Magnetic polarizability of a charged pion from four-point functions

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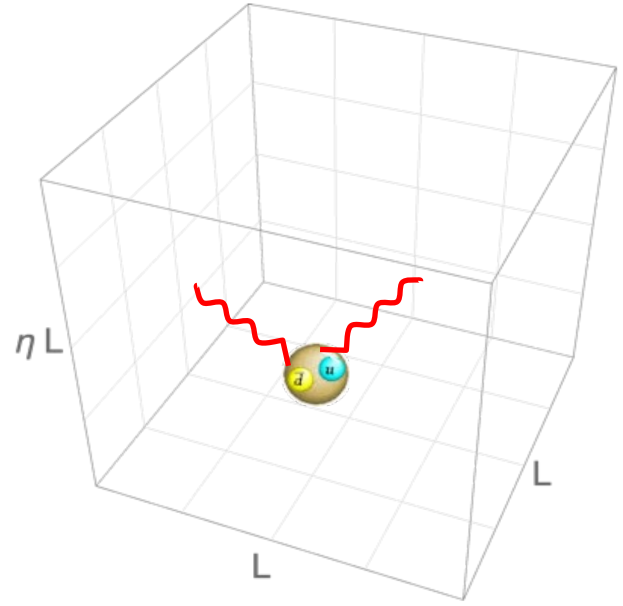
Outline

1) Motivation
2) Background field method
3) Four-point function method
4) Lattice simulations and results
5) Conclusion



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Lattice 2023, August 4, Fermilab

Hadron polarizabilities (in units of 10^{-4} fm^3)

- Polarizabilities encode information on charge and current distributions inside hadrons at low energies.
- An active community in nuclear physics is engaged in the effort (experiment, theory, lattice QCD)

1) Hadrons are stiff
2) QCD+QED

Charged pion ($\pi^\pm$) $\alpha_E = 2.0(6)(7) = -\beta_M$ (PDG)

$\alpha_E = 2.93(5),\ \beta_M = -2.77(11)$ (ChPT)

Neutral pion (π^0) $\alpha_E = -0.69(7)(4) = -\beta_M$ (PDG)

$\alpha_E = -0.40(18)\ ,\ \beta_M = 1.50\ (27)$ (ChPT)

Charged kaon ($K^\pm$) $\alpha_E = 0.58 = -\beta_M$ (ChPT)

IJMPA34 (2019) , Moinester and Scherer

Proton $\alpha_{E1} = 11.2(0.4),\quad \beta_{M1} = 2.5(1.2)$ (PDG)

$\alpha_{E1} = 11.2(0.7),\quad \beta_{M1} = 3.9(0.7)$ (ChPT)

$\gamma_{E1E1} = -3.3(0.8),\quad \gamma_{M1M1} = 2.9(1.5),$
$\gamma_{E1M2} = -0.2(0.2),\quad \gamma_{M1E2} = 1.1(0.3)$ (ChPT)

Neutron $\alpha_{E1} = 11.8(1.1),\quad \beta_{M1} = 3.7(1.2)$ (PDG)

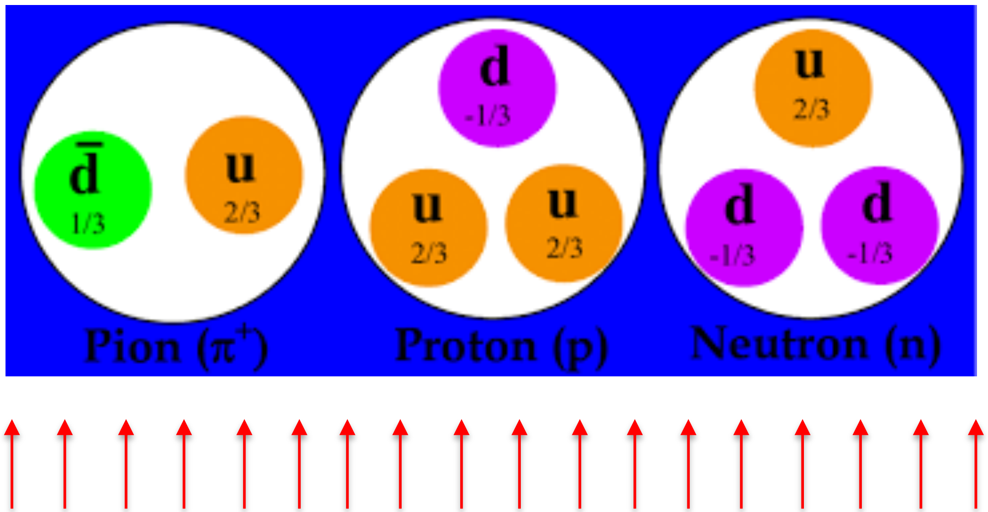
$\alpha_{E1} = 13.7(3.1),\quad \beta_{M1} = 4.6(2.7)$ (ChPT)

