# TOWARD A PRECISION CALCULATION OF GENERALIZED PARTON DISTRIBUTION FUNCTIONS.

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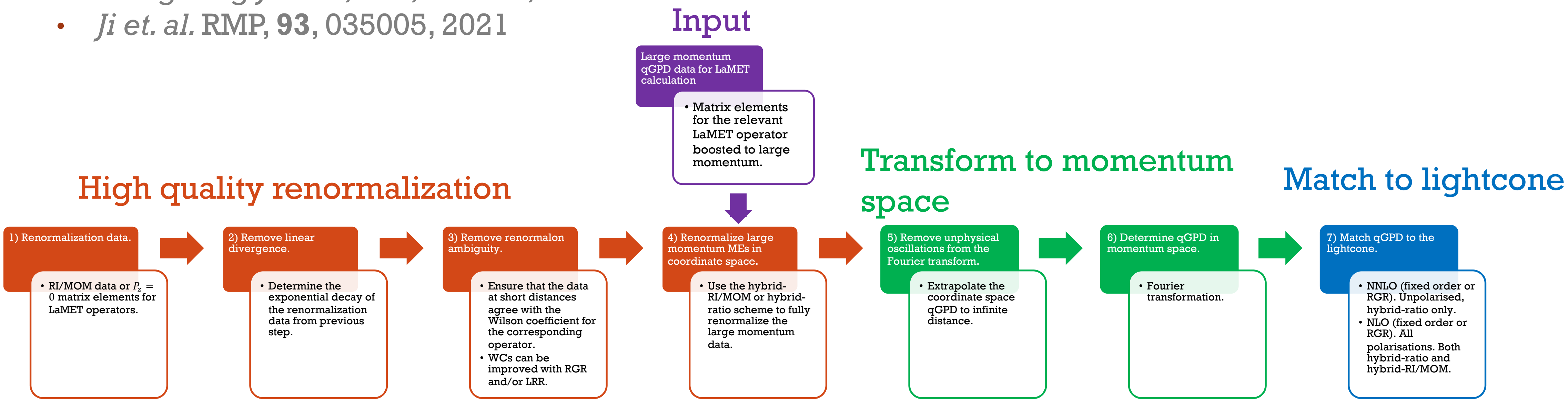
3rd Aug 2023.

# OUTLINE

- A walkthrough of the large-momentum effective theory (LaMET) process for a high-precision calculation of hadron GPDs.
  - *Xiangdong Ji.* PRL, **110**, 262002, 2013
  - *Ji et. al.* RMP, **93**, 035005, 2021




- Preliminary results.

# WHY STUDY GPDS?

- GPDs encode information about the internal- and spin-structure of a hadron. They are a hybrid of parton distribution functions (PDFs), form factors and distribution amplitudes. All of these serve as inputs to scattering experiments and theoretical calculations.
- A precise calculation of GPDs is, thus, of great interest.

- Methods of renormalization group resummation (RGR) have been applied to the pion PDF
  - *Su, JH* et. al. NPB, **991**, 116201, 2023
- RGR and leading renormalon resummation (LRR) have been applied to the pion DA.
  - *JH et. al.* NPB, **993**, 116282, 2023
- We apply these methods to GPDs.

# LATTICE DATA

- Lattice configurations from the MILC collaboration.
  - *Bazavov et. al.* PRD, **87**, 054505. 2013.
- Lattice matrix elements:

$$\tilde{h}^B(z,\xi,t) = \langle P_f | \overline{\psi}(0)\Gamma W(0,z)\psi(z) | P_i \rangle.$$
$$z^\mu = (0,0,0,z)^\mu$$

- Define the quantities:
  - $\Delta^\mu = P_f^\mu - P_i^\mu$, "momentum transfer", $t = \Delta^2$,

$\xi = \frac{P_f^z - P_i^z}{P_f^z + P_i^z}$, "(quasi-)skewness"

- When $\xi = 0$, GPDs reduce to the more familiar PDFs.
- We use the symmetric frame.

- Lattice spacing, $a = 0.09$ fm.
- Volume = $64^3 \times 96$.
- 2+1+1 flavors of highly improved staggered quarks.
- One-loop Symanzik improved gauge action.
- One step of HYP smearing on gauge links.
- Physical pion mass.
- ~500,000 matrix elements from ~1000 configurations.

# RENORMALIZATION DATA

1) Renormalization data.

RI/MOM data or $P_z = 0$ matrix elements for LaMET operators.

- We renormalize our data using the hybrid-RI/MOM scheme.
  - *Ji et. al.* NPB, **964**, 115311. 2021
- At short distances ($z \lesssim 0.2$ fm), we use the familiar RI/MOM scheme:

$$\tilde{h}^R(z,\xi,t) \sim \frac{\tilde{h}^B(z,\xi,t)}{Z(z,a,p_R=0)}.$$

- At large distances ($z \gtrsim 0.2$ fm), we remove the linear divergence and renormalon ambiguity:

