Perturbative study of renormalization and mixing for asymmetric staple-shaped Wilson-line operators on the lattice

M. Constantinou[1], H. Panagopoulos[2], **G. Spanoudes**[2,*]

1. Temple University, 2. University of Cyprus, * Speaker

The 40th International Symposium on Lattice Field Theory,
Fermilab, Batavia, Illinois, USA, 31 July - 04 August 2023

Outline

A. Introduction

B. Study of operator mixing through symmetries

C. Construction of regularization-independent (RI$'$) renormalization prescriptions

D. One-loop perturbative study in dimensional regularization (DR)

E. One-loop perturbative study in lattice regularization (LR)

F. Conclusions and future prospects

Introduction

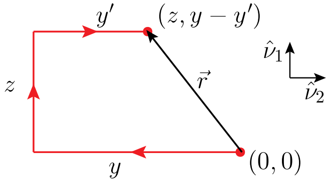
- **TMD quasi-PDFs:** [E.g., X. Ji et al., Rev. Mod. Phys. 93, 035005 (2021)]

$$\tilde{f}(x,z,\mu,\zeta) = \lim_{y\to\infty} \int \frac{d\delta y}{2\pi} e^{-i\delta y \zeta} \frac{2P_{\nu_2}}{N_\Gamma} \frac{\langle h(P_{\nu_2})|\mathcal{O}_\Gamma(x,z,y,y')|h(P_{\nu_2})\rangle}{\sqrt{\langle L(z,y+y')\rangle}}, \quad \delta y = y' - y$$

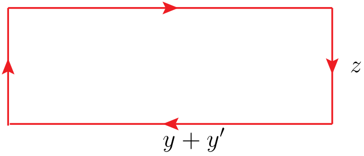
(See our conventions below)

- **Asymmetric staple-shaped Wilson-line quark bilinear operators:**

$$\mathcal{O}_\Gamma(x,z,y,y') \equiv \bar{\psi}(x)\,\Gamma\,\mathcal{W}_{\rm staple}(x,z,y,y')\,\psi(x+z\hat{\nu}_1+(y-y')\hat{\nu}_2), \qquad \Gamma = 1, \gamma_5, \gamma_\mu, \gamma_5\gamma_\mu, \sigma_{\mu\nu}$$



$\mathcal{W}_{\rm staple}(x,z,y,y')$



$L(z,y+y')$

- **Renormalization challenges:**
  1. Linear divergences
  2. Logarithmic divergences at cusp, end points, contact points
  3. Pinch-pole singularity (in the limit $y \to \infty$)
  5. Operator mixing

Operator mixing

- **Different studies:**

1. Based on **one-loop perturbation theory**: [M. Constantinou et al., PRD 99, 074508 (2019)]

Mixing between $(\Gamma, \pm[\Gamma, \gamma_{\nu_2}]/2)$ for non-chiral fermions
Mixing sets: $(\gamma_5, \gamma_5\gamma_{\nu_2})$, $(\gamma_{\nu_1}, \sigma_{\nu_1\nu_2})$, $(\gamma_{\nu_3}, \sigma_{\nu_3\nu_2})$, $(\gamma_{\nu_4}, \sigma_{\nu_4\nu_2})$
Multiplicative renormalization: $1, \gamma_{\nu_2}, \gamma_5\gamma_{\nu_1}, \gamma_5\gamma_{\nu_3}, \gamma_5\gamma_{\nu_4}, \sigma_{\nu_3\nu_1}, \sigma_{\nu_4\nu_1}, \sigma_{\nu_4\nu_3}$

where $(\nu_1, \nu_2, \nu_3, \nu_4)$ are all different and $\varepsilon_{\nu_1\nu_2\nu_3\nu_4} = +1$.

2. **Maximal prescription**: 16×16 mixing [P. Shanahan et al., PRD 101, 074505 (2020)]

one mixing set: $(1, \gamma_5, \gamma_{\nu_1}, \gamma_{\nu_2}, \gamma_{\nu_3}, \gamma_{\nu_4}, \gamma_5\gamma_{\nu_1}, \gamma_5\gamma_{\nu_2}, \gamma_5\gamma_{\nu_3}, \gamma_5\gamma_{\nu_4}, \sigma_{\nu_1\nu_2}, \sigma_{\nu_3\nu_1}, \sigma_{\nu_4\nu_1}, \sigma_{\nu_3\nu_2}, \sigma_{\nu_4\nu_2}, \sigma_{\nu_4\nu_3})$