$\gamma_{E1E1} = -4.7(1.1),\quad \gamma_{M1M1} = 2.9(1.5),$
$\gamma_{E1M2} = 0.2(0.2),\quad \gamma_{M1E2} = 1.6(0.4)$ (ChPT)

Eur. Phys. J. C75 (2015) Lensky, McGovern, Pascalutsa

Symmetry (2020), Hagelstein.

Background field method in QCD



E or B field

Interaction Hamiltonian for weak fields:

$$
\begin{aligned}
H \;=\; & -\vec{p}\cdot\vec{E} - \vec{\mu}\cdot\vec{B} - \frac{1}{2}\alpha E^2 - \frac{1}{2}\beta B^2 \\
& -\frac{1}{2}\left(\gamma_{E1}\vec{\sigma}\cdot\vec{E}\times\dot{\vec{E}} + \gamma_{M1}\vec{\sigma}\cdot\vec{B}\times\dot{\vec{B}} - 2\gamma_{E2}E_{ij}\sigma_i B_j + 2\gamma_{M2}B_{ij}\sigma_i E_j\right) \\
& -\frac{1}{2}\left(\alpha_{E\nu}\dot{\vec{E}}^2 + \beta_{M\nu}\dot{\vec{B}}^2\right) - \frac{1}{12}4\pi\left(\alpha_{E2}E_{ij}^2 + \beta_{M2}B_{ij}^2\right) + \cdots .
\end{aligned}
$$

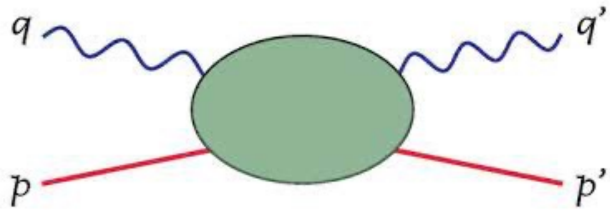
It works well for neutral hadrons (π^0, K^0, n)

New challenges arise for charged particles in background field method:

- Acceleration in electric fields
- Landau levels in magnetic field
- They come at leading order (polarizabilities at 2nd order)
- Their energies must be disentangled from the total to obtain the deformation energy on which polarizabilities are defined.

Alternative approach: four-point functions

- Mimics the Compton scattering process on the lattice
- Instead of background field, electromagnetic currents couple to quarks
- All photon, gluon, and quark interactions are included
- Charged and neutral hadrons are on equal footing



Charged pion polarizability formulas

$$\alpha_E = \frac{\alpha\langle r_E^2\rangle}{3m_\pi} + \frac{2\alpha}{\boldsymbol{q}^2}\int_0^\infty dt\left[Q_{44}(\boldsymbol{q},t) - Q_{44}^{elas}(\boldsymbol{q},t)\right]$$

$$\beta_M = -\frac{\alpha\langle r_E^2\rangle}{3m_\pi} + \frac{2\alpha}{\boldsymbol{q}^2}\int_0^\infty dt\left[Q_{11}^{inel}(\boldsymbol{q},t) - Q_{11}^{inel}(\boldsymbol{0},t)\right]$$

Charge radius can be extracted from elastic part of the same Q_{44} ,

$$Q_{44}^{elas}(\boldsymbol{q},t) = \frac{(E_\pi + m_\pi)^2}{4E_\pi m_\pi} F_\pi^2(\boldsymbol{q}^2) e^{-a(E_\pi(\boldsymbol{q})-m_\pi)t}$$