$$\tilde{h}^R(z,\xi,t) = e^{(\delta m + m_0)z}\tilde{h}^B(z,\xi,t).$$

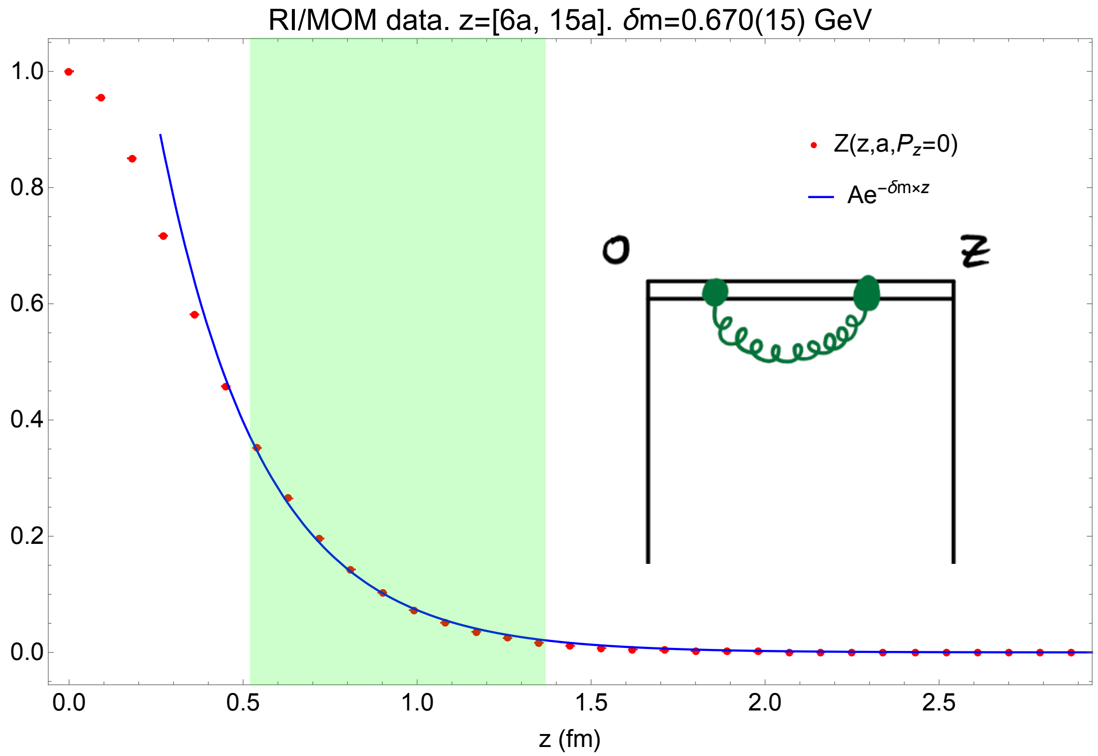
(Impose continuity at $z = z_s$.)

The parameters $\delta m$ and $m_0$ must be determined carefully.

# REMOVE LINEAR DIVERGENCE ($\delta m$)

2) Remove linear divergence.

- Determine the exponential decay of the renormalization data from previous step.




- The parameter $\delta m$ accounts for the linear divergence that occurs in the Wilson line, $W(0, z)$.
- Fit the data to $Ae^{-\delta m \times z}$.
  - *Ji et. al.* NPB, **964**, 115311, 2021
- We select the value $\delta m = 0.670(15)$ GeV.

# REMOVE RENORMALON AMBIGUITY $(m_0)$

3) Remove the renormalon ambiguity.

- Ensure that the data at short distances agree with the Wilson coefficient for the corresponding operator.

- We demand that our renormalized matrix elements agree with the operator product expansion (OPE) for $z \leq 0.3$ fm.
- The functions are Wilson coefficients at distance $z$ and energy scale $\mu$: $C_0(z,\mu)$
  - *Yao Ji et. al*. arXiv:2212.14415.

$$C_0(z,\mu) = 1 + \frac{\alpha_s(\mu) C_F}{2\pi}\left(\frac{3}{2}\ln\left(\frac{z^2\mu^2 e^{2\gamma_E}}{4}\right) + \frac{5}{2}\right)$$

$$C_0(z,\mu) \sim e^{(\delta m + m_0) z} Z(z, a, P_z = 0)$$

- We improve $m_0$ calculation using leading renormalon resummation (LRR).
  - *Zhang et. al.* PLB, **844**, 138081
  - Yushan Su's talk. "Leading power accuracy in lattice calculations of parton distribution functions." 1st Aug, 16:20 CDT.
  - Jianhui Zhang's talk. "Renormalons in the renormalization of quasi-PDF matrix elements." 3rd Aug, 11:00 CDT.
  - Andreas Kronfeld's talk. "More minimal renormalon subtraction." 3rd Aug, 13:50 CDT.

