leibler divergence) w.r.t. Haar measure $\rightarrow$ full thermodynamic integration

- Less ambitious: $-S + \log |J| = -S' \rightarrow$ partial thermodynamic integration
  - $S' \equiv S_{\text{defect}} \rightarrow$ restoration of topological ergodicity (cf. Dan Hackett’s talk)
  - $S' \equiv S_{\Lambda} \rightarrow$ renormalization group interpretation (cf. backup slides)

Lüscher [arXiv:0907.5491]

(f) Renormalization group. By composing the trivializing map $U = \mathcal{F}_1(V)$ in the Wilson theory with its inverse at another value of the gauge coupling, one obtains a group of transformations whose only effect on the action is a shift of the coupling. The locality properties of these transformations are not transparent, however, and could be quite different from the ones of a Wilsonian “block spin” transformation.
Wilson vs Kadanoff

Huang, Statistical Mechanics

Fig. 18.5 Coarse-graining in momentum space and in real space. In the former, one effectively lowers the cutoff. In the latter, one blots out finer details, enlarging the effective lattice spacing.
Archeological survey of trivializing maps for LGT
Recursion equations in gauge field theories

A. A. Migdal

L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences
(Submitted April 28, 1975)
Zh. Eksp. Teor. Fiz. 69, 810-822 (September 1975)

An approximate recursion equation is formulated, describing the scale transformation of the effective action of a gauge field. In two-dimensional space-time the equation becomes exact. In four-dimensional theories it reproduces asymptotic freedom to an accuracy of 30% in the coefficients of the B-function. In the strong-coupling region the B-function remains negative and this results in an asymptotic prison in the infrared region. Possible generalizations and applications to the quark-gluon gauge theory are discussed.

PACS numbers: 11.10.Np

Phase transitions in gauge and spin-lattice systems

A. A. Migdal

L. D. Landau Theoretical Physics Institute, USSR Academy of Sciences
(Submitted June 11, 1975)
Zh. Eksp. Teor. Fiz. 69, 1457-1465 (October 1975)

A simple recursion equation giving an approximate description of critical phenomena in lattice systems is proposed. The equations for a d-dimensional spin system and a 2d-dimensional gauge system coincide. An interesting consequence is the zero transition temperature in the two-dimensional Heisenberg model and four-dimensional Yang-Mills model; this corresponds to asymptotic freedom in field theory.

Notes on Migdal's Recursion Formulas*

LEO P. KADANOFF†

The James Franck Institute, The University of Chicago, Chicago, Illinois 60637
Received March 24, 1976

A set of renormalization group recursion formulas which were proposed by Migdal are rederived, reinterpreted, and critically analyzed. The new derivation shows the connection between these formulas and previous work on renormalization via decimation.

MIGDAL-KADANOFF RECURRENCE RELATIONS IN SU(2) AND SU(3) GAUGE THEORIES

Michael NAUENBERG

Physics Department, University of California, Santa Cruz, California 95060 and Institute for Theoretical Physics, University of California, Santa Barbara, California 93106, USA

Doug TOUSSAINT

Physics Department and Institute for Theoretical Physics, University of California, Santa Barbara, California 93106, USA

Received 23 October 1980

We study the Migdal recursion relations and the reformulation due to Kadanoff for SU(2) and SU(3) lattice gauge theory, using analytic approximations for large and small couplings and numerical methods for all couplings. In SU(2) we obtain the beta function and the expectation value of the plaquette, which is compared with recent Monte Carlo results. In analogy to U(1), we find that a Villain form (periodic gaussian) for the exponential of the plaquette action is a good approximation to the result of the Migdal renormalization transformation. We also perform some calculations in SU(3) and find that its behavior is similar to SU(2).

