Trivializing Flow in 2D-$O(3)$ Model

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Trivializing Map Overview

- A map can be used to connect two distinct probability distributions to each other.
- Trivializing Map is a bijective map between a trivial distribution and one that is difficult to sample.
- Typical methods for sampling these ‘difficult‘ distributions is via Markov Chain Monte Carlo. MCMC has its own issues, long autocorrelation times
- With a trivializing map, it would then be possible to sample the trivial distribution to generate unique uncorrelated samples from the target distribution

Trivial Case

$$\psi \sim \mathcal{N}(0,1) \leftrightarrow \phi \sim \mathcal{N}(100,1)$$

Thus we define a map, $\mathcal{F} : \phi \to \psi$

$$\psi = \mathcal{F}(\phi) = \phi - 100$$

which is bijective

$$\phi = \mathcal{F}^{-1}(\psi) = \psi + 100$$

Trivializing Flow

A trivializing flow, is a trivializing map defined by a solution to a differential equation defined by a generating functional, $\tilde{S}$.

$$\dot{\phi}(x, y) = \partial_x \tilde{S}(\phi(t)) \qquad \partial_x \equiv \frac{\partial}{\partial \phi(x)}$$

We may then consider 2 probability distributions, $p(\phi), q(\psi)$, given by:

$$p(\phi) \equiv \frac{e^{-S(\phi)}}{Z} \qquad q(\psi) \equiv \frac{e^{-S_0(\psi)}}{Z_0}$$

After applying the flow, which is a change of variables, where $J(\phi)$ is the determinate of the Jacobian matrix, $\mathcal{J}_{xy}$.

$$q(\psi)\mathcal{D}\psi = q(\mathcal{F}(\phi))\mathcal{J}(\phi)\mathcal{D}\phi$$

Thus our matching condition is that this is equal to our trivial distribution.

$$p(\phi) = q(\mathcal{F}(\phi))\mathcal{J}(\phi)$$

$$\frac{e^{-S(\phi)}}{Z} = \frac{e^{-S_0(\mathcal{F}(\phi))}}{Z_0}$$

$$S(\phi) = S_0(\mathcal{F}(\phi)) - \ln[\mathcal{J}(\phi)] + \ln[\frac{Z_0}{Z}]$$

Time Dependent Flow

Consider a time-dependent flow such that, $\phi(t) \equiv \mathcal{F}_t^{-1}(\psi, t)$ must satisfy a time-dependent probability distribution, $p_t(\phi(t), t)$. Where,

$$p_t(\phi(t), t) = \frac{e^{-S_t(\phi(t), t)}}{Z_t(t)}$$

Here the time-dependent map, $\mathcal{F}_t(\phi, t)$, is defined such that at $t = 1$, the distribution is trivial and at $t = 0$ is our target distribution.

$$p_t(\phi(1), 1) = q(\psi) \qquad p_t(\phi(0), 0) = p(\phi)$$

$$\mathcal{F}_t(\phi(1), 1) = \mathcal{F}(\phi) = \psi \qquad \mathcal{F}_t^{-1}(\phi(0), 0) = \phi$$

$$S_t(\phi(1), 1) = S_0(\psi) \qquad S_t(\phi(0), 0) = \mathcal{S}(\phi)$$

$$Z_t(1) = Z_0 \qquad Z_t(0) = Z$$

This construction leads to the same condition as before, now with time-dependent pieces.

$$S_t(\phi(t), t) = S(\mathcal{F}(\phi(t), t)) - \ln[\mathcal{J}(\phi)] + C(t)$$

As Z and $Z_t(t)$ are independent of the fields, we may replace them with a time-dependent constant, $C(t)$. Note $C(0) = 0$

Time Dependent Flow

Consider the full time derivative of the previous equation.

$$\frac{d}{dt}\left[S_t(\phi(t),t) = \mathcal{S}(\mathcal{F}(\phi(t),t)) - \ln[\mathcal{J}(\phi)] + C(t)\right]$$

$$\sum_x \partial_x S_t(\phi(t),t)\dot{\phi}(x,t) + \partial_t S_t(\phi(t),t) = \sum_x \partial_x \mathcal{S}(\mathcal{F}(\phi(t),t))\dot{\phi}(x,t) - \frac{d\ln[\mathcal{J}(t)]}{dt} + \dot{C}(t)$$