# REMOVE RENORMALON AMBIGUITY $(m_0)$

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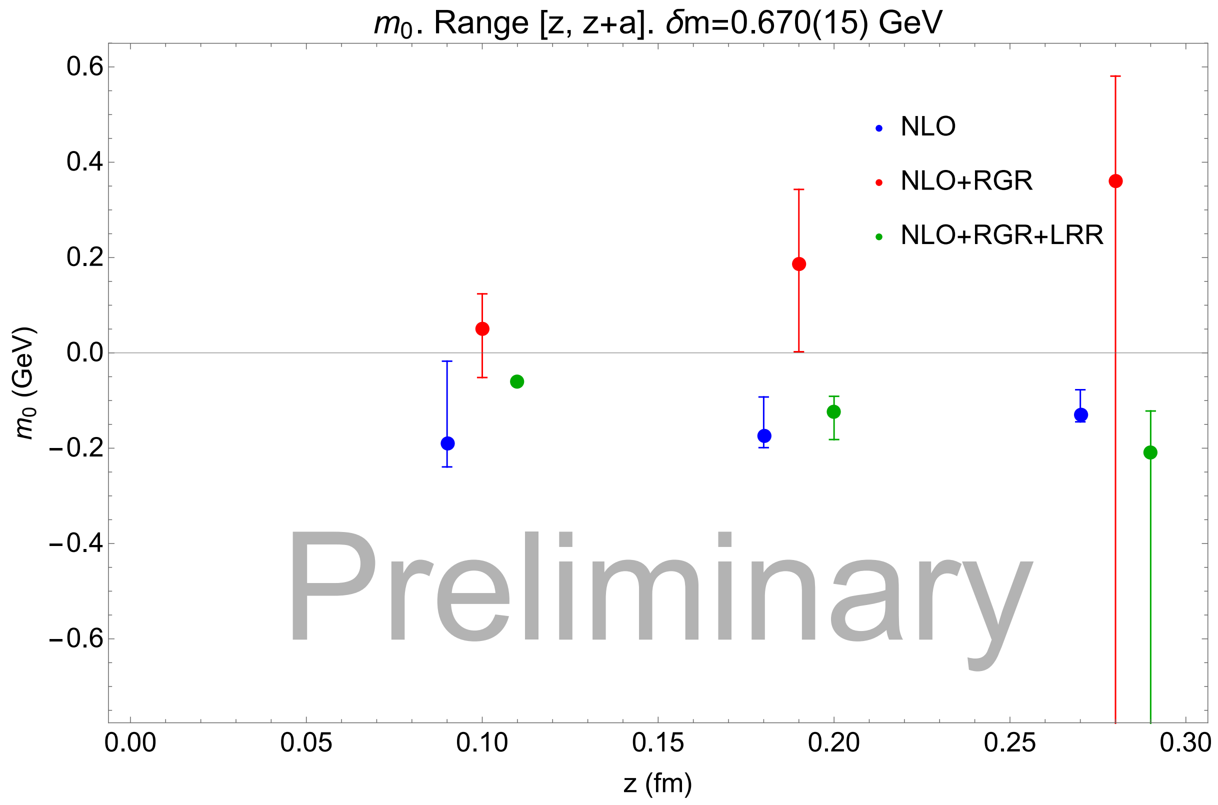
- Ensure that the data at short distances agree with the Wilson coefficient for the corresponding operator.




(Points shifted slightly to improve readability.)

- The $m_0$ parameter determined from the linear fit

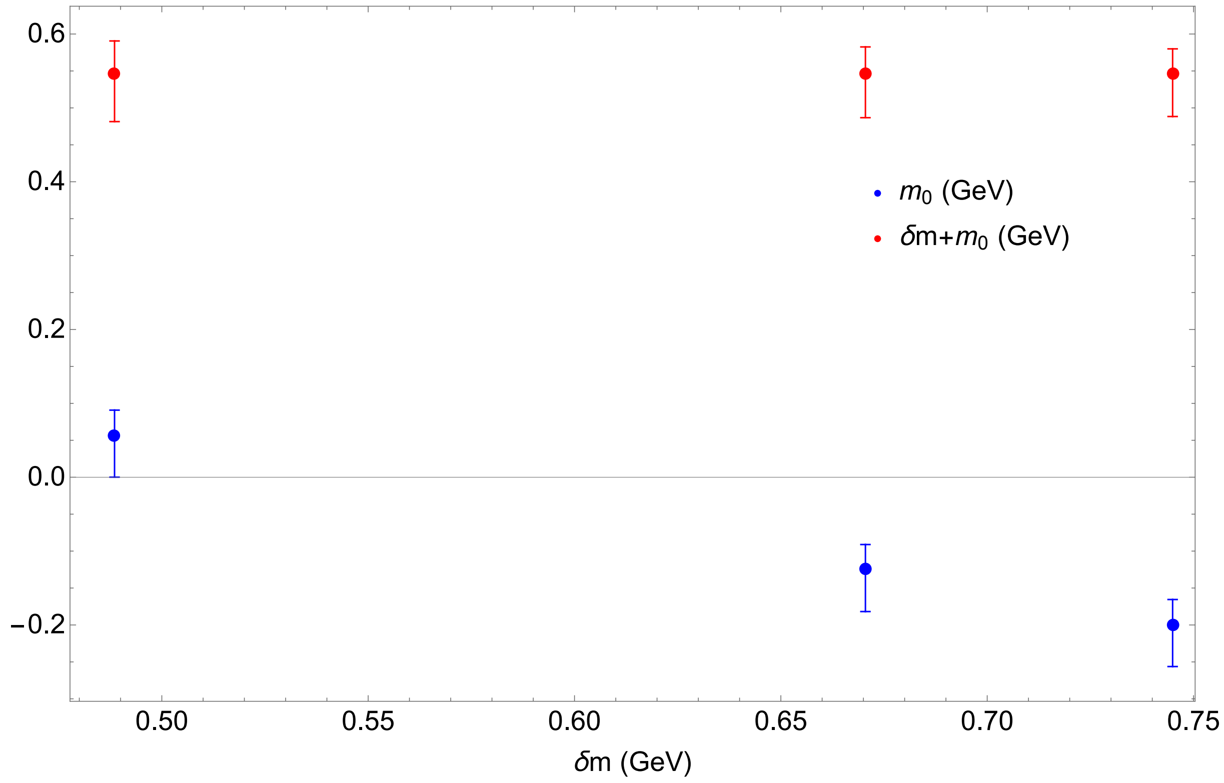
$$m_0 z + I_0 = \ln\left(\frac{e^{-\delta m \times z} C_0(z,\mu)}{Z(z,a,p_R=0)}\right)$$