GROUP INTEGRATION FOR LATTICE GAUGE THEORY AT LARGE N AND AT SMALL COUPLING*

Richard C. BROWER and Michael NAUENBERG

Physics Department, University of California, Santa Cruz, California 94064, USA

Received 15 July 1980

We consider the fundamental SU(N) invariant integrals encountered in Wilson's lattice QCD with an eye to analytical results for N \to \infty and approximations for small g^2 at fixed N. We develop a new semiclassical technique starting from the Schwinger-Dyson equations cast in differential form to give an exact solution to the single-link integral for N \to \infty. The third-order phase transition discovered by Gross and Witten for two-dimensional QCD occurs here for any.
Contemporary survey of trivializing maps for LGT

Trivializing maps, the Wilson flow and the HMC algorithm

Martin Lüscher

Tackling critical slowing down using global correction steps with equivariant flows: the case of the Schwinger model

Jacob Finkenrath

Use of Schwinger-Dyson equation in constructing an approximate trivializing map

Decimation Map in 2D for accelerating HMC

Monday, 16:00

Peter Boyle, Taku Izubuchi, Luchang Jin, Chulwoo Jung, Christoph Lehner, and Akio Tomiya

Nobuyuki Matsumoto

Learning Trivializing Gradient Flows for Lattice Gauge Theories

Simone Bacchio, Pan Kessel, Stefan Schaefer, and Lorenz Vait

Equivariant flow-based sampling for lattice gauge theory

Gurtej Kanwar, Michael S. Albergo, Denis Boyd, Kyle Cranmer, Daniel C. Hackett, Sébastien Racanière, Danilo Jimenez Rezende, and Phiala E. Shanahan

Sampling using SU(N) gauge equivariant flows

Denis Boyd, Gurtej Kanwar, Sébastien Racanière, Danilo Jimenez Rezende, Michael S. Albergo, Kyle Cranmer, Daniel C. Hackett, and Phiala E. Shanahan

Flow-based sampling in the lattice Schwinger model at criticality

Michael S. Albergo, Denis Boyd, Kyle Cranmer, Daniel C. Hackett, Gurtej Kanwar, Sébastien Racanière, Danilo J. Rezende, Fernando Romero-López, Phiala E. Shanahan, and Julian M. Urban

Gauge-equivariant flow models for sampling in lattice field theories with pseudofermions

Ryan Abbott, Michael S. Albergo, Denis Boyd, Kyle Cranmer, Daniel C. Hackett, Gurtej Kanwar, Sébastien Racanière, Danilo J. Rezende, Fernando Romero-López, Phiala E. Shanahan, Betsy Tian, and Julian M. Urban

Sampling QCD field configurations with gauge-equivariant flow models


Normalizing flows for lattice gauge theory in arbitrary space-time dimension


To be continued ...
\((\text{Un-})\text{trivializing } (1+1)d\text{ U}(1)\text{ LGT}\)

\[ \phi_k \in [0, 2\pi), \quad S = -\beta \cos \left( \sum_{k=1}^{4} \phi_k \right) \]

\[ Z = \prod_{k=1}^{4} \left( \int_{0}^{2\pi} d\phi_k \right) \exp \left( -S \left( \sum_{k=1}^{4} \phi_k \right) \right) \]
(Un-)trivializing \((1+1)d\) U(1) LGT

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- Change of variables: \( \chi(\phi_1) = \phi_1 + \sum_{k=2}^{4} \phi_k \in \left[ \sum_{k=2}^{4} \phi_k, 2\pi + \sum_{k=2}^{4} \phi_k \right] \equiv [\chi, \bar{\chi}] \rightarrow \frac{\partial \chi}{\partial \phi_1} = 1 \)

\[ \rightarrow Z = \int_{\chi}^{{\bar{\chi}}} d\chi \int_{0}^{2\pi} d\phi_2 d\phi_3 d\phi_4 \exp \left( \beta \cos(\chi) \right) \equiv \int_{0}^{2\pi} d\chi d\phi_2 d\phi_3 d\phi_4 \exp \left( \beta \cos(\chi) \right) \]
\( (\text{Un-})\text{trivializing (1+1)d U(1) LGT} \)

