Fuzzy Qubitization of Gauge Theories

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in collaboration with

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new paper out soon; previous work:
arXiv:1903.06577
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arXiv:2209.00098

Finite Gauge Theories: History

There's a long history of proposing quantum gauge theories with finite local Hilbert space:

- Horn, 1981: SU(2), $\dim \mathscr{H}_\ell = 5$
- Hamer, 1981: SU(2), $\dim \mathscr{H}_\ell = 5, 14, \ldots$
- Orland & Rohrlich, 1989, "gauge magnets": SU(2), $\dim \mathscr{H}_\ell = 4, 10, \ldots$
- Brower, Chandrasekharan, & Wiese, 1997, "quantum links": U(N), SU(N), $\dim \mathscr{H}_\ell = 2N$
- ...

All of which are candidates to qubitize lattice gauge theory. But...

Continuum limits?

Universality?

Probably a tough problem; recall relation between sigma models and quantum spin chains.

(We'll see that O.R.'s model is suitable for fuzzification.)

Outline

1. Fuzzification of Gauge Theories
   - Field space representation
   - General fuzzy theories
   - Explicit example: $SU(2)$
2. Continuum Limits
   - Sigma models
   - Perturbation theory; the continuum limit
   - Small volumes
3. Summary and Future Work

Gauge theory: field space representation

States are wave functions

$$\psi(U) = \psi_0 + \psi_{ij} U_{ij} + \widetilde{\psi}_{ij} U^*_{ij} + O(U^2), \quad \text{links} \quad U \in SU(N)$$

Gauge transformations, $g = \mathrm{e}^{-i\theta_a T_a} \in SU(N)$:

$$\psi(U) \mapsto \psi(g^\dagger U) \quad \text{(left)}, \quad \psi(U) \mapsto \psi(Ug) \quad \text{(right)}$$

Generated by ("conjugate momentum operators"), e.g.,

$$\mathbb{T}^L_a \psi(U) = \Big(-(T_a U)_{ij} \frac{\partial}{\partial U_{ij}} + (T_a U)^*_{ij} \frac{\partial}{\partial U^*_{ij}} \Big) \psi(U)$$

Symmetry algebra:

$$\begin{aligned}
[\mathbb{T}^L_a, \mathbb{T}^L_b] &= \mathrm{i} f_{abc} \mathbb{T}^L_c, & \qquad [\mathbb{T}^L_a, \mathbb{U}_{ij}] &= -(T_a)_{ii'} \mathbb{U}_{i'j}, \\
[\mathbb{T}^R_a, \mathbb{T}^R_b] &= \mathrm{i} f_{abc} \mathbb{T}^R_c, & \qquad [\mathbb{T}^R_a, \mathbb{U}_{ij}] &= +\mathbb{U}_{ij'} (T_a)_{j'j}, \\
[\mathbb{T}^L_a, \mathbb{T}^R_b] &= 0
\end{aligned}$$

Commutativity of the U_{ij}:

$$[\mathbb{U}_{ij}, \mathbb{U}_{kl}] = [\mathbb{U}_{ij}, \mathbb{U}^*_{kl}] = 0$$

Kogut-Susskind Hamiltonian

$$\mathbb{H} = \frac{g^2}{2} \sum_{\ell,a} \left[\mathbb{T}_a^L(\ell)^2 + \mathbb{T}_a^R(\ell)^2 \right] - \frac{1}{g^2} \sum_P \square(P) + \text{h.c.}$$

where $\ell = (x, \mu)$ label the links of the lattice, and $P = (x, \mu, \nu)$ label the plaquettes,

$$\square(P) = \mathbb{U}_{ij}(\ell_1)\, \mathbb{U}_{jk}(\ell_2)\, \mathbb{U}_{lk}(\ell_3)^*\, \mathbb{U}_{il}(\ell_4)^*$$

Kinetic energy operator (per link) is

$$\mathbb{K}(\ell) = \sum_a \left((\mathbb{T}_a^L)^2 + (\mathbb{T}_a^R)^2 \right)$$

Laplacian truncation strategy: For $N = 2$, $\mathbb{K}$ is the Laplacian on SU(2)

Eigenstates of $\mathbb{K}$ are Wigner matrices:

$$\sum_{k=1}^{3} \mathbb{T}_k^2\, \mathscr{D}^j_{mm'}(U) = j(j+1)\, \mathscr{D}^j_{mm'}(U) \ \Rightarrow \ \textit{truncate to } j \leq j_{\max}$$