in the range of $z$-values $[z, z+a]$. $I_0$ is a constant from the renormalization group.
  - *Zhang et. al.* PLB, **844**, 138081.
- Error bars are obtained by varying the energy scale $\mu = 1 \rightarrow 4$ GeV.
- RGR alone, makes the error bars too large. Need leading renormalon resummation.
- Data follow same trend as in *ibid*.

# REMOVE RENORMALON AMBIGUITY $(m_0)$

3) Remove the renormalon ambiguity.

- Ensure that the data at short distances agree with the Wilson coefficient for the corresponding operator.




- Different $\delta m$ values yield different $m_0$ values.
- Their sum $\delta m' = \delta m + m_0$ used in the final renormalization remains the same.
- $m_0$ determined with NLO+RGR+LRR

# RENORMALIZE LARGE MOMENTUM QGPD

Large momentum ME for LaMET calculation

- Matrix elements for the relevant LaMET operator boosted to large momentum.

4) Renormalize large momentum MEs in coordinate space.

- Use the hybrid-RI/MOM or hybrid-ratio scheme to fully renormalize the large momentum data.

- The full renormalization of $\tilde{h}^B(z,\xi,t)$ is

$$\tilde{h}^R(z,\xi,t) = \frac{\tilde{h}^B(z,\xi,t)}{Z(z,a,p_R=0)}\theta(z_s - z) + \frac{e^{(\delta m + m_0)(z-z_s)}\tilde{h}^B(z,\xi,t)}{Z(z_s,a,p_R=0)}\theta(z-z_s)$$

We choose $z_s = 3a = 0.27$ fm.

# EXTRAPOLATE AND FOURIER TRANSFORM

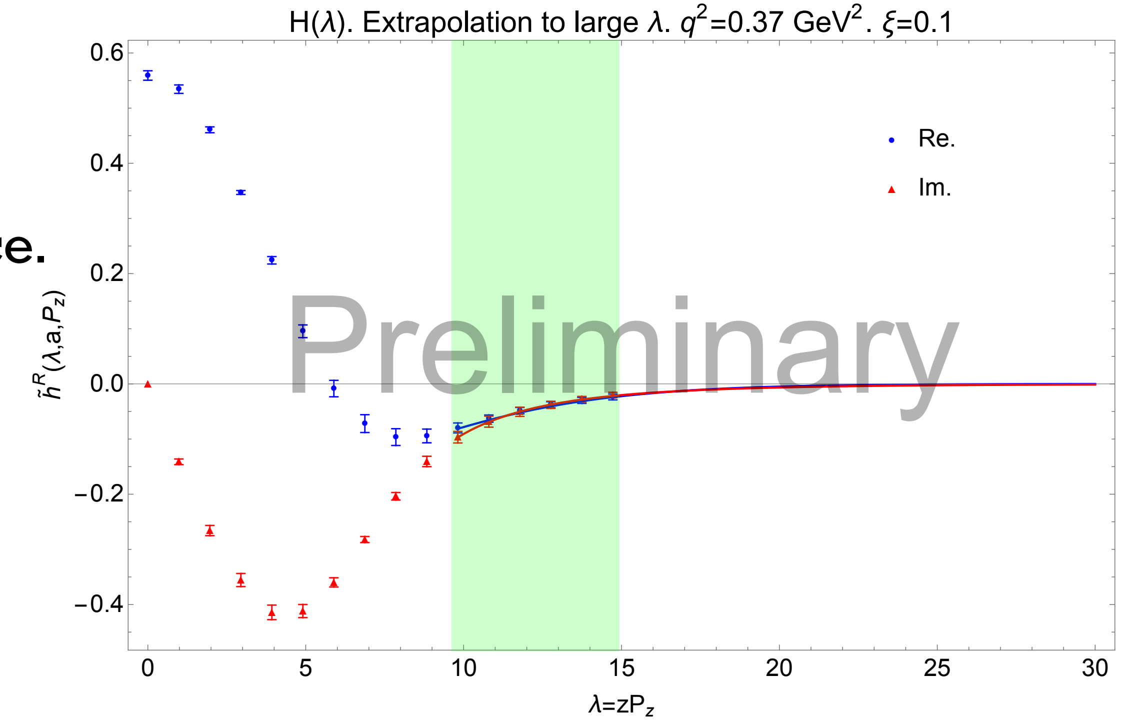
5) Remove unphysical oscillations from the Fourier transform.

- Extrapolate the coordinate space qGPD to infinite distance.

6) Determine qPDF in momentum space.

- Fourier transformation.