The Jacobian Term can be simplified,

$$\begin{aligned}
\frac{d\ln(\mathcal{J}(t))}{dt} &= \mathrm{Tr}\left[J^{-1}_{xy}\dot{J}_{yx}\right] = \sum_{x,y}\left[\left(\frac{\partial\phi(y,t)}{\partial\phi(x)}\right)^{-1}\frac{\partial\dot{\phi}(y,t)}{\partial\phi(x)}\right] \\
&= \sum_{x,y}\left[\frac{\partial\phi(x)}{\partial\phi(y,t)}\frac{\partial\dot{\phi}(y,t)}{\partial\phi(x)}\right] = \sum_y \frac{\partial\dot{\phi}(y,t)}{\partial\phi(y,t)} \\
&= \sum_y \frac{\partial}{\partial\phi(y,t)}\dot{\phi}(y,t) = \sum_y \frac{\partial}{\partial\phi(y,t)}\partial_y\tilde{S} = \sum_y \partial_y\partial_y\tilde{S} \\
\frac{d\ln(\mathcal{J}(t))}{dt} &= \partial^2\tilde{S}
\end{aligned}$$

In which case the full equation is,

$$-\partial^2\tilde{S} + \sum_x \partial_x\mathcal{S}\,\partial_x\tilde{S} - \sum_x \partial_x S_t\,\partial_x\tilde{S} = \partial_t S_t - \dot{C}(t)$$

Model Freedom

At this point we have an equation related two unknown quantities, S_t and $\tilde{S}$. The choice of one determines the other. Consider a choice of S_t that linear interpolates to the trivial action.

$$S_t(\phi(t), t) = \mathcal{S}(\phi(t)) + t\Big(S_0(\phi(t)) - \mathcal{S}(\phi(t))\Big)$$

Then using this form,

$$-\partial^2\tilde{S} + \sum_x \partial_x\mathcal{S}\,\partial_x\tilde{S} - \sum_x \partial_x(\mathcal{S} + t(S_0 - \mathcal{S}))\partial_x\tilde{S} = \partial_t(\mathcal{S} + t(S_0 - \mathcal{S})) - \dot{C}(t)$$

$$-\partial^2\tilde{S} + \sum_x \partial_x\mathcal{S}\,\partial_x\tilde{S} - \sum_x \partial_x\mathcal{S}\,\partial_x\tilde{S} - t\sum_x \partial_x(S_0 - \mathcal{S})\partial_x\tilde{S} = (S_0 - \mathcal{S}) - \dot{C}(t)$$

$$-\partial^2\tilde{S} - t\sum_x \partial_x(S_0 - \mathcal{S})\partial_x\tilde{S} = S_0 - \mathcal{S} - \dot{C}(t)$$

As S_0 is the trivial distribution, $S_0 = 0$, as long as the elements are part of a compact group.

$$-\partial^2\tilde{S} + t\sum_x \partial_x\mathcal{S}\,\partial_x\tilde{S} = -\mathcal{S} - \dot{C}(t)$$

Small Flow Time Expansion

By expanding $\tilde{S}$, $\dot{C}$ in flow time,

$$\tilde{S} \equiv \sum_n t^n S^{(n)} \qquad\qquad \dot{C} \equiv \sum_n t^n \dot{C}^{(n)}$$

we may find a perturbative solution for $\tilde{S}$.

$$-\partial^2 \big(\sum_{n=0}^{\infty} t^n S^{(n)} \big) + t \sum_x \left[\partial_x \mathcal{S} \left(\partial_x \big(\sum_{n=0}^{\infty} t^n S^{(n)} \big) \right) \right] = -\mathcal{S} - \sum_{n=0}^{\infty} t^n \dot{C}^{(n)}$$

$$\sum_{n=0}^{\infty} \left[-t^n \partial^2 S^{(n)} + t^{n+1} \sum_x \left(\partial_x \mathcal{S} \, \partial_x S^{(n)} \right) \right] = -\mathcal{S} - \sum_{n=0}^{\infty} t^n \dot{C}^{(n)}$$

Then collecting powers of t. There are 2 classes of functions.[1]

$$n = 0: \qquad -\partial^2 S^{(0)} = -\mathcal{S} - \dot{C}^{(0)}$$

$$n \geq 1: \qquad -\partial^2 S^{(n)} + \sum_x \partial_x \mathcal{S} \, \partial_x S^{(n-1)} = -\dot{C}^{(n)}$$

[1] Luscher, "Trivializing maps, the Wilson flow and the HMC algorithm".