PRD104 (2021), Wilcox, Lee

PRD108 (2023), Lee, Alexandru, Culver, Wilcox

arXiv:2307.08620 (2023), Lee, Wilcox , Alexandru, Culver

Proton formulas

$$\alpha_E = \frac{\alpha\langle r_E^2\rangle}{3m_p} + \frac{\alpha(1+\kappa^2)}{4m_p^3} + \frac{2\alpha}{\boldsymbol{q}^2}\int_0^\infty dt\,\left[Q_{44}(\boldsymbol{q},t) - Q_{44}^{elas}(\boldsymbol{q},t)\right]$$

$$\beta_M = -\frac{\alpha\langle r_E^2\rangle}{3m_p} - \frac{\alpha(1+\kappa+\kappa^2)}{2m_p^3} + \frac{2\alpha}{\boldsymbol{q}^2}\int_0^\infty dt\,\left[Q_{11}(\boldsymbol{q},t) - Q_{11}^{elas}(\boldsymbol{q},t) - Q_{11}(\boldsymbol{0},t)\right]$$

$$Q_{44}^{elas}(\boldsymbol{q},t) \xrightarrow[t\gg1]{} \left[1 - \boldsymbol{q}^2\left(\frac{1}{4m_p^2} + \frac{\langle r_E^2\rangle}{3}\right)\right] e^{-(E_p-m_p)t}$$

$$Q_{11}^{elas}(\boldsymbol{q},t) \xrightarrow[t\gg1]{} \frac{(1+\kappa)^2}{4m_p^2}\boldsymbol{q}^2 e^{-(E_p-m_p)t}$$

PRD104 (2021), Wilcox, Lee

Neutron formulas

$$\alpha_E = \frac{\alpha\kappa^2}{4m_p^3} + \frac{2\alpha}{\boldsymbol{q}^2}\int_0^\infty dt\,Q_{44}(\boldsymbol{q},t)$$

$$\beta_M = -\frac{\alpha\kappa^2}{2m_p^3} + \frac{2\alpha}{\boldsymbol{q}^2}\int_0^\infty dt\,\left[Q_{11}(\boldsymbol{q},t) - Q_{11}^{elas}(\boldsymbol{q},t) - Q_{11}(\boldsymbol{0},t)\right]$$

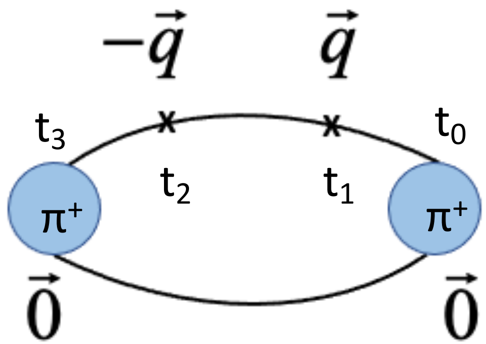
$$Q_{11}^{elas}(\boldsymbol{q},t) \xrightarrow[t\gg1]{} \frac{\kappa^2}{4m_p^2}\boldsymbol{q}^2 e^{-(E_p-m_p)t}$$

Four-point function in lattice QCD

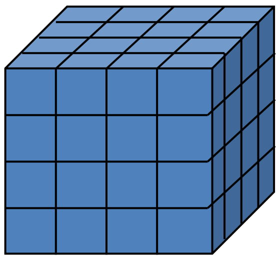
$$\frac{\sum_{\boldsymbol{x}_3,\boldsymbol{x}_2,\boldsymbol{x}_1,\boldsymbol{x}_0} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}_2} e^{i\boldsymbol{q}\cdot\boldsymbol{x}_1} \langle\Omega|\psi^\dagger(x_3) : j^L_\mu(x_2) j^L_\nu(x_1) : \psi(x_0)|\Omega\rangle}{\sum_{\boldsymbol{x}_3,\boldsymbol{x}_0} \langle\Omega|\psi^\dagger(x_3)\psi(x_0)|\Omega\rangle}$$

$$\equiv Q_{\mu\nu}(\boldsymbol{q}, t_3, t_2, t_1, t_0)$$

Kinematics
(zero-momentum Breit frame)



Path integrals in Euclidean spacetime



Proof-of-concept simulation:

- Quenched Wilson action on 24^3x48 lattice with spacing a=0.085 fm.
- Dirichlet boundary condition in time, periodic in space.
- Quark mass parameter κ=0.1520, 0.1543, 0.1555, 0.1565 corresponding to pion mass m_π=1100, 800, 600, 370 MeV. Analyzed 1000 configurations for each mass.
- 5 momenta **q**={0,0,0}, {0,0,1}, {0,1,1}, {1,1,1}, {0,0,2} per mass