- With a view to Fourier transforming to momentum space, we extrapolate to infinite distance.
- $\tilde{h}^R(z, P_z) \to \frac{A e^{-\frac{m\lambda}{P_z}}}{|\lambda|^d}$ as $\lambda \to \infty$.
  - *Ji* et. al. NPB, **964**, 115311. 2021



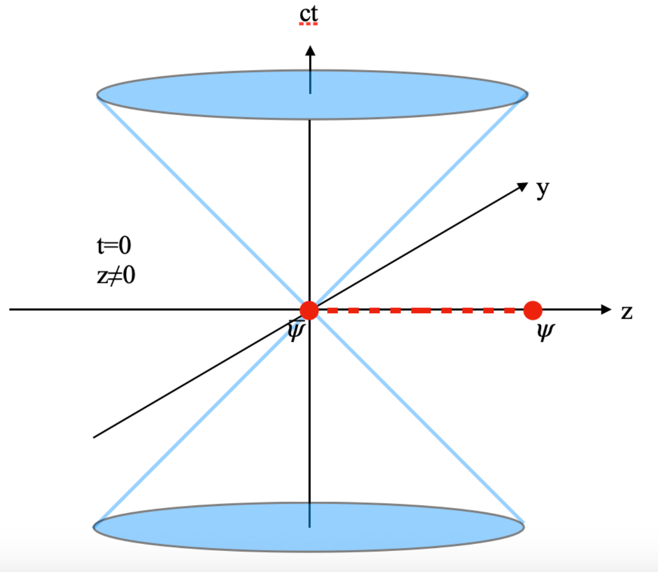

- We then Fourier transform to momentum space:

$$\tilde{F}(x, \xi, t) = \int_{-\infty}^{\infty} \frac{P_z dz}{2\pi} e^{ixzP_z} \tilde{h}^R(z, \xi, t)$$

# LIGHTCONE MATCHING

7) Match qGPD to the lightcone.

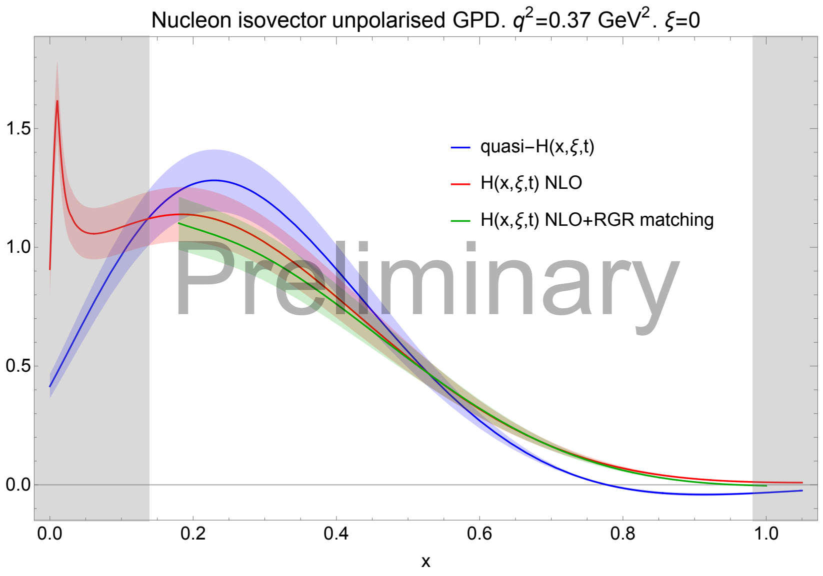
- NNLO (fixed order or RGR). Unpolarised, hybrid-ratio only.
- NLO (fixed order or RGR). All polarisations. Both hybrid-ratio and hybrid-RI/MOM.

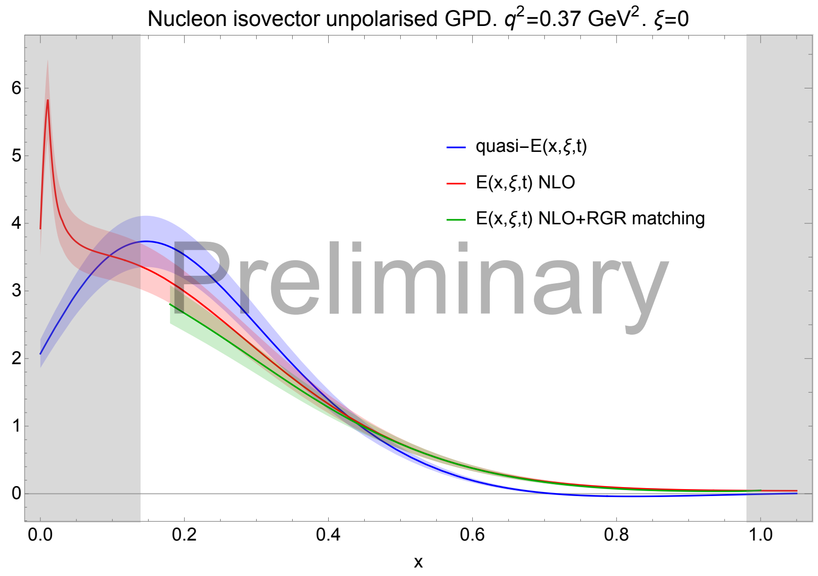



- The qGPD can be matched to the lightcone via

$$F(x,\xi,t) = \int_{-\infty}^{\infty} \frac{dy}{|y|} \mathcal{C}^{-1}(x,y,\mu,P_z)\tilde{F}(y,\xi,t) + \mathcal{O}\left(\frac{\Lambda_{QCD}^2}{x^2 P_z^2}, \frac{\Lambda_{QCD}^2}{(1-x)^2 P_z^2}\right)$$