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- Trivialization: \( \mathcal{I}(\eta) = \int_{0}^{\eta} dx \exp \left( \beta \cos(x) \right), \quad \chi' = \frac{\mathcal{I}(\chi) \mathcal{I}(\chi')}{\mathcal{I}(2\pi)} \rightarrow \frac{\partial \chi'}{\partial \chi} = \frac{2\pi}{\mathcal{I}(2\pi)} \exp \left( \beta \cos(\chi) \right) \)

\[ \rightarrow Z = \int_{0}^{2\pi} d\chi' d\phi_2 d\phi_3 d\phi_4 \left| \frac{\partial \chi'}{\partial \chi} \right|^{-1} \exp \left( \beta \cos(\chi) \right) = \frac{\mathcal{I}(2\pi)}{2\pi} \int_{0}^{2\pi} d\chi' d\phi_2 d\phi_3 d\phi_4 \]
(Un-)trivializing $(1+1)d$ U(1) LGT

\[ \phi_k \in [0, 2\pi), \quad S = -\beta \cos \left( \sum_{k=1}^{4} \phi_k \right) \]

\[ Z = \prod_{k=1}^{4} \left( \int_{0}^{2\pi} d\phi_k \right) \exp(-S \left( \sum_{k=1}^{4} \phi_k \right)) \]

- Change of variables: \( \chi(\phi_1) = \phi_1 + \sum_{k=2}^{4} \phi_k \in \left[ \sum_{k=2}^{4} \phi_k, 2\pi + \sum_{k=2}^{4} \phi_k \right] \equiv [\chi, \bar{\chi}] \rightarrow \frac{\partial \chi}{\partial \phi_1} = 1 \)

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\[ \rightarrow Z = \int_{0}^{2\pi} d\chi' d\phi_2 d\phi_3 d\phi_4 \left| \frac{\partial \chi'}{\partial \chi} \right|^{-1} \exp \left( \beta \cos(\chi) \right) = \frac{\mathcal{I}(2\pi)}{2\pi} \int_{0}^{2\pi} d\chi' d\phi_2 d\phi_3 d\phi_4 \]

- Change of variables: \( \phi_1'(\chi') = \chi' - \sum_{k=2}^{4} \phi_k \in \left[ -\sum_{k=2}^{4} \phi_k, 2\pi - \sum_{k=2}^{4} \phi_k \right] \equiv [\phi'_1, \bar{\phi}_1'] \rightarrow \frac{\partial \phi_1'}{\partial \chi'} = 1 \)

\[ \rightarrow Z = \frac{\mathcal{I}(2\pi)}{2\pi} \int_{\phi'_1}^{\bar{\phi}_1'} d\phi_1' \int_{0}^{2\pi} d\phi_2 d\phi_3 d\phi_4 \equiv \frac{\mathcal{I}(2\pi)}{2\pi} \int_{0}^{2\pi} d\phi_1' d\phi_2 d\phi_3 d\phi_4 \]
(Un-)trivializing $(1+1)d\ U(1)\ LGT$

\[
\phi_k \in [0, 2\pi), \quad S = -\beta \cos \left( \sum_{k=1}^{4} \phi_k \right)
\]

\[
Z = \prod_{k=1}^{4} \left( \int_{0}^{2\pi} d\phi_k \right) \exp \left( -\beta \sum_{k=1}^{4} \phi_k \right)
\]

- Change of variables: \( \chi(\phi_1) = \phi_1 + \sum_{k=2}^{4} \phi_k \in \left[ \sum_{k=2}^{4} \phi_k, 2\pi + \sum_{k=2}^{4} \phi_k \right] \equiv [\chi, \chi'] \rightarrow \frac{\partial \chi}{\partial \phi_1} = 1 \)

  \text{change of variables: gauge fields (links) } \longleftrightarrow \text{ invariants (plaquettes)}

  \rightarrow Z = \int_{\chi}^{\chi'} d\chi \int_{0}^{2\pi} d\phi_2 d\phi_3 d\phi_4 \exp \left( \beta \cos(\chi) \right) = \int_{0}^{2\pi} d\chi d\phi_2 d\phi_3 d\phi_4 \exp \left( \beta \cos(\chi) \right)