The $O(3)$ Model

The $O(3)$ model is defined by the 2D Euclidean lattice action

$$\mathcal{S}[s] = -\frac{1}{2}\beta \sum_{x,\hat{\mu}} \langle s_x | s_{x+\hat{\mu}} \rangle$$

where $s \in \mathcal{S}^2$. Thus they have a probability distribution defined by this action

$$p[s] = \frac{e^{-\mathcal{S}[s]}}{\mathcal{Z}}, \qquad \mathcal{Z} = \int \mathcal{D}s\, e^{-\mathcal{S}[s]}$$

and the integration measure, $\mathcal{D}s$, is over $\mathcal{S}^2$.

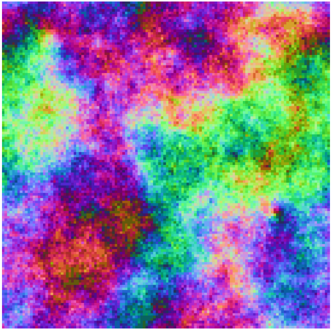
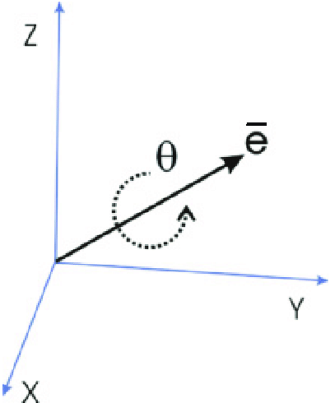


Figure: Example distribution (x,y,z) $\rightarrow$ (r,g,b)

[1]Bietenholz et al., "Topological susceptibility of the 2D O(3) model under gradient flow".

Defining a Flow Equation



When determining a gradient flow it is important to choose dynamics that preserve properties of the elements.

$$\forall R \in SO(3) \quad \& \quad \forall s \in \mathcal{S}^2$$
$$R * s \in \mathcal{S}^2$$

Therefore, a general map that takes an S^2 valued vector field s and maps to an S^2 valued vector field σ, can be written as

$$\sigma(x) = R(x)\, s(x)\,,$$

where $R(x)$ is an element of SO(3) which is determined up to an SO(2) rotation.

The quotient SO(3)/SO(2) subgroup of SO(3) is diffeomorphic to the sphere $\mathcal{S}^2$ manifold.

Unit Vectors to Rotations

By parameterizing the field with respect to rotation of a reference vector, s_0,

$$s_x = R(x) s_{0,x}$$

the fields may be represented in the Lie algebra, $\mathfrak{so}(3)$ of SO(3), which is isomorphic to $\mathbb{R}^3$ with cross product. The Lie derivative, ∂^a_x, is well defined on these elements.

$$R(x) = \sum_\alpha \omega^\alpha_x T_\alpha$$
$$\partial^\beta_y R(x) = \sum_\alpha \omega^\alpha_x T_\alpha \; \delta_{\alpha,\beta} \; \delta_{x,y}$$

Using a scalar functional of the fields, $\tilde{S}$, whose gradient defines the generator of the flow,

$$\dot{s}(x,t) = -\sum_a T_a \partial^\alpha_x \tilde{S}[s(x,t)] \, s_x,$$

we may preserve the symmetries of the system.

0^{th} Order

We want to solve:

$$-\partial^2 S^{(0)} = -\mathcal{S}$$

with,

$$\mathcal{S} = \frac{-\beta}{2} \sum_x \sum_\mu \langle s_x | s_{x+\mu} \rangle \qquad\qquad S^{(0)} = \gamma_0 \sum_x \sum_\mu \langle s_x | s_{x+\mu} \rangle$$