Fuzzy gauge theories

Replace the links U_{ij} by **finite matrices** $\mathscr{U}_{ij} = D \times D$

States are matrix-valued, and have a series rep:

$$\Psi = \Psi_0 \mathbb{1} + \Psi_{ij} \mathscr{U}_{ij} + \widetilde{\Psi}_{ij} (\mathscr{U}_{ij})^\dagger + O(\mathscr{U}^2)$$

with inner product $\langle \Psi | \Phi \rangle := \mathrm{tr}[\Psi^\dagger \Phi]$. Fuzzy $\mathscr{H}$ dimension: D^2.

Gauge transformations: $\Psi \to \mathrm{e}^{-i\theta_a T_a^L} \Psi \, \mathrm{e}^{i\theta_a T_a^L}$

Generators:

$$\mathcal{T}_a^L = [T_a^L, \bullet], \qquad \mathcal{T}_a^R = [T_a^R, \bullet]$$

$\mathscr{U}_{ij}$ and $\mathcal{T}_a^{L,R}$ obey the gauge symmetry algebra owing to the Leibniz rule of commutators:

$$\mathcal{T}_a^L(\Psi\Phi) = \Psi \, \mathcal{T}_a^L(\Phi) + \mathcal{T}_a^L(\Psi) \, \Phi$$

Fuzzy unitarity constraints: analogues of $U_{ij} U_{kj}^* = U_{ji}^* U_{jk} = \delta_{ik}$ and $\det U = 1$,

$$\tfrac{1}{2} \{ \mathscr{U}_{ij}, (\mathscr{U}_{kj})^\dagger \} = \delta_{ik} \mathbb{1}, \qquad \tfrac{1}{2} \{ (\mathscr{U}_{ji})^\dagger, \mathscr{U}_{jk} \} = \delta_{ik} \mathbb{1},$$

$$\frac{1}{N!} \varepsilon_{i_1 \cdots i_N} \varepsilon_{j_1 \cdots j_N} \mathscr{U}_{i_1 j_1} \cdots \mathscr{U}_{i_N j_N} = \mathbb{1}$$

So, the fuzzy rep closely parallels the field space rep, except that $[\mathscr{U}_{ij}, \mathscr{U}_{kl}] \neq 0$.

Fuzzy SU(2) gauge theory

Orland-Rohrlich "gauge magnets" use 4×4 Gamma matrices to define finite links:

$$\mathscr{U}_{ij} = \delta_{ij}\Gamma_4 - i\sum_k (\sigma_k)_{ij}\Gamma_k$$

$$T_k^L = \frac{1}{2}\begin{pmatrix} \sigma_k & 0 \\ 0 & 0 \end{pmatrix}, \quad T_k^R = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_k \end{pmatrix}$$

These matrices satisfy the fuzzy gauge theory algebra, with $\mathcal{T}_a^{L,R} = [T_a^{L,R}, \bullet]$.

They furthermore satisfy the *fuzzy unitarity* constraints!

$$\tfrac{1}{2}\left\{(\mathscr{U}_{ji})^\dagger, \mathscr{U}_{jk}\right\} = \delta_{ik}\mathbb{1}, \qquad \tfrac{1}{2}\left\{\mathscr{U}_{ij}, (\mathscr{U}_{kj})^\dagger\right\} = \delta_{ik}\mathbb{1}$$

$$\frac{1}{2}\Big(\{\mathscr{U}_{11}, \mathscr{U}_{22}\} - \{\mathscr{U}_{12}, \mathscr{U}_{21}\}\Big) = \mathbb{1}$$

The fuzzy states $\Psi(\mathscr{U})$ are 4×4 matrices, so $\dim \mathscr{H} = 16$ per link.

Fuzzy Hamiltonian

Hamiltonian by analogy with Kogut-Susskind:

$$\mathcal{H} = \frac{g^2}{2} \sum_{\ell} \mathcal{K}(\ell) \pm \frac{1}{g^2} \sum_{P} \square(P) + \text{h.c.}$$

with

$$\square(P) = \mathscr{U}_{ij}(\ell_1)\mathscr{U}_{jk}(\ell_2)\mathscr{U}_{lk}(\ell_3)^\dagger \mathscr{U}_{il}(\ell_4)^\dagger$$

we choose the plaquette to act by **left-multiplication** (reasons soon).