- Trivialization: \( I(\eta) = \int_{0}^{\eta} dx \exp \left( \beta \cos(x) \right), \quad \chi' = 2\pi \frac{I(\chi)}{I(2\pi)} \rightarrow \frac{\partial \chi'}{\partial \chi} = \frac{2\pi}{I(2\pi)} \exp \left( \beta \cos(\chi) \right) \)

  (un-)trivialization via (inverse) cumulative distribution function

  \rightarrow Z = \int_{0}^{2\pi} d\chi' d\phi_2 d\phi_3 d\phi_4 \left| \frac{\partial \chi'}{\partial \chi} \right|^{-1} \exp \left( \beta \cos(\chi) \right) = \frac{I(2\pi)}{2\pi} \int_{0}^{2\pi} d\chi' d\phi_2 d\phi_3 d\phi_4

- Change of variables: \( \phi_1'(\chi') = \chi' - \sum_{k=2}^{4} \phi_k \in \left[ -\sum_{k=2}^{4} \phi_k, 2\pi - \sum_{k=2}^{4} \phi_k \right] \equiv [\phi_1', \phi_1] \rightarrow \frac{\partial \phi_1'}{\partial \chi'} = 1 \)

  \text{change of variables: invariants (plaquettes) } \longleftrightarrow \text{ gauge fields (links)}

  \rightarrow Z = \frac{I(2\pi)}{2\pi} \int_{\phi_1'}^{\phi_1} d\phi_1' \int_{0}^{2\pi} d\phi_2 d\phi_3 d\phi_4 \equiv \frac{I(2\pi)}{2\pi} \int_{0}^{2\pi} d\phi_1' d\phi_2 d\phi_3 d\phi_4
(Un-)trivializing (1+1)d U(1) LGT

$$\phi_k \in [0, 2\pi), \ S = -\beta \cos \left( \sum_{k=1}^{4} \phi_k \right)$$

$$Z = \prod_{k=1}^{4} \left( \int_{0}^{2\pi} d\phi_k \right) \exp(-S\left( \sum_{k=1}^{4} \phi_k \right))$$

---

inverse transform sampling the von Mises distribution

rejection sampling

inverse CDF
(Un-)trivializing \((1+1)d\) U(1) LGT
(Un-)trivializing $1+1d$ U(1) LGT

$\cos(\chi)$
(Un-)trivializing (1+1)d U(1) LGT
(Un-)trivializing (1+1)d U(1) LGT

$\cos(\chi)$

[Image of a color-coded matrix with 7 rows and columns, showing a pattern with colors ranging from yellow to purple, indicating a phase transition or state transition in a 1+1 dimensional U(1) LGT model.]
(Un-)trivializing (1+1)d U(1) LGT

\[ \cos(\chi) \]
(Un-)trivializing (1+1)d U(1) LGT

\[ \cos(\chi) \]

7

\[ +1 \]

\[ -1 \]
(Un-)trivializing (1+1)d U(1) LGT

\[ \cos(\chi) \]
(Un-)trivializing $(1+1)d$ U(1) LGT

\[ \cos(\chi) \]
(Un-)trivializing (1+1)d U(1) LGT

\[ \cos(\chi) \]
(Un-)trivializing $(1+1)d$ U(1) LGT

$\cos(\chi)$
(Un-)trivializing $(1+1)d$ $U(1)$ LGT

\[ \cos(\chi) \]
(Un-)trivializing (1+1)d U(1) LGT

\[ \cos(\chi) \]

Denis Boyda
Thursday, 14:50
(Un-)trivializing $(1+1)d$ U(1) LGT

16 × 16

Metropolizing defect:
- baseline
- 1st order improvement (effective 2 × 1 loop action)

acceptance rate
(indipendence sampling)

0 2 4 6 8 10 12
\( \beta \)