Thus expanding on the Left Hand Side:

$$\begin{aligned}
-\partial^2 S^{(0)} &= -\sum_x \sum_a \partial_x^a \partial_x^a \left[\gamma_0 \sum_y \sum_\mu \langle s_y | s_{y+\mu} \rangle \right] \\
&= -\gamma_0 \sum_x \sum_a \partial_x^a \left[\sum_y \sum_\mu - \langle s_y | T^a | s_{y+\mu} \rangle \delta_{x,y} + \langle s_y | T^a | s_{y+\mu} \rangle \delta_{x,y+\mu} \right] \\
&= -\gamma_0 \sum_x \sum_a \left[\sum_\mu \langle s_x | T^a T^a | s_{x+\mu} \rangle + \langle s_{x-\mu} | T^a T^a | s_x \rangle \right] \\
&= \gamma_0 \sum_x \left[\sum_\mu \langle s_x | C_F | s_{x+\mu} \rangle + \langle s_{x-\mu} | C_F | s_x \rangle \right] \\
-\partial^2 S^{(0)} &= 2 C_F \gamma_0 \sum_x \sum_\mu \langle s_x | s_{x+\mu} \rangle = 2 C_F S^{(0)}
\end{aligned}$$

Therefore

$$S^{(0)} = \frac{\beta}{8} \sum_x \sum_\mu \langle s_x | s_{x+\mu} \rangle$$

1^{st} Order - Gradient Product

We want to solve

$$-\partial^2 S^{(1)} + \sum_x \sum_a \left(\partial_x^a \mathcal{S}\right)\left(\partial_x^a S^{(0)}\right) = 0$$

First lets consider the term

$$\begin{aligned}
\sum_x \sum_a \left(\partial_x^a \mathcal{S}\right)\left(\partial_x^a S^{(0)}\right) &= \frac{-\beta^2}{16} \sum_{x,a} \Big[\big(\partial_x^a \sum_{y,\mu} \langle s_y | s_{y+\mu} \rangle \big) \big(\partial_x^a \sum_{z,\nu} \langle s_z | s_{z+\nu} \rangle \big) \Big] \\
&= \frac{-\beta^2}{16} \sum_{x,a} \Big[\big(\sum_{\mu} - \langle s_x | T^a | s_{x+\mu} \rangle + \langle s_{x-\mu} | T^a | s_x \rangle \big) \\
&\qquad \big(\sum_{\nu} - \langle s_x | T^a | s_{x+\nu} \rangle + \langle s_{x-\nu} | T^a | s_x \rangle \big) \big) \Big] \\
&= \frac{-\beta^2}{16} \sum_{x,a} \Big[\big(\sum_{\mu} -2 \langle s_x | T^a | s_{x+\mu} \rangle \big) \big(\sum_{\nu} -2 \langle s_x | T^a | s_{x+\nu} \rangle \big) \Big] \\
&= \frac{-\beta^2}{4} \sum_{x,a} \Big[\sum_{\mu,\nu} \langle s_x | T^a | s_{x+\mu} \rangle \langle s_x | T^a | s_{x+\nu} \rangle \Big] \\
&= \frac{-\beta^2}{4} \sum_{x} \Big[\sum_{\mu,\nu} \langle s_x | s_x \rangle \langle s_{x+\mu} | s_{x+\nu} \rangle - \langle s_x | s_{x+\mu} \rangle \langle s_x | s_{x+\nu} \rangle \Big]
\end{aligned}$$

$$\sum_x \sum_a \left(\partial_x^a \mathcal{S}\right)\left(\partial_x^a S^{(0)}\right) = \frac{-\beta^2}{4} \sum_{x} \Big[\sum_{\mu,\nu} \langle s_x | s_{x+\mu+\nu} \rangle - \langle s_x | s_{x+\mu} \rangle \langle s_x | s_{x+\nu} \rangle \Big]$$

1^{st} Order - Solution

Again we choose $S^{(1)}$ to be parameterized by terms we are trying to cancel.

$$S^{(1)} = \gamma_1 \Psi^{(2)} + \gamma_2 \tilde{\Psi}^{(1,1)}$$

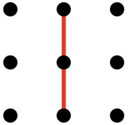
where,

$$\Psi^{(2)} = \sum_x \sum_{\mu,\nu} \langle s_x | s_{x+\mu+\nu} \rangle \qquad \tilde{\Psi}^{(1,1)} = \sum_x \sum_{\mu,\nu} \langle s_x | s_{x+\mu} \rangle \langle s_x | s_{x+\nu} \rangle$$