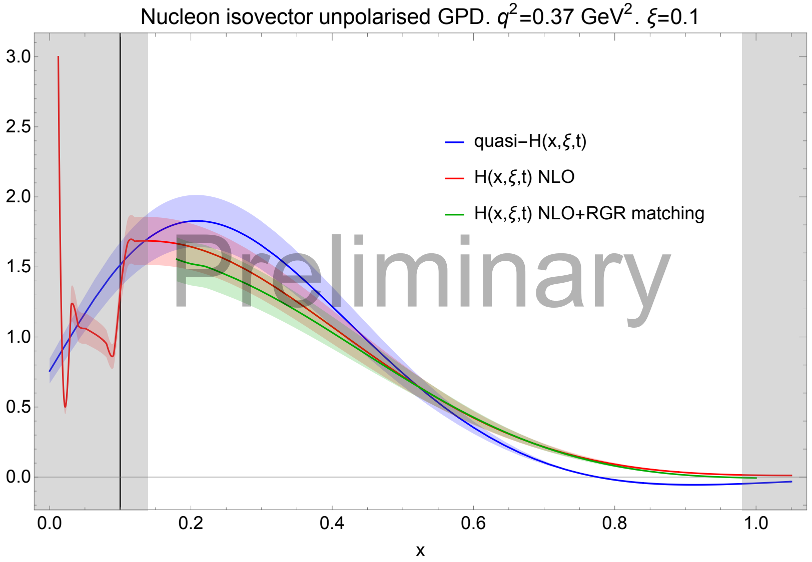
- $\mathcal{C}^{-1}(x,y,\mu,P_z)$ is the matching kernel.
  - Yao Ji, et. al. arXiv:2212.14415.
- We improve the matching process with renormalization group resummation.
  - *Su, JH et. al.* NPB, **991**, 116201. 2023
  - *Chen et. al.* 2208.08008. Lattice parton collaboration

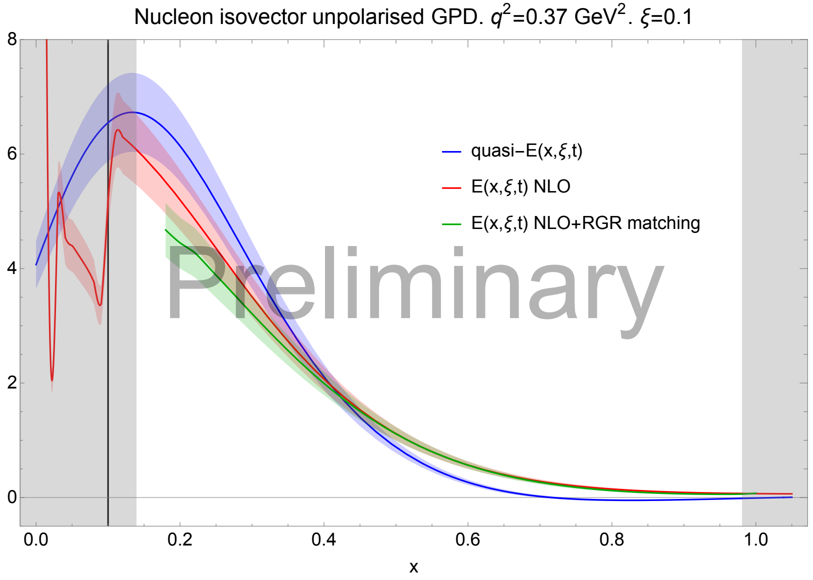



# PRELIMINARY RESULTS

- Top: $H(x,\xi,t)$. Bottom: $E(x,\xi,t)$.
- $q^2 = 0.37\ GeV^2, \xi = 0.$
- LaMET expansion breaks down in the small- and large-$x$ regions: $x \to 0$ and $|x| \to 1$ "endpoint regions".
- Corrections are $\mathcal{O}\left(\frac{\Lambda_{QCD}^2}{P_z^2 x(1-x)^2}\right)$
  - *Braun et. al.* PRD, **99**, 014013. 2019
  - *Gao, JH et. al.* PRD, **107**, 074509. 2023
- Systematic errors are taken as 10% to reflect preliminary plots.




# PRELIMINARY RESULTS

- Top: $H(x,\xi,t)$. Bottom: $E(x,\xi,t)$.
- $q^2 = 0.37$ GeV², $\xi = 0.1$.
- LaMET expansion breaks down in the small- and large-$x$ regions: $x \to 0$ and $|x| \to 1$ "endpoint regions".

- Corrections are $\mathcal{O}\left(\frac{\Lambda_{QCD}^2}{P_z^2 x(1-x)^2}\right)$
  - *Braun et. al.* PRD, **99**, 014013. 2019
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- Systematic errors are taken as 10% to reflect preliminary plots.