3. Based on **symmetries**: [C. Alexandrou et al., arXiv:2305.11824], [Y. Ji et al., PRD 104, 094510 (2021)]

- Generalized time reversal $\mathcal{T}$ (in each direction)
- Generalized parity $\mathcal{P}$ (in each direction)
- Charge conjugation $\mathcal{C}$
- Chiral transformations $\mathcal{A}$

Operator mixing based on symmetries

- **Basis of operators:** We take 8 independent linear combinations of $(\mathcal{O}_\Gamma(x,+z,+y,+y^{'}),\mathcal{O}_\Gamma(x,-z,+y,+y^{'}),\ \mathcal{O}_\Gamma(x,+z,-y,-y^{'}),\ \mathcal{O}_\Gamma(x,-z,-y,-y^{'}),$ $\mathcal{O}_\Gamma(x,+z,+y^{'},+y),\mathcal{O}_\Gamma(x,-z,+y^{'},+y),\ \mathcal{O}_\Gamma(x,+z,-y^{'},-y),\ \mathcal{O}_\Gamma(x,-z,-y^{'},-y)),$ which are odd/even under $\mathcal{T},\mathcal{P},\mathcal{C}$.

- **Mixing pattern for chiral fermions:** $(\Gamma, \pm\Gamma\gamma_{\nu_1}\gamma_{\nu_2})$

  Mixing sets: $(1,\sigma_{\nu_1\nu_2}),\ (\gamma_{\nu_1},\gamma_{\nu_2}),\ (\gamma_5,\sigma_{\nu_4\nu_3}),\ (\gamma_5\gamma_{\nu_1},\gamma_5\gamma_{\nu_2}),$
  $(\gamma_{\nu_3},\gamma_5\gamma_{\nu_4}),\ (\sigma_{\nu_3\nu_1},\sigma_{\nu_3\nu_2}),\ (\gamma_{\nu_4},\gamma_5\gamma_{\nu_3}),\ (\sigma_{\nu_4\nu_1},\sigma_{\nu_4\nu_2})$

  In symmetric staples: $(\Gamma, \pm[\Gamma,\gamma_{\nu_1}\gamma_{\nu_2}]/2)$

  Mixing sets: $(\gamma_{\nu_1},\gamma_{\nu_2}),\ (\gamma_5\gamma_{\nu_1},\gamma_5\gamma_{\nu_2}),(\sigma_{\nu_3\nu_1},\sigma_{\nu_3\nu_2}),\ (\sigma_{\nu_4\nu_1},\sigma_{\nu_4\nu_2})$

  Multiplicative renormalization: $1,\gamma_5,\gamma_{\nu_3},\gamma_{\nu_4},\gamma_5\gamma_{\nu_3},\gamma_5\gamma_{\nu_4},\sigma_{\nu_1\nu_2},\sigma_{\nu_4\nu_3}$

- **Mixing pattern for non-chiral fermions:** $(\Gamma, \pm\Gamma\gamma_{\nu_1}\gamma_{\nu_2}, \pm\Gamma\gamma_{\nu_1}, \pm\Gamma\gamma_{\nu_2})$

  Mixing sets: $(1,\sigma_{\nu_1\nu_2},\gamma_{\nu_1},\gamma_{\nu_2}),\ (\gamma_5,\sigma_{\nu_4\nu_3},\gamma_5\gamma_{\nu_1},\gamma_5\gamma_{\nu_2}),$
  $(\gamma_{\nu_3},\gamma_5\gamma_{\nu_4},\sigma_{\nu_3\nu_1},\sigma_{\nu_3\nu_2}),\ (\gamma_{\nu_4},\gamma_5\gamma_{\nu_3},\sigma_{\nu_4\nu_1},\sigma_{\nu_4\nu_2})$

  In symmetric staples: $(\Gamma, \pm[\Gamma,\gamma_{\nu_1}\gamma_{\nu_2}]/2, \pm[\Gamma,\gamma_{\nu_1}]/2, \pm[\Gamma,\gamma_{\nu_2}]/2)$

  Mixing sets: $(\sigma_{\nu_1\nu_2},\gamma_{\nu_1},\gamma_{\nu_2}),\ (\gamma_5,\gamma_5\gamma_{\nu_1},\gamma_5\gamma_{\nu_2}),$
  $(\gamma_{\nu_3},\sigma_{\nu_3\nu_1},\sigma_{\nu_3\nu_2}),\ (\gamma_{\nu_4},\sigma_{\nu_4\nu_1},\sigma_{\nu_4\nu_2})$

  Multiplicative renormalization: $1,\gamma_5\gamma_{\nu_3},\gamma_5\gamma_{\nu_4},\sigma_{\nu_4\nu_3}$