And applying the ∂^2, we find a new term to arise,

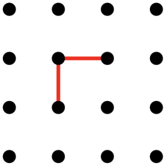
$$\begin{aligned}
&= -\sum_{x,a} \partial_x^a \partial_x^a \sum_y \sum_{\mu,\nu} \Big[\langle s_y | s_{y+\mu} \rangle \langle s_y | s_{y+\nu} \rangle \\
&\quad - \langle s_y | s_{y+\mu} \rangle \langle s_y | T^a | s_{y+\nu} \rangle \delta_{x,y} + \langle s_y | s_{y+\mu} \rangle \langle s_y | T^a | s_{y+\nu} \rangle \delta_{x,y+\nu} \Big] \\
&= -\sum_{x,a} \sum_{\mu,\nu} \partial_x^a \Big[\langle s_{x-\mu} | T^a | s_x \rangle \langle s_{x-\mu} | s_{x-\mu+\nu} \rangle + ... \Big] \\
&= -\sum_{x,a} \sum_{\mu,\nu} \Big[\langle s_{x-\mu} | T^a | s_x \rangle \langle s_{x-\mu} | T^a | s_{x-\mu+\nu} \rangle \delta_{x,x-\mu+\nu} + ... \Big] \\
&= -\sum_x \sum_\mu \Big[\langle s_{x-\mu} | s_{x-\mu} \rangle \langle s_x | s_x \rangle - \langle s_{x-\mu} | s_x \rangle \langle s_{x-\mu} | s_x \rangle + ... \Big] \\
&= -\sum_x \sum_\mu \Big[1 - \langle s_x | s_{x+\mu} \rangle^2 + ... \Big] = \sum_x \sum_\mu \langle s_x | s_{x+\mu} \rangle^2 + ...
\end{aligned}$$

Therefore, we can define this new term, and add it to our parameterization of $S^{(1)}$.



$$\Psi^{(1,1f)} = \sum_x \sum_\mu \langle s_x | s_{x+\mu} \rangle^2$$

$$S^{(1)} = \gamma_1 \Psi^{(2)} + \gamma_2 \tilde{\Psi}^{(1,1)} + \gamma_3 \Psi^{(1,1f)}$$

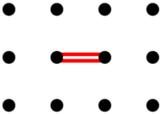


then including this term, we may find a solution,

$$S^{(1)} = \frac{\beta^2}{40} \left[2\Psi^{(2)} - \tilde{\Psi}^{(1,1)} + \frac{1}{6} \Psi^{(1,1f)} \right]$$

Combining with the previous order,

$$\tilde{S} = \frac{\beta}{8} \Psi^{(1)} + \frac{\beta^2}{40} \left[2\Psi^{(2)} - \tilde{\Psi}^{(1,1)} + \frac{1}{6} \Psi^{(1,1f)} \right] t + \mathcal{O}(t^2)$$



2^{nd} Order - Terms

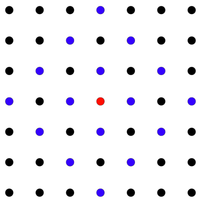


Figure: $\Psi^{(3)}$

$$\Psi^{(1)} \equiv \sum_x \sum_\mu \langle s_x | s_{x+\mu} \rangle$$

$$\Psi^{(3)} \equiv \sum_x \sum_{\mu,\nu,\lambda} \left\langle s_x \middle| s_{x+\mu+\nu+\lambda} \right\rangle$$

$$\tilde{\Psi}^{(2,1)} \equiv \sum_x \sum_{\mu,\nu,\lambda} \langle s_x | s_{x+\mu+\nu} \rangle \langle s_x | s_{x+\lambda} \rangle$$

$$\tilde{\Psi}^{(1,2)}_{disc} \equiv \sum_x \sum_{\mu,\nu,\lambda} \langle s_x | s_{x+\mu} \rangle \langle s_{x+\nu} | s_{x+\lambda} \rangle$$

$$\tilde{\Psi}^{(1,1,1)}_{branch} \equiv \sum_x \sum_{\mu,\nu,\lambda} \langle s_x | s_{x+\mu} \rangle \langle s_x | s_{x+\nu} \rangle \langle s_x | s_{x+\lambda} \rangle$$

$$\tilde{\Psi}^{(1,1,1)}_{chain} \equiv \sum_x \sum_{\mu,\nu,\lambda} \langle s_x | s_{x+\mu} \rangle \langle s_{x+\mu} | s_{x+\mu+\nu} \rangle \left\langle s_{x+\mu+\nu} \middle| s_{x+\mu+\nu+\lambda} \right\rangle$$

$$\tilde{\Psi}^{(1,2f)} \equiv \sum_x \sum_{\mu,\nu} \langle s_x | s_{x+\mu} \rangle \langle s_x | s_{x+\mu+\nu} \rangle$$

$$\tilde{\Psi}^{(1,1f,1)} \equiv \sum_x \sum_{\mu,\nu} \langle s_x | s_{x+\mu} \rangle^2 \langle s_x | s_{x+\nu} \rangle$$

$$\tilde{\Psi}^{(1,1f,1f)} \equiv \sum_x \sum_\mu \langle s_x | s_{x+\mu} \rangle^3$$