Denis Boyda
Thursday, 14:50
(Un-)trivializing (1+1)d U(1) LGT

16 × 16, β = 8

#sweeps
(Un-)trivializing (1+1)d SU(3) LGT

\[ U_k \in \text{SU}(3), \quad S = -\frac{\beta}{3} \text{Re} \text{Tr} \left( \prod_{k=1}^{4} U_k \right) \]

\[ Z = \prod_{k=1}^{4} \left( \int dU_k \right) \exp \left( -S \left( \prod_{k=1}^{4} U_k \right) \right) \]

- Change of variables: \( P(U_1) = U_1 \prod_{k=2}^{4} U_k \rightarrow Z = \int dP dU_2 dU_3 dU_4 \exp(-S(P)) \)

- Weyl integration formula for compact connected Lie group \( G \) in terms of a maximal torus \( T \):

\[ \int_{G} dU f(U) = \int_{T} d\mu(\theta) f(\theta) \quad \text{with} \quad d\mu(\theta) = \prod_{m>n} |e^{i\theta_m} - e^{i\theta_n}| \prod_{k} d\theta_k , \]

where \( f \) is a class function, i.e. \( f(U) = f(\Omega U \Omega^\dagger) \) (conjugation-invariant), and \( e^{i\theta_k} \) are unique eigenvalues (\( N - 1 \) for SU(\( N \))).

\[ \rightarrow \text{ reduces the eight-dim. map for the complete parameterization of SU(3)} \]
\[ \text{to a two-dim. map for the unique eigenvalue angles } \theta_k \]
(Un-)trivializing \((1+1)\text{d SU}(3)\) LGT
(Un-)trivializing (1+1)d SU(3) LGT

\[
\begin{align*}
\theta_2 \\
\beta = 6
\end{align*}
\]
(Un-)trivializing $(1+1)d$ SU$(3)$ LGT

- Approximate solution with tractable Jacobian using differentiable quadrature:

\[ \rightarrow \text{acceptance rate } \sim 0.15 \text{ at } 16 \times 16, \beta = 6 \]
Higher dimensions?
(Un-)trivializing (n+1)d SU(3) LGT with ML

Abbott et al [arXiv:2305.02402]

- Replace local conditional CDF with rational quadratic spline (RQS)
  - finite interval with fixed endpoints $\rightarrow$ compactness
  - monotonicity $\rightarrow$ invertibility
  - differentiability $\rightarrow$ Jacobian
  - bounded derivative $\rightarrow$ stability
- Locally compute spline parameters from surrounding features using neural networks
- Global invertibility from alternating masking patterns $\rightarrow$ coupling layers

- Variational optimization by minimizing reverse Kullback-Leibler divergence
Neural RQS eigenvalue flow

- Choose suitable parameterization of canonical cell, e.g. polar coordinates → best results so far
- Choose set of features preserving gauge covariance, e.g. loops:

  [Diagram of plaquette, 2x1, chair, crown]

- Choose local coupling geometry, e.g.

  [Diagram of direction coupling and location coupling]
Masking pattern algorithm

- Flexible parameterization for automated construction of suitable masks in $(n+1)d$
- Simple alternation scheme via cyclic permutations of parameters
  \[\rightarrow\] cover all links in one cycle to avoid blind spots
- Iterate over loop orientations \[\rightarrow\] cover all plaquettes

- Full implementation and interactive visualizations in supplementary jupyter notebook
  [arXiv:2305.02402]
Training and evaluation

- Variance reduction in gradient estimates using path gradients + control variates

Estimated Sample Size (ESS)

\[ \sim \left( \frac{1}{N} \sum_{k=1}^{N} w(U_k) \right)^2 \leq \frac{1}{N} \sum_{k=1}^{N} w(U_k)^2 \leq \frac{1}{N} \]

with \( N \) model samples \( U_k \sim q(U_k) \)

and \( w(U_k) = \frac{\exp(-S(U_k))}{q(U_k)} \).