There's multiple options for the kinetic term, however:

$$\begin{aligned}
\mathcal{K}_1 &= (\mathcal{T}^L)^2 + (\mathcal{T}^R)^2 = [T^L_a, [T^L_a, \bullet]] + [T^R_a, [T^R_a, \bullet]] \\
\mathcal{K}_2 &= -\mathscr{U}_{ij} \bullet \mathscr{U}^\dagger_{ij} - \mathscr{U}^\dagger_{ij} \bullet \mathscr{U}_{ij}, \\
\mathcal{K}_3 &= \Gamma_5
\end{aligned}$$

We will argue that this fuzzy model exhibits promising indications of universality.

To understand why, we return to the sigma model, where things are more tractable...

Fuzzy sigma model

Recall the fuzzy Hamiltonian

$$\mathcal{H} = g^2 \sum_x \mathcal{T}(x)^2 \pm \frac{1}{g^2} \sum_x \mathfrak{n}(x) \cdot \mathfrak{n}(x+1)$$

$\mathcal{T}_k \propto [\sigma_k, \bullet]$ and $\mathfrak{n}_k \propto \sigma_k$, and the neighbor operator acts on states $\Psi(\mathfrak{n}) = 2 \times 2$ from the left.

Choose basis $E_{ab} = \mathrm{e}_a \otimes \mathrm{e}_b$ for matrices, $\mathrm{e}_1 = [1,0]^\top$, $\mathrm{e}_2 = [0,1]^\top$. Then all-left-multiplying ops **factorize**

$$\sum_x \mathfrak{n}_k(x)\mathfrak{n}_k(x+1) \quad \longrightarrow \quad \Big(\sum_x \sigma_k(x)\sigma_k(x+1)\Big) \otimes \mathbb{1}$$

i.e., the left factor is a spin-1/2 Heisenberg chain.[1]

Eigenstates of fuzzy neighbor operator then factorize:

$$|\Psi_n\rangle = |n\rangle \otimes |\chi\rangle, \qquad |\chi\rangle = \text{any } 2^N\text{−dimensional state}$$

and $|n\rangle$ = eigenstate of Heisenberg chain (solvable via Bethe ansatz).

$\Rightarrow$ each eigenvalue v_n is *massively* degenerate.

Spectrum of $|n\rangle$ is gapless: $\omega(p) \propto p$, $p = 0, 2\pi/L, \ldots$

[1] In fact, this basis reveals an equivalence with the Heisenberg comb of Bhattacharya *et al.* (2020)

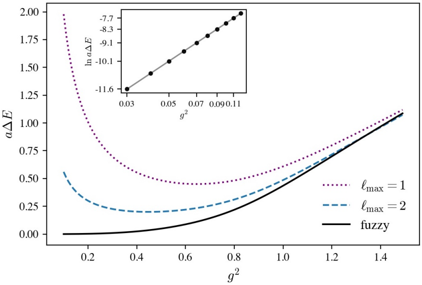
Continuum limit of the fuzzy sigma model

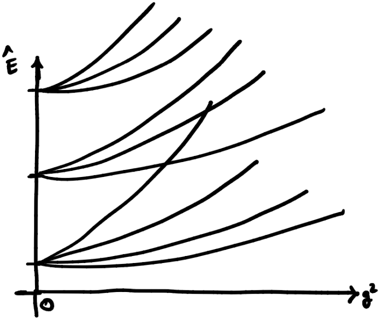
$$\mathcal{H} = g^2 \sum_x \mathcal{T}(x)^2 \pm \frac{1}{g^2} \sum_x \mathfrak{n}(x) \cdot \mathfrak{n}(x+1)$$

First: The kinetic op $\mathcal{K}$ breaks the degeneracy of each $|\Psi_n\rangle$. But for small g^2, $\mathcal{V}$ dominates.

As $g^2 \to 0$, the states associated with $|n\rangle$ merge together and become 2^N-degenerate.

$\Rightarrow$ this strongly suggests the existence of a critical point as $g^2 \to 0$, with exponentially large Hilbert space.





Continuum limit of the fuzzy sigma model

Next: Insight from perturbation theory.