Renormalization prescriptions

$$\mathcal{O}^R_\Gamma(x,z,y,y') = Z^{R,X}_{\Gamma\Gamma'}\mathcal{O}_{\Gamma'}(x,z,y,y'), \qquad \psi^R(x) = (Z^{R,X}_\psi)^{1/2}\psi(x),$$

$$\Gamma \in \{S_1 \equiv (1,\sigma_{\nu_1\nu_2},\gamma_{\nu_1},\gamma_{\nu_2}), S_2 \equiv (\gamma_5,\sigma_{\nu_4\nu_3},\gamma_5\gamma_{\nu_1},\gamma_5\gamma_{\nu_2}),$$
$$S_3 \equiv (\gamma_{\nu_3},\gamma_5\gamma_{\nu_4},\sigma_{\nu_3\nu_1},\sigma_{\nu_3\nu_2}), S_4 \equiv (\gamma_{\nu_4},\gamma_5\gamma_{\nu_3},\sigma_{\nu_4\nu_1},\sigma_{\nu_4\nu_2})\}$$

- **Standard RI$'$ scheme:**

$$\boxed{\frac{1}{4N_c}(Z^{\mathrm{RI}',X}_\psi)^{-1} Z^{\mathrm{RI}',X}_{\Gamma\Gamma'}\mathrm{Tr}[\Lambda_{\Gamma'}(q,z,y,y')\mathcal{P}_{\Gamma''}]\Big|_{q=\bar{q}} = \delta_{\Gamma\Gamma''}},$$

where $\Lambda_\Gamma(q,z,y,y') = \sum_x \langle\psi(q)|\mathcal{O}_\Gamma(x,z,y,y')|\bar{\psi}(q)\rangle_{\mathrm{amp.}}$,

$$Z^{\mathrm{RI}',X}_\psi = \frac{1}{4N_c}\,\mathrm{Tr}\left[\langle\psi(q)\bar{\psi}(q)\rangle^{-1}\cdot\frac{i\not{q}}{q^2}\right]\Big|_{q=\bar{q}},\ X\text{: regularization}$$

$$\mathcal{P}^{[1]}_\Gamma = e^{-i\mathsf{q}\cdot\mathsf{r}}\,\Gamma^\dagger,$$

$$\mathcal{P}^{[2]}_\Gamma = \begin{cases} e^{-i\mathsf{q}\cdot\mathsf{r}}\left(1-\frac{\not{q}_T\ \not{q}_L}{q_T^2}\right)\Gamma^\dagger, & \Gamma\in S_1,S_2 \\ e^{-i\mathsf{q}\cdot\mathsf{r}}\left(1-\frac{(\not{q}_T-\not{q}_{\nu_3})(\not{q}_L+\not{q}_{\nu_3})}{q_T^2-q_{\nu_3}^2}\right)\Gamma^\dagger, & \Gamma\in\{\gamma_{\nu_3},\gamma_5\gamma_{\nu_3},\sigma_{\nu_3\nu_1},\sigma_{\nu_3\nu_2}\} \\ e^{-i\mathsf{q}\cdot\mathsf{r}}\left(1-\frac{(\not{q}_T-\not{q}_{\nu_4})(\not{q}_L+\not{q}_{\nu_4})}{q_T^2-q_{\nu_4}^2}\right)\Gamma^\dagger, & \Gamma\in\{\gamma_{\nu_4},\gamma_5\gamma_{\nu_4},\sigma_{\nu_4\nu_1},\sigma_{\nu_4\nu_2}\} \end{cases},$$

where $\vec{q}_L \equiv q_{\nu_1}\hat{\nu}_1 + q_{\nu_2}\hat{\nu}_2$ and $\vec{q}_T \equiv \vec{q} - \vec{q}_L = q_{\nu_3}\hat{\nu}_3 + q_{\nu_4}\hat{\nu}_4$.

Renormalization prescriptions

- **Alternative RI$'$ scheme (RI$'$-bar):** [M. Ebert et al., JHEP 03 (2020) 099], [Y. Ji et al., PRD 104, 094510 (2021)]

$$\boxed{\frac{1}{4N_c}\left(Z_\psi^{\mathrm{RI}',X}\right)^{-1}\bar{Z}_{\Gamma\Gamma'}^{\mathrm{RI}',X}\,\mathrm{Tr}\left[\frac{\Lambda_{\Gamma'}(q,z,y,y')}{\sqrt{\langle L(z,y+y')\rangle}}\mathcal{P}_{\Gamma''}\right]\Bigg|_{\substack{q=\bar{q},\\ z=\bar{z},\\ y'-y=\delta\bar{y}}} = \delta_{\Gamma\Gamma''}},$$

$*$ **Main advantage:** Set small values of $\bar{z}$ and $\delta\bar{y}$ and use $\bar{Z}_{\Gamma\Gamma'}^{\mathrm{RI}',X}$ to renormalize correlators at all values of z and $y'-y$.