- Easy target \( (8^4, \beta = 1) \), testing heatbath prior with \( 0 < \beta_{\text{heatbath}} < \beta_{\text{target}} \)

<table>
<thead>
<tr>
<th>flow</th>
<th>ESS</th>
<th>prior</th>
<th></th>
<th>prior</th>
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<tbody>
<tr>
<td>spectral</td>
<td>0.75</td>
<td>( \beta = 0 )</td>
<td>0.82</td>
<td>( \beta = 0.5 )</td>
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<tr>
<td>residual</td>
<td>0.09</td>
<td>0.18</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Summary

- (1+1)d LGT can be trivialized almost trivially with (semi-)analytic methods
- Proof-of-principle results for machine-learned maps applied to (3+1)d SU(3) LGT

Outlook

- Gauge invariants represent only the most basic data features from the perspective of geometric deep learning
  → need covariant information to achieve expressivity
- Naive (un-)trivialization in (3+1)d leads to proliferation of defects due to the geometric properties of Wilson loop actions
  → need hierarchical / multi-scale architectures
- Machine-learned maps are already capable of partial trivialization / thermodynamic integration on small volumes
  → practical applications are within reach

Denis Boyda
Thursday, 14:50

Ryan Abbott
Monday, 16:20

Daniel Hackett
Thursday, 15:10
Thanks!
Wilsonian RG interpretation (smoothing)

- Regularized stochastic quantization (Langevin equation): \( \frac{\partial \phi}{\partial \tau} = -\frac{\delta S}{\delta \phi} + r_\Lambda(\Delta) \eta \)

- Corresponding Fokker-Planck equation: \( \frac{\partial p(\phi, \tau)}{\partial \tau} = \int d^d x \frac{\delta}{\delta \phi} \left( \frac{\delta S}{\delta \phi} + r_\Lambda^2(\Delta) \frac{\delta}{\delta \phi} \right) p(\phi, \tau) \)

- \( \partial \tau p = 0 \rightarrow p_\Lambda(\phi) \propto \exp(-S - \Delta S_\Lambda) \) with \( \Delta S_\Lambda = \frac{1}{2} \int d^d p \phi(p) \Lambda^2 \left( \frac{1}{r_\Lambda(p^2)} - 1 \right) \phi(p) \)

- Sharp cutoff: \( r_\Lambda(p^2) = \theta(\Lambda^2 - p^2) \rightarrow r_\Lambda(\Delta) \eta(x) = \frac{1}{(2\pi)^2} \int d^d p e^{-ipx} \eta(p) \theta(\Lambda^2 - p^2) \)

Pawlowski et al [arXiv:1705.06231]
Wilsonian RG interpretation (smoothing)


The functional RG combines this functional approach with the RG idea of treating the fluctuations not all at once but successively from scale to scale [9, 10]. Instead of studying correlation functions after having averaged over all fluctuations, only the \textit{change} of the correlation functions as induced by an infinitesimal momentum shell of fluctuations is considered. From a structural viewpoint, this allows to transform the functional-integral structure of standard field theory formulations into a functional-differential structure [11, 12, 13, 14]. This goes along not only with a better analytical and numerical accessibility and stability, but also with a great flexibility of devising approximations adapted to a specific physical system. In addition, structural investigations of field theories from first principles such as proofs of renormalizability can more elegantly and efficiently be performed with this strategy [13, 15, 16, 17].

Cotler et al [arXiv:2202.11737]

One of our main results is that Polchinski’s equation can be written as

\begin{equation}
-L \frac{d}{d\Lambda} P_\Lambda[\phi] = -\nabla_{W_2} S(P_\Lambda[\phi] \| Q_\Lambda[\phi])
\end{equation}

(1.2)

where $\nabla_{W_2}$ is a gradient with respect to a functional generalization of the Wasserstein-2 metric, $S(P \| Q) := \int [d\phi] P[\phi] \log(P[\phi]/Q[\phi])$ is a functional version of the relative entropy, and $Q_\Lambda[\phi]$ is a background probability functional which essentially defines our RG scheme. We emphasize that
Kadanoffian RG interpretation (blocking)

Huang, *Statistical Mechanics*

**Fig. 18.1** Block-spin transformation: averaging the spins in a block, and then rescaling the lattice to the original size. In more than one dimension, the indirect interaction between $B$ and $C$ gives rise to next-to-nearest-neighbor interactions of the block spins.