Effective hamiltonian in the degenerate groundstate subspace $\{|\Psi_\alpha\rangle = |0\rangle \otimes |\chi_\alpha\rangle,\ \alpha = 1, ..., 2^N\}$

$$\mathcal{H}_{\text{eff}} = \mathcal{H}_{\text{eff}}^{(0)} + \mathcal{H}_{\text{eff}}^{(1)} + \mathcal{H}_{\text{eff}}^{(2)} + \cdots$$

where $\mathcal{H}_{\text{eff}}^{(0)} = \frac{1}{g^2}\mathcal{V}$. First order:

$$\mathcal{H}_{\text{eff}}^{(1)} = g^2 \mathcal{K} \propto g^2 \mathbb{1}$$

Second order: $\mathcal{H}_{\text{eff}}^{(2)} = g^6 \mathcal{K} \Delta_0 \mathcal{K}$ or

$$(\mathcal{H}_{\text{eff}}^{(2)})_{\alpha\beta} = \sum_{x,y} \Big[\sum_{m\geq 1} \frac{2g^6}{v_m - v_0} \langle 0| T_i(x) |m\rangle \langle m| T_j(y) |0\rangle \Big] \langle \chi_\alpha | T_i(x) T_j(y) |\chi_\beta\rangle$$

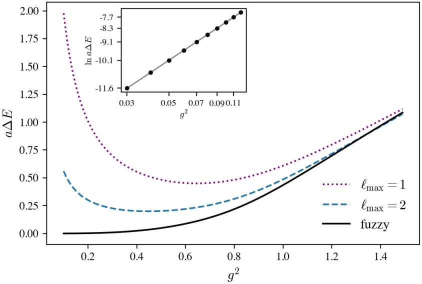
(excited subspaces $\{|m\rangle \otimes |\chi\rangle\},\ m \geq 1$)

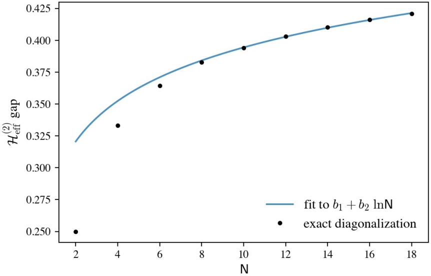
The sum over m includes a piece like $\int \mathrm{d}p/p \sim \log N$: so pert. theory breaks down in large volumes!

$\Rightarrow$ *a hallmark of asymptotically free theories*

Small volumes

(left: $N = 4$ site chains in each case)





Gauge theory case

In the $SU(2)$ fuzzy gauge theory, one can again use the product basis $E_{ab} = \mathrm{e}_a \otimes \mathrm{e}_b$ for 4×4 matrices.

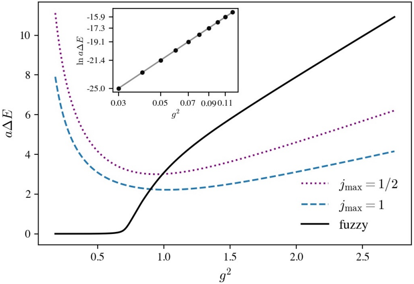
The plaquette operator factorizes just as before; it has 4^N-degenerate eigenvalues. The analogue of the spin-1/2 Heisenberg chain is an Orland-Rohrlich gauge magnet.

Only the kinetic term $\mathcal{K}_2 = -\mathscr{U}_{ij} \bullet \mathscr{U}_{ij}^\dagger - \mathscr{U}_{ij}^\dagger \bullet \mathscr{U}_{ij}$ completely breaks the degeneracy.

$\Rightarrow$ with $\mathcal{K}_2$ we expect the model to have a continuum limit as $g^2 \to 0$

Dynamics of gauge magnets is not well-understood; spin-wave analysis suggests the pure-plaquette system is gapless, with dispersion $\omega(p) \sim p^2$.

$\Rightarrow$ this would allow for infrared divergences in $\mathcal{H}_{\mathrm{eff}}$ coefficients.



Summary / Future

- We proposed some general properties for "fuzzy gauge theories" and gave an explicit example in the $SU(2)$ case.
- We identified salient features of fuzzy models that indicate a continuum limit with non-perturbative properties.
- A full numerical check of universality (e.g., computing a FSS scaling curve) remains to be done...
- **But:** preliminary Monte Carlo simulations suggest a severe sign problem. Viable simulation methods are still being explored / sought.
  (tensor networks? neural nets? suggestions?)

Thanks for listening!