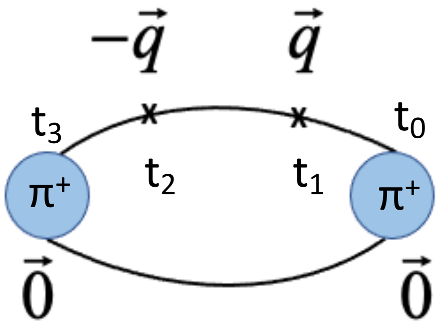
Operators

$$\frac{\sum_{\boldsymbol{x}_3,\boldsymbol{x}_2,\boldsymbol{x}_1,\boldsymbol{x}_0} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}_2} e^{i\boldsymbol{q}\cdot\boldsymbol{x}_1} \langle\Omega|\psi^\dagger(x_3) : j^L_\mu(x_2) j^L_\nu(x_1) : \psi(x_0)|\Omega\rangle}{\sum_{\boldsymbol{x}_3,\boldsymbol{x}_0} \langle\Omega|\psi^\dagger(x_3)\psi(x_0)|\Omega\rangle}$$

$$\equiv Q_{\mu\nu}(\boldsymbol{q}, t_3, t_2, t_1, t_0)$$

Charged pion: $\psi_{\pi^+}(x) = \bar{d}(x)\gamma_5 u(x)$

Local current: $j^{(PC)}_\mu = Z_V \left(q_u \bar{u}\gamma_\mu u + q_d \bar{d}\gamma_\mu d\right)$



Conserved current (Z_V=1):

$$j^{(PS)}_\mu(x) = q_u\kappa[-\bar{u}(x)(1-\gamma_\mu)U_\mu(x)u(x+a\hat{\mu}) + \bar{u}(x+a\hat{\mu})(1+\gamma_\mu)U^\dagger_\mu(x)u(x)]$$
$$+ q_d\kappa[-\bar{d}(x)(1-\gamma_\mu)U_\mu(x)d(x+a\hat{\mu}) + \bar{d}(x+a\hat{\mu})(1+\gamma_\mu)U^\dagger_\mu(x)d(x)]$$

Current conservation at momentum **q**=0 for Q_{44}:

$$\frac{\sum_{\boldsymbol{x}_3,\boldsymbol{x}_2,\boldsymbol{x}_1,\boldsymbol{x}_0} \langle\Omega|\psi(x_3) j^L_4(x_2) j^L_4(x_1)\psi^\dagger(x_0)|\Omega\rangle}{\sum_{\boldsymbol{x}_3,\boldsymbol{x}_0} \langle\Omega|\psi(x_3)\psi^\dagger(x_0)|\Omega\rangle} = q_1 q_2$$

(used for numerical validation of the diagrams)

Wick contractions

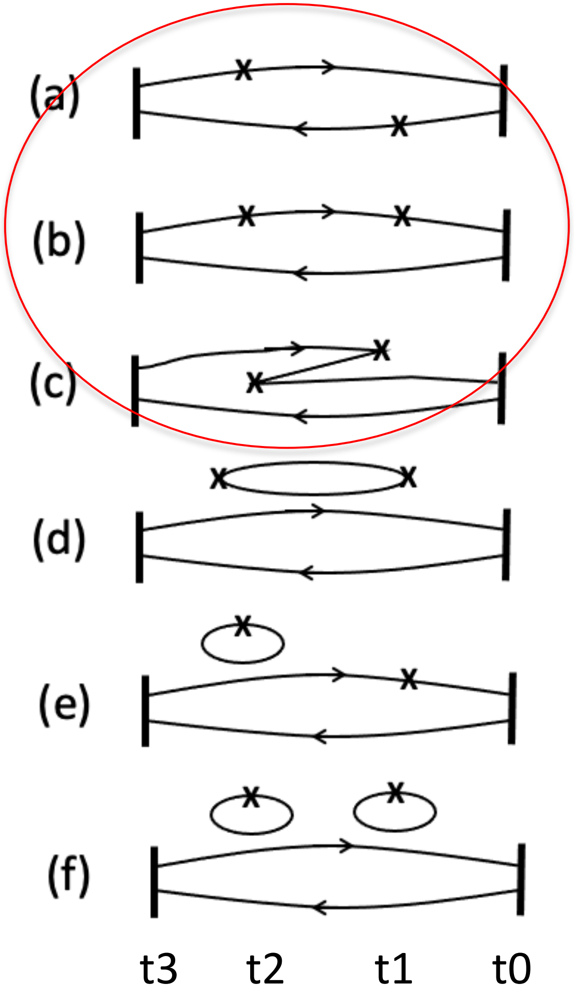
$$\frac{\sum_{\boldsymbol{x}_3,\boldsymbol{x}_2,\boldsymbol{x}_1,\boldsymbol{x}_0} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}_2} e^{i\boldsymbol{q}\cdot\boldsymbol{x}_1} \langle\Omega|\psi^\dagger(x_3) : j^L_\mu(x_2) j^L_\nu(x_1) : \psi(x_0)|\Omega\rangle}{\sum_{\boldsymbol{x}_3,\boldsymbol{x}_0} \langle\Omega|\psi^\dagger(x_3)\psi(x_0)|\Omega\rangle}$$