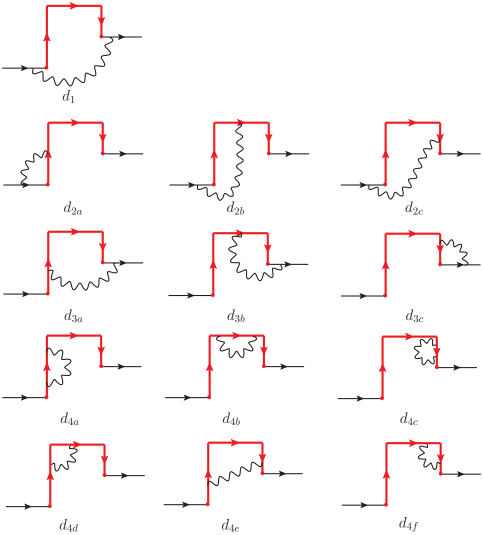
$\Rightarrow$ Avoid non-perturbative effects at large distances and residual linear divergences [LPC, PRL 129, 082002 (2022)].

$*$ **Further improvement:** Remove one-loop artifacts (for the lattice regularization) from $Z_\psi, \Lambda_\Gamma, L$ (similarly to [M. Constantinou, H. Panagopoulos, PRD 107, 014503 (2023)] in the case of non-local straight Wilson-line operators).

$\Rightarrow$ Reduce sizable contributions from finite lattice spacing effects due to the use of small values of $\bar{z}/a$ and $\delta\bar{y}/a$.

One-loop calculation in DR

- **Feynman diagrams:** $\Lambda_\Gamma(q,z,y,y') = \sum_x \langle \psi(q) | \mathcal{O}_\Gamma(x,z,y,y') | \bar{\psi}(q) \rangle_{\text{amp.}}$



Divergent terms:

1. End-point divergences:
   $\Lambda_\Gamma^{d2(a)}|_{\frac{1}{\varepsilon}} = \Lambda_\Gamma^{d3(c)}|_{\frac{1}{\varepsilon}} =$
   $\Gamma e^{iq\cdot r} \frac{g^2 C_F}{16\pi^2} \frac{1}{\varepsilon}(1-\beta),$
2. Contact divergences:
   $\Lambda_\Gamma^{d4(a)}|_{\frac{1}{\varepsilon}} = \Lambda_\Gamma^{d4(b)}|_{\frac{1}{\varepsilon}} =$
   $\Lambda_\Gamma^{d4(c)}|_{\frac{1}{\varepsilon}} =$
   $\Gamma e^{iq\cdot r} \frac{g^2 C_F}{16\pi^2} \frac{1}{\varepsilon}(2+\beta),$
3. Cusp divergences:
   $\Lambda_\Gamma^{d4(d)}|_{\frac{1}{\varepsilon}} = \Lambda_\Gamma^{d4(f)}|_{\frac{1}{\varepsilon}} =$
   $\Gamma e^{iq\cdot r} \frac{g^2 C_F}{16\pi^2} \frac{1}{\varepsilon}(-\beta),$
4. Pinch-pole divergences:
   $\Lambda_\Gamma^{d4(e)}|_{y} = \Gamma e^{iq\cdot r} \frac{g^2 C_F}{16\pi^2} 4 \times$

$$\left[\frac{y}{z} \tan^{-1}\left(\frac{y}{z}\right) + \frac{y'}{z} \tan^{-1}\left(\frac{y'}{z}\right) \right]$$

One-loop calculation in DR

- **Dirac structures:** (See our forthcoming paper)
  E.g.,

$$\begin{aligned}\Lambda_{\gamma_\mu}(q,z,y,y') &= \Sigma_1\ \gamma_\mu + \Sigma_2\ \varepsilon_{\nu_1\nu_2\mu\rho}\gamma_5\gamma_\rho + \Sigma_3\ q_\mu\gamma_{\nu_1} + \Sigma_4\ q_\mu\gamma_{\nu_2} \\ &+ \Sigma_5\ \sigma_{\mu\nu_1}\not{q} + \Sigma_6\ \sigma_{\mu\nu_2}\not{q} + \Sigma_7\ q_\mu 1\not{q},\end{aligned}$$

  where $\mu \neq \nu_1, \nu_2$ and $\Sigma_i \equiv \Sigma_i(\bar{\mu}^2, q_{\nu_1}, q_{\nu_2}, q^2, z, y, y')$.