# CONCLUSION AND POSSIBLE FUTURE STEPS

- Sharper tools applied to pion PDF and DA calculations are applied to GPDs.
  - Resummation of large logarithms.
  - Improved handling of renormalon ambiguity.
- LRR process greatly reduces uncertainty in $m_0$ parameter.
- RGR matching process makes sense (greater modification at small-$x$).

- Check LRR modification to matching kernel.
- Use of $P_z = 0$ data as opposed to RI/MOM for comparison of systematic errors.

# THANK YOU

# RGR WILSON COEFFICIENT

- Set $\mu = \frac{2e^{-\gamma_E}}{z} \equiv z_0^{-1}$ so the logarithms disappear. Then evolve to the desired energy scale using the renormalization group.

- $C_0(z,\mu) = 1 + \frac{\alpha(\mu)C_F}{2\pi}\left(\frac{3}{2}\ln\left(\frac{z^2\mu^2e^{2\gamma_E}}{4}\right) + \frac{5}{2}\right)$
- $C_0(z,z_0^{-1}) = 1 + \frac{\alpha(z_0^{-1})C_F}{2\pi}\frac{5}{2}$
- $\frac{dC_0(z,\mu)}{d\ln(\mu^2)} = \gamma(\mu)C_0(z,\mu).$

- $\gamma(\mu)$ is the anomalous dimension.

# LRR WILSON COEFFICIENT

- We modify the "naïve" Wilson coefficients with the LRR method.

- $C_0^{LRR}(z, z_0^{-1}) = C_0(z, z_0^{-1}) + 2e^{-\gamma_E} N_m (C_0(z, z_0^{-1})_{PV} - \sum_i \alpha_s^{n+1}(z_0^{-1}) r_i)$

$$C_0(z, z_0^{-1})_{PV} = \frac{4\pi}{\beta_0} \int_{0,PV}^{\infty} du\, e^{\frac{-4\pi u}{\alpha(z_0^{-1})\beta_0}} \frac{1}{(1-2u)^{1+b}} (1 + c_1(1-2u) + \cdots)$$

  - *Zhang, JH et. al.* PLB, **884**, 138081. 2023

- Then evolve from $z_0^{-1}$ to $\mu$ with the RG.

# RGR MATCHING

- Large logarithms infect the matching kernel, too.

- The method of RGR matching combines fixed order matching with the DGLAP evolution equation:

$$\frac{d\mathcal{C}^{-1}\left(\frac{x}{y},\mu\right)}{d\ln(\mu^2)} = \int_x^1 \frac{dz}{|z|} P(z,\mu)_+ \mathcal{C}^{-1}\left(\frac{x}{zy},\mu\right).$$ $P(z,\mu)$ is the DGLAP kernel.

- We apply the method of RGR matching.
  - *Su, JH et. al.* NPB, **991**, 116201. 2023