$$\equiv Q_{\mu\nu}(\boldsymbol{q}, t_3, t_2, t_1, t_0)$$

$\bar{u}d$ $\quad u\bar{u} + d\bar{d}$ $\quad u\bar{u} + d\bar{d}$ $\quad u\bar{d}$

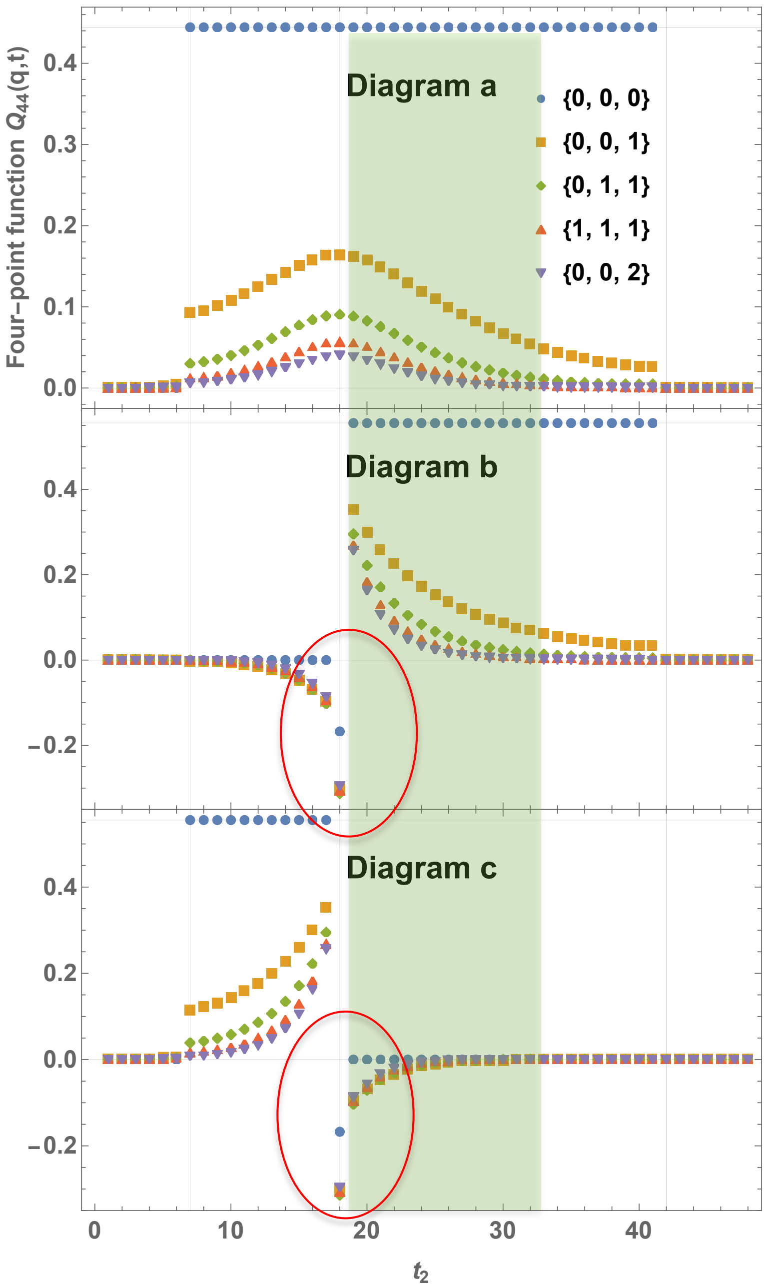
Connected contributions

$$d_4^{\mathrm{A}} = -2\,\mathrm{tr}\left[S(t_1,t_3)\gamma_5 S(t_3,t_2)\gamma_\mu e^{-i\mathbf{q}} S(t_2,t_0)\gamma_5 S(t_0,t_1)\gamma_\nu e^{i\mathbf{q}}\right]$$
$$d_2^{\mathrm{A\text{-}bwd}} = -2\,\mathrm{tr}\left[S(t_2,t_3)\gamma_5 S(t_3,t_1)\gamma_\nu e^{i\mathbf{q}} S(t_1,t_0)\gamma_5 S(t_0,t_2)\gamma_\mu e^{-i\mathbf{q}}\right]$$
$$d_1^{\mathrm{B}} = 4\,\mathrm{tr}\left[S(t_2,t_3)\gamma_5 S(t_3,t_0)\gamma_5 S(t_0,t_1)\gamma_\nu e^{i\mathbf{q}} S(t_1,t_2)\gamma_\mu e^{-i\mathbf{q}}\right]$$
$$d_7^{\mathrm{B\text{-}bwd}} = 1\,\mathrm{tr}\left[S(t_0,t_3)\gamma_5 S(t_3,t_2)\gamma_\mu e^{-i\mathbf{q}} S(t_2,t_1)\gamma_\nu e^{i\mathbf{q}} S(t_1,t_0)\gamma_5\right]$$
$$d_0^{\mathrm{C}} = 4\,\mathrm{tr}\left[S(t_1,t_3)\gamma_5 S(t_3,t_0)\gamma_5 S(t_0,t_2)\gamma_\mu e^{-i\mathbf{q}} S(t_2,t_1)\gamma_\nu e^{i\mathbf{q}}\right]$$