2^{nd} Order - Laplacians

$$
\begin{aligned}
-\partial^2 \Psi^{(1)} &= 4\Psi^{(1)} \\
-\partial^2 \Psi^{(3)} &= 4\Psi^{(3)} \\
-\partial^2 \tilde{\Psi}^{(2,1)} &= 10\tilde{\Psi}^{(2,1)} - 2\Psi^{(3)} - 16\Psi^{(1)} \\
-\partial^2 \tilde{\Psi}^{(1,2)}_{d} &= 8\tilde{\Psi}^{(1,2)}_{d} - 32\Psi^{(1)} + 4\tilde{\Psi}^{(1,2f)} \\
-\partial^2 \tilde{\Psi}^{(1,1,1)}_{branch} &= 18\tilde{\Psi}^{(1,1,1)}_{branch} - 6\tilde{\Psi}^{(1,2)}_{d} - 24\Psi^{(1)} + 6\tilde{\Psi}^{(1,1f,1)} \\
-\partial^2 \tilde{\Psi}^{(1,1,1)}_{chain} &= 16\tilde{\Psi}^{(1,1,1)}_{chain} - 16\Psi^{(1)} + 8\tilde{\Psi}^{(1,1f,1)} - 4\tilde{\Psi}^{(2,1)} - 4\tilde{\Psi}^{(1,2f)} \\
-\partial^2 \tilde{\Psi}^{(1,2f)} &= 10\tilde{\Psi}^{(1,2f)} - 12\Psi^{(1)} \\
-\partial^2 \tilde{\Psi}^{(1,1f,1)} &= 20\tilde{\Psi}^{(1,1f,1)} - 20\Psi^{(1)} - 4\tilde{\Psi}^{(1,2f)} + 4\tilde{\Psi}^{(1,1f,1f)} \\
-\partial^2 \tilde{\Psi}^{(1,1f,1f)} &= 24\tilde{\Psi}^{(1,1f,1f)} - 12\Psi^{(1)}
\end{aligned}
$$

Our goal is to solve,

$$
-\partial^2 S^{(1)} + \sum_x \sum_a \left(\partial_x^a \mathcal{S}\right)\left(\partial_x^a S^{(0)}\right) = 0
$$

$$
\partial^2 S^{(1)} = \sum_x \sum_a \left(\partial_x^a \mathcal{S}\right)\left(\partial_x^a S^{(0)}\right)
$$

which can be translated to,

$$
\hat{A}\vec{x} = \vec{b} \implies \vec{x} = \hat{A}^{-1}\vec{b}
$$

2^{nd} Order - Solution

Solving the next order gives:

$$S^{(2)} = \frac{\beta^3}{72000}\Big[-4467\Psi^{(1)} + 1305\Psi^{(3)} - 990\tilde{\Psi}^{(2,1)} - 300\tilde{\Psi}_d^{(1,2)} + 200\tilde{\Psi}_{branch}^{(1,1,1)} + 225\tilde{\Psi}_{chain}^{(1,1,1)} + 246\tilde{\Psi}^{(1,2f)} - 210\tilde{\Psi}^{(1,1f,1)} + 35\tilde{\Psi}^{(1,1f,1f)}\Big]$$

Procedure can be repeated to any order but this is where our analytic approach ends.
2nd Order has > 30 unique terms, 3rd Order has > 100 unique terms