$$\begin{aligned}\mathrm{Tr}\left[\Lambda_{\gamma_\mu}\mathcal{P}^{[1]}_{\gamma_\mu}\right] &= \Sigma_1 + \Sigma_5\ q_{\nu_1} + \Sigma_6\ q_{\nu_2} + \Sigma_7\ q_\mu^2, \\ \mathrm{Tr}\left[\Lambda_{\gamma_\mu}\mathcal{P}^{[2]}_{\gamma_\mu}\right] &= \Sigma_1 \qquad \text{independent of } q_\mu\end{aligned}$$

- **Expectation value of Wilson loop:**

$$\langle L(z,y+y')\rangle = 1 + \frac{g^2}{16\pi^2}C_F\ 8\left\{2 + \frac{1}{\varepsilon} + 2\gamma_E + \frac{y+y'}{z}\tan^{-1}(\frac{y+y'}{z}) + \frac{z}{y+y'}\tan^{-1}(\frac{z}{y+y'}) + \ln(\frac{\bar{\mu}^2 z^2}{4}) - \ln(1+\frac{z^2}{(y+y')^2})\right\} + \mathcal{O}(g^4)$$

  Agree with [e.g., LPC, PRL 129, 082002 (2022)].

One-loop calculation in DR

- **Conversion matrices between RI$'$-type and $\overline{\text{MS}}$ schemes:** (See our forthcoming paper)

$$Z^{\overline{\text{MS}},X}_{\Gamma\Gamma'} = C^{\overline{\text{MS}},\text{RI}'}_{\Gamma\Gamma''} Z^{\text{RI}',X}_{\Gamma''\Gamma'},$$

$$C^{\overline{\text{MS}},\text{RI}'} = \begin{pmatrix} C^{\overline{\text{MS}},\text{RI}'}_{\Gamma,\Gamma} & C^{\overline{\text{MS}},\text{RI}'}_{\Gamma,\Gamma\gamma_{\nu_1}\gamma_{\nu_2}} & 0 & 0 \\ C^{\overline{\text{MS}},\text{RI}'}_{\Gamma\gamma_{\nu_1}\gamma_{\nu_2},\Gamma} & C^{\overline{\text{MS}},\text{RI}'}_{\Gamma\gamma_{\nu_1}\gamma_{\nu_2},\Gamma\gamma_{\nu_1}\gamma_{\nu_2}} & 0 & 0 \\ 0 & 0 & C^{\overline{\text{MS}},\text{RI}'}_{\Gamma\gamma_{\nu_1},\Gamma\gamma_{\nu_1}} & C^{\overline{\text{MS}},\text{RI}'}_{\Gamma\gamma_{\nu_1},\Gamma\gamma_{\nu_2}} \\ 0 & 0 & C^{\overline{\text{MS}},\text{RI}'}_{\Gamma\gamma_{\nu_2},\Gamma\gamma_{\nu_1}} & C^{\overline{\text{MS}},\text{RI}'}_{\Gamma\gamma_{\nu_2},\Gamma\gamma_{\nu_2}} \end{pmatrix} + \mathcal{O}(g^4_{\overline{\text{MS}}}).$$

∗ Block diagonal

∗ Example: ($\mu \neq \nu_1, \nu_2$, F_i, G_i, $\bar{G}_i$, H_i, $\bar{H}_i$, I_i are integrals over Bessel functions)

$$C^{\overline{\text{MS}},\text{RI}'_1}_{\gamma_\mu,\gamma_\mu} = 1 + \frac{g_{\overline{\text{MS}}}{}^2}{16\pi^2} C_F \times$$

$$\Big\{15 - 4F_2 + \beta(-1 + 2F_1 - F_4) + s_0 + 2i(\bar{\mathsf{q}}\cdot\mathsf{r})\beta(F_1 - F_2)$$

$$-2iz\bar{q}_{\nu_1}(G_1 + \bar{G}_1 + G_2 + \bar{G}_2) + \bar{q}_\mu^2/\bar{q}^2(2\beta F_4 - 2(2F_1 - 4F_2 + F_4))$$

$$-(r^2\bar{q}_\mu^2)\beta F_3 + 2iy\bar{q}_{\nu_2}(-H_1 - H_2 + H_3 + H_4) - 4i(y - y')\bar{q}_{\nu_2}(I_1 + I_2)$$