# RGR MATCHING APPLIED TO PION

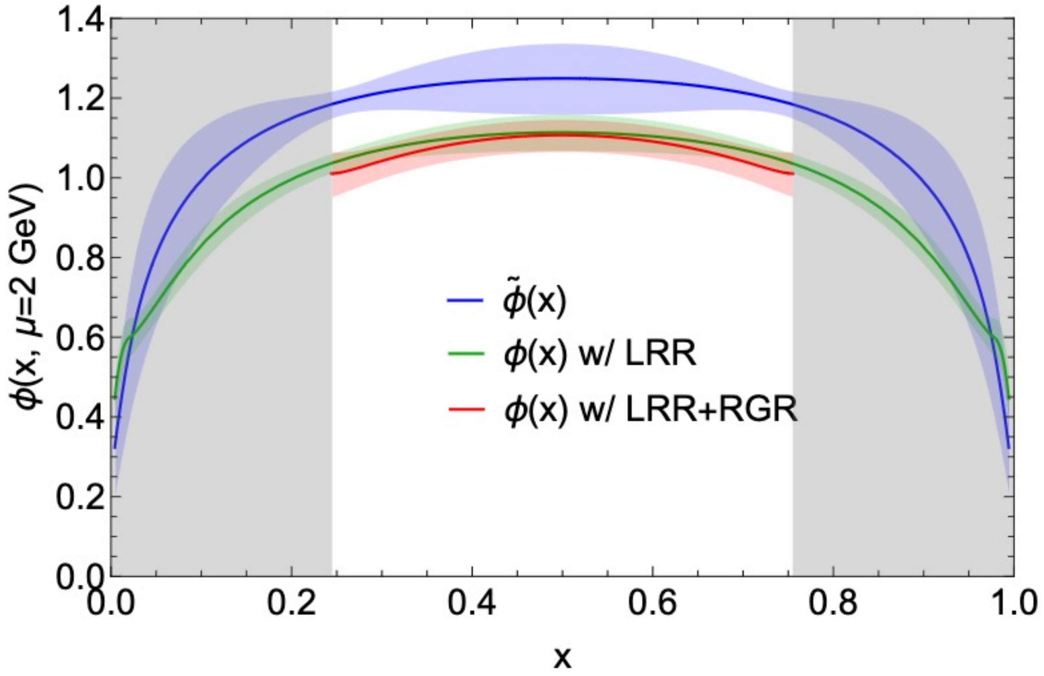



Fig.7 of *JH et. al.* NPB, **993**, 116282, 2023

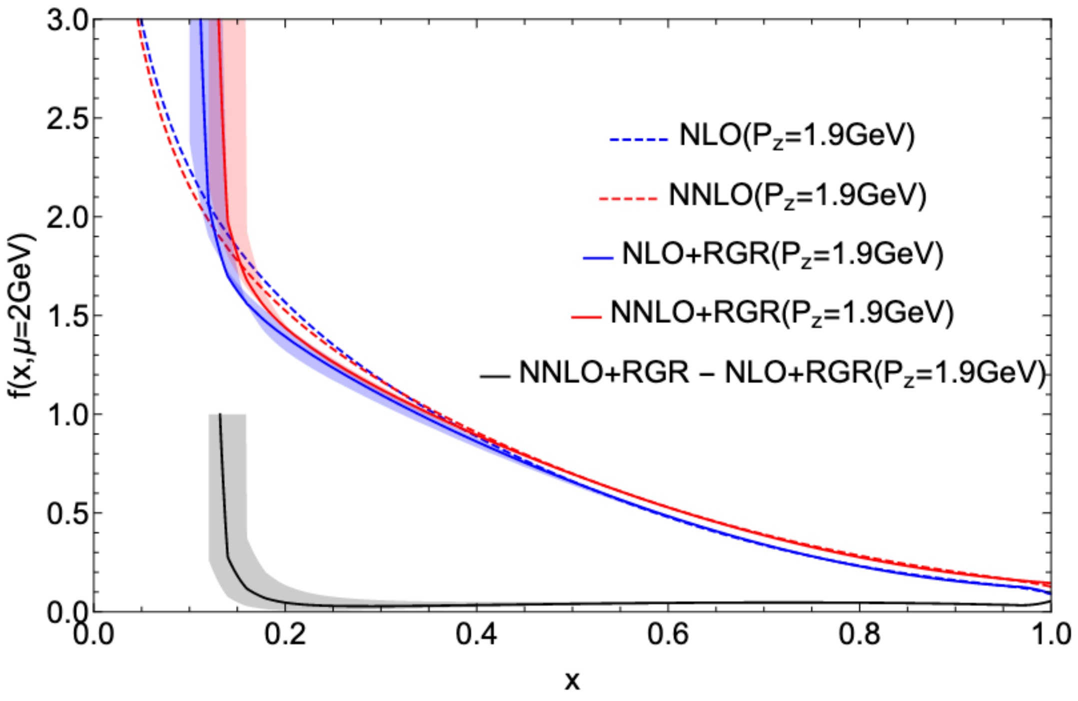
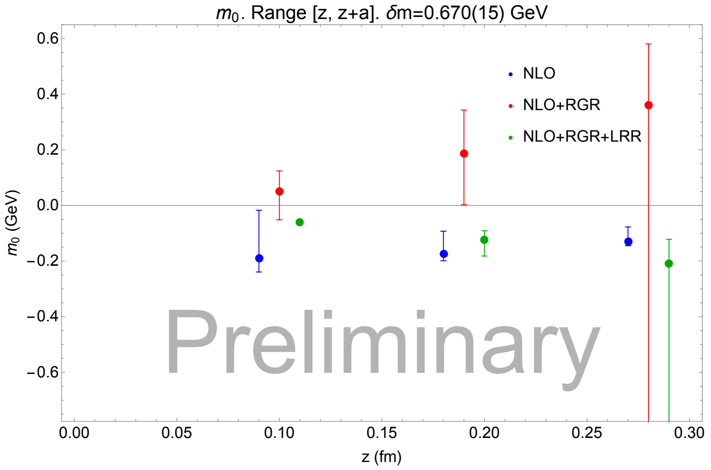
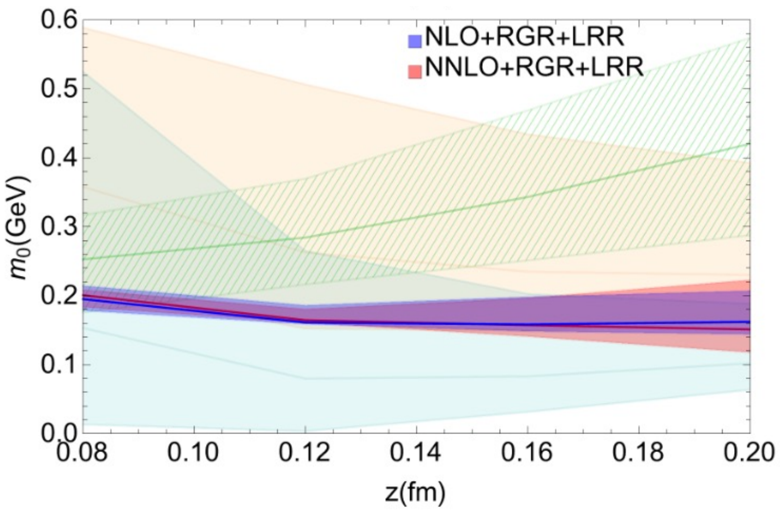



Fig. 4 of *Su, JH* et. al. NPB, **991**, 116201, 2023

# COMPUTATION OF $m_0$




- Bottom image: calculation of $m_0$ performed with pion PDF matrix elements.
  - Fig. 2 of *Zhang et. al.* PLB, **844**, 138081

- Values of $m_0$ follow the same trend between the two calculations.