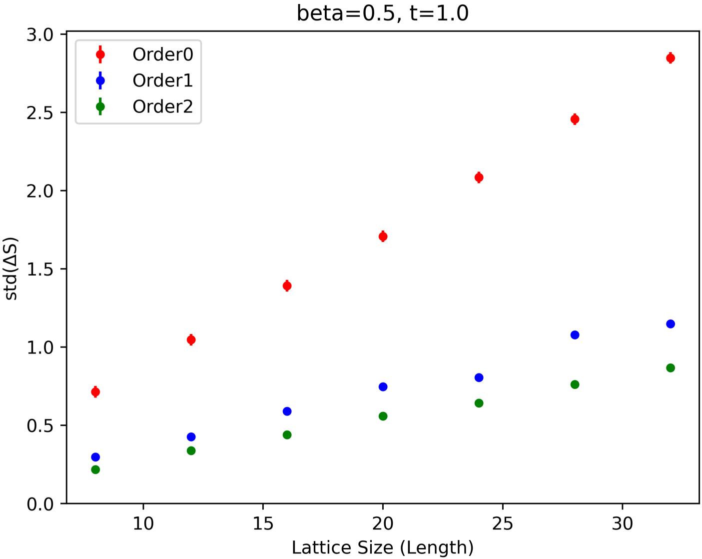
$$\tilde{S} = \frac{\beta}{8}\Psi^{(1)} + \frac{\beta^2}{40}\Big[2\Psi^{(2)} - \tilde{\Psi}^{(1,1)} + \frac{1}{6}\Psi^{(1,1f)}\Big]t + \frac{\beta^3}{72000}\Big[-4467\Psi^{(1)} + 1305\Psi^{(3)} - 990\tilde{\Psi}^{(2,1)} - 300\tilde{\Psi}_d^{(1,2)} + 200\tilde{\Psi}_{branch}^{(1,1,1)} + 225\tilde{\Psi}_{chain}^{(1,1,1)} + 246\tilde{\Psi}^{(1,2f)} - 210\tilde{\Psi}^{(1,1f,1)} + 35\tilde{\Psi}^{(1,1f,1f)}\Big]t^2 + \mathcal{O}(t^3)$$

By applying the flow, and observing the variance of the error between, the action and our flowed ensemble, we may determine how well the flow preforms

$$\Delta S \equiv \mathcal{S}(\phi) - \Big[S_0(\mathcal{F}(\phi)) - \ln \mathcal{J}(\phi)\Big] + C(\mathcal{Z}, \mathcal{Z}_0)$$

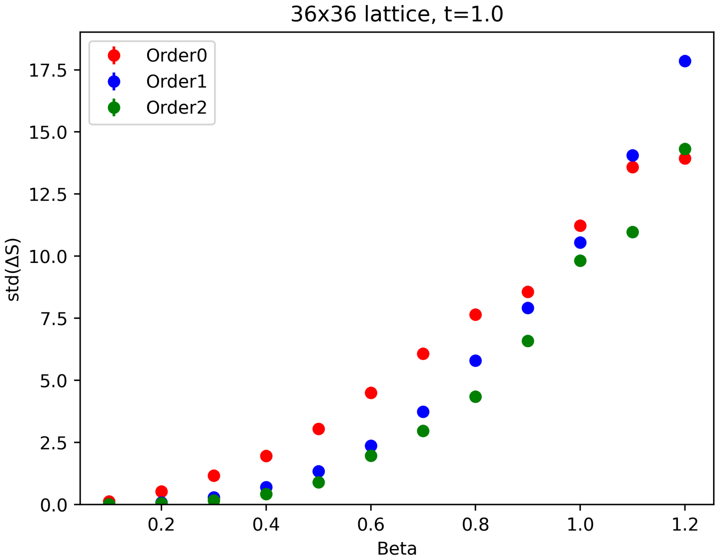
Scaling with Lattice Size

We observe linear scaling with the lattice side length



Scaling with Beta

We observe quadratic scaling with beta with the method breaking as $\beta > 1$



Future Steps

It is worth noting $S^{(0)}$, appears in $S^{(2)}$, therefore all terms of $S^{(1)}$ will appear in $S^{(3)}$. So a subset of the higher order terms is already known.

$$
\begin{aligned}
S^{(2n)} =& \beta^{2n+1} \sum_a \gamma_a^{(2n)} \Psi_{even}^a \\
S^{(2n+1)} =& \beta^{2(n+1)} \sum_a \gamma_a^{(2n+1)} \Psi_{odd}^a
\end{aligned}
$$

By non-perturbative tuning of the γ_a, it might be possible to increase performance further with little cost to preforming the flow.

Conclusion

- It is possible to solve the Luscher equation analytically
- Scaling with lattice volume is inevitable
- Non-perturbative tuning may still improve solution
- Preforming the flow may be just as computationally expensive as HMC