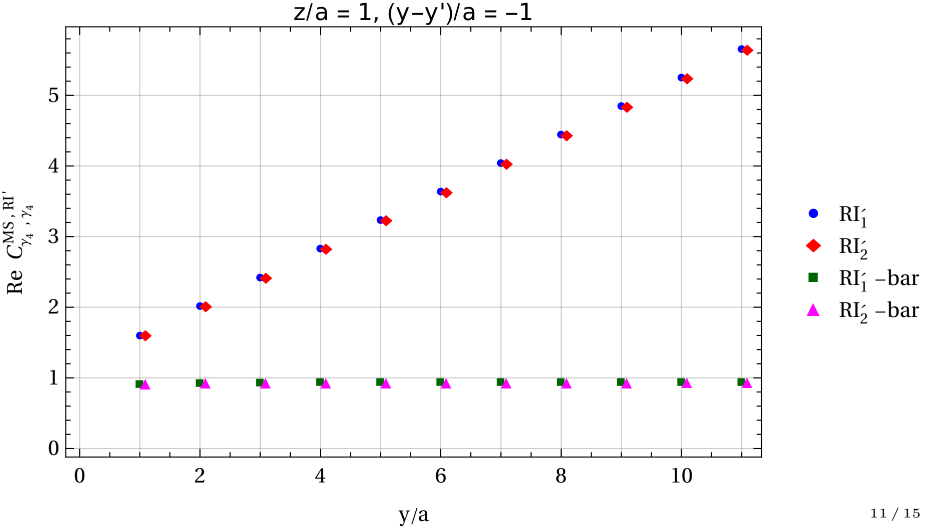
$$+2iy'\bar{q}_{\nu_2}(\bar{H}_1 + \bar{H}_2 - \bar{H}_3 - \bar{H}_4) + 4i(\bar{\mathsf{q}}\cdot\mathsf{r})\bar{q}_\mu^2/\bar{q}^2 F_3\Big\} + \mathcal{O}(g^4_{\overline{\text{MS}}}),$$

$$s_0 = 2(6+\beta)\gamma_E + 4\left[y/z\tan^{-1}(y/z) + y'/z\tan^{-1}\left(y'/z\right) - (y-y')/z\tan^{-1}((y-y')/z)\right] + (1-\beta)\ln\left(\bar{\mu}^2/q^2\right)$$

$$+(2+\beta)\ln\left(\bar{\mu}^2 r^2/4\right) + 4\ln\left(\bar{\mu}^2 z^2/4\right) + 2\left[\ln\left(1 + z^2/y^2\right) + \ln\left(1 + z^2/y'^2\right)\right].$$

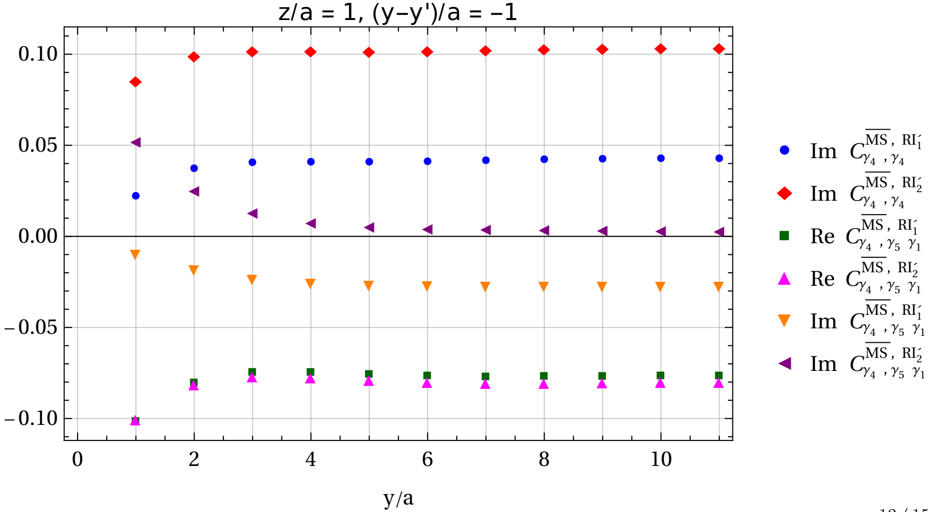
One-loop calculation in DR

- **Example: Unpolarized TMD PDF** $\Gamma = \gamma_4$, $(\nu_1, \nu_2, \nu_3, \nu_4) = (2, 3, 1, 4)$
  ETMC ensembles: $\beta = 1$, $\bar{\mu} = 2$ GeV, $a \simeq 0.09$ fm, $L/a = 24$, $T/a = 48$,
  $(a\bar{q}) = 2\pi a(n_1/L, n_2/L, n_3/L, (n_4 + 0.5)/T)$, $(n_1, n_2, n_3, n_4) = (3, 3, 3, 5)$



One-loop calculation in DR

- **Example: Unpolarized TMD PDF** $\Gamma = \gamma_4$, $(\nu_1, \nu_2, \nu_3, \nu_4) = (2, 3, 1, 4)$
  ETMC ensembles: $\beta = 1$, $\bar{\mu} = 2$ GeV, $a \simeq 0.09$ fm, $L/a = 24$, $T/a = 48$,
  $(a\bar{q}) = 2\pi a(n_1/L, n_2/L, n_3/L, (n_4 + 0.5)/T)$, $(n_1, n_2, n_3, n_4) = (3, 3, 3, 5)$



One-loop calculation on the lattice

- **Green's function of staple-shaped operator with n cusps and total length ℓ using Wilson/clover fermions and Symanzik-improved gluons:**

$$\Lambda_\Gamma^{\rm LR}(q,\ell_1,\ell_2,\dots\ell_{n+1}) = \Lambda_\Gamma^{\overline{\rm MS}}(q,\ell_1,\ell_2,\dots\ell_{n+1},\bar\mu) - \frac{g^2\,C_F}{16\,\pi^2}\,e^{i\,\mathsf{q}\cdot\mathsf{r}}\times$$

$$\Big\{2\,\Gamma\Big[\boldsymbol{\alpha_1} + 16\pi^2\boldsymbol{P_2}\,\beta + (1-\beta)\ln(a^2\bar\mu^2)\Big] + (\Gamma\hat{\psi}_i + \hat{\psi}_f\Gamma)(\boldsymbol{\alpha_2}\,r + \boldsymbol{\alpha_3}\,c_{SW})$$

$$+(n+1)\Gamma\Big[\boldsymbol{\alpha_4} - 16\pi^2\boldsymbol{P_2}\,\beta + (2+\beta)\ln(a^2\bar\mu^2)\Big] + \Gamma\boldsymbol{\alpha_5}\,\frac{\ell}{a},$$

$$+n\Gamma\Big[\boldsymbol{\alpha_6} + 16\pi^2\boldsymbol{P_2}\,\beta - \beta\ln(a^2\bar\mu^2)\Big]\Big\} + \mathcal{O}\left(g^4\right),$$

where $P_2 = 0.02401318111946489(1)$, $r(c_{\rm SW})$ is the Wilson (clover) parameter, ℓ_i is the length of the $i^{\rm th}$ segment ($\sum_{i=1}^{n+1}\ell_i = \ell$), r is the vector connecting the two end points, and $\hat{\nu}_i$ ($\hat{\nu}_f$) is the direction of the Wilson line in the initial (final) end point.

* end-point divergences, mixing, contact divergences, linear divergences, cusp divergences

Gluon action	$\boldsymbol{\alpha_1}$	$\boldsymbol{\alpha_2}$	$\boldsymbol{\alpha_3}$	$\boldsymbol{\alpha_4}$
Wilson	-4.464066(5)	7.224955(7)	-4.142333(4)	-4.52575(1)
Tree-Level Symanzik	-4.341269(5)	6.377911(6)	-3.836778(4)	-3.93028(1)
Iwasaki	-4.163735(5)	4.968266(5)	-3.263819(3)	-1.90532(1)

One-loop calculation on the lattice

- **Green's function of rectangular Wilson loop with lengths ℓ_1, ℓ_2 ($\ell_1 + \ell_2 = \ell$) using Symanzik-improved gluons:**

$$\langle L(\ell_1,\ell_2)\rangle^{\rm LR} = \langle L(\ell_1,\ell_2,\bar\mu)\rangle^{\overline{\rm MS}} - \frac{g^2}{16\pi^2} C_F \left\{ \boldsymbol{b_1} + [4(2+\beta) - 4\beta] \ln(a^2\bar\mu^2) + \boldsymbol{b_2}\ \frac{\ell}{a} \right\} + \mathcal{O}\left(g^4\right).$$

* contact divergences, cusp divergences, linear divergences, $b_1 = 4(\alpha_4 + \alpha_6)$, $b_2 = 2\alpha_5$

Gluon action	$\boldsymbol{\alpha_5}$	$\boldsymbol{\alpha_6}$	$\boldsymbol{b_1}$	$\boldsymbol{b_2}$
Wilson	19.95484(2)	0	-18.10303(1)	39.90968(4)
Tree-Level Symanzik	17.29374(2)	-0.809890(1)	-18.96069(3)	34.58748(3)
Iwasaki	12.97809(1)	-2.101083(2)	-16.02564(3)	25.95618(3)

- **Renormalization functions in $\overline{\rm MS}$:** (See our forthcoming paper)

$$Z^{\overline{\rm MS},{\rm LR}}_{\Gamma,\Gamma} = 1 - \frac{g^2_{\overline{\rm MS}} C_F}{16\pi^2}\Big[\boldsymbol{e^\Gamma(n)} - \boldsymbol{\alpha_5}\frac{\ell}{a} + \boldsymbol{e_2^\psi}\ c_{SW} + \boldsymbol{e_3^\psi}\ c_{SW}^2 - (2n+3)\log\left(a^2\bar\mu^2\right)\Big] + \mathcal{O}(g^4_{\overline{\rm MS}}),$$

$$Z^{\overline{\rm MS},{\rm LR}}_{\Gamma,(\Gamma\hat{\not\psi}_i + \hat{\not\psi}_f\Gamma)} = \frac{g^2_{\overline{\rm MS}} C_F}{16\pi^2}\Big[\boldsymbol{\alpha_2} r + \boldsymbol{\alpha_3} c_{SW}\Big] + \mathcal{O}(g^4_{\overline{\rm MS}}),$$

$$\bar Z^{\overline{\rm MS},{\rm LR}}_{\Gamma,\Gamma} = 1 - \frac{g^2_{\overline{\rm MS}} C_F}{16\pi^2}\Big[\boldsymbol{e^\Gamma(0)} + \boldsymbol{e_2^\psi}\ c_{SW} + \boldsymbol{e_3^\psi}\ c_{SW}^2 - 3\log\left(a^2\bar\mu^2\right)\Big] + \mathcal{O}(g^4_{\overline{\rm MS}}), \qquad (n=2),$$

$$\bar Z^{\overline{\rm MS},{\rm LR}}_{\Gamma,(\Gamma\hat{\not\psi}_i + \hat{\not\psi}_f\Gamma)} = Z^{\overline{\rm MS},{\rm LR}}_{\Gamma,(\Gamma\hat{\not\psi}_i + \hat{\not\psi}_f\Gamma)} + \mathcal{O}(g^4_{\overline{\rm MS}}), \qquad (n=2),$$

where $\boldsymbol{e^\Gamma(n)} = \Big[\boldsymbol{e_1^\psi} + 1 - 2\boldsymbol{\alpha_1} - (n+1)\boldsymbol{\alpha_4} - n\boldsymbol{\alpha_6}\Big].$

Conclusions and future prospects

- One-loop perturbative study of asymmetric staple-shaped operators in both DR and LR.
- Provide different RI$'$-type prescriptions for renormalization (Different projectors, standard RI$'$ vs RI$'$-bar).
- Identify mixing sets through symmetries:
  chiral fermions: $(\Gamma, \pm\Gamma\gamma_{\nu_1}\gamma_{\nu_2})$
  non-chiral fermions: $(\Gamma, \pm\Gamma\gamma_{\nu_1}\gamma_{\nu_2}, \pm\Gamma\gamma_{\nu_1}, \pm\Gamma\gamma_{\nu_2})$
- Confirm (in one-loop perturbation theory) that RI$'$-bar addresses all divergences: linear, cusp, end-point, contact and pinch-pole singularities
- Extract one-loop conversion functions for all 16 quark bilinear operators
- **Future plans:** One-loop calculation of lattice artifacts,
  Extend our study to 2 loops
- **Acknowledgements:**
  This work is funded by the European Regional Development Fund and the Republic of Cyprus through the Research and Innovation Foundation (Project: EXCELLENCE/0421/0025).

Conclusions and future prospects

- One-loop perturbative study of asymmetric staple-shaped operators in both DR and LR.
- Provide different RI$'$-type prescriptions for renormalization (Different projectors, standard RI$'$ vs RI$'$-bar).
- Identify mixing sets through symmetries:
  chiral fermions: $(\Gamma, \pm\Gamma\gamma_{\nu_1}\gamma_{\nu_2})$
  non-chiral fermions: $(\Gamma, \pm\Gamma\gamma_{\nu_1}\gamma_{\nu_2}, \pm\Gamma\gamma_{\nu_1}, \pm\Gamma\gamma_{\nu_2})$
- Confirm (in one-loop perturbation theory) that RI$'$-bar addresses all divergences: linear, cusp, end-point, contact and pinch-pole singularities
- Extract one-loop conversion functions for all 16 quark bilinear operators
- **Future plans:** One-loop calculation of lattice artifacts,
  Extend our study to 2 loops
- **Acknowledgements:**
  This work is funded by the European Regional Development Fund and the Republic of Cyprus through the Research and Innovation Foundation (Project: EXCELLENCE/0421/0025).

THANK YOU!