$$d_9^{\mathrm{C\text{-}bwd}} = 1\,\mathrm{tr}\left[S(t_0,t_3)\gamma_5 S(t_3,t_1)\gamma_\nu e^{i\mathbf{q}} S(t_1,t_2)\gamma_\mu e^{-i\mathbf{q}} S(t_2,t_0)\gamma_5\right]$$
$$d_{10}^{\mathrm{D}} = -5\,\mathrm{tr}\left[S(t_0,t_3)\gamma_5 S(t_3,t_0)\gamma_5\right]\mathrm{tr}\left[S(t_1,t_2)\gamma_\mu e^{-i\mathbf{q}} S(t_2,t_1)\gamma_\nu e^{i\mathbf{q}}\right]$$
$$d_5^{\mathrm{El}} = -2\,\mathrm{tr}\left[S(t_1,t_3)\gamma_5 S(t_3,t_0)\gamma_5 S(t_0,t_1)\gamma_\nu e^{i\mathbf{q}}\right]\mathrm{tr}\left[S(t_2,t_2)\gamma_\mu e^{-i\mathbf{q}}\right]$$
$$d_6^{\mathrm{El\text{-}bwd}} = 1\,\mathrm{tr}\left[S(t_0,t_3)\gamma_5 S(t_3,t_1)\gamma_\nu e^{i\mathbf{q}} S(t_1,t_0)\gamma_5\right]\mathrm{tr}\left[S(t_2,t_2)\gamma_\mu e^{-i\mathbf{q}}\right]$$
$$d_3^{\mathrm{Er}} = -2\,\mathrm{tr}\left[S(t_2,t_3)\gamma_5 S(t_3,t_0)\gamma_5 S(t_0,t_2)\gamma_\mu e^{-i\mathbf{q}}\right]\mathrm{tr}\left[S(t_1,t_1)\gamma_\nu e^{i\mathbf{q}}\right]$$
$$d_8^{\mathrm{Er\text{-}bwd}} = 1\,\mathrm{tr}\left[S(t_0,t_3)\gamma_5 S(t_3,t_2)\gamma_\mu e^{-i\mathbf{q}} S(t_2,t_0)\gamma_5\right]\mathrm{tr}\left[S(t_1,t_1)\gamma_\nu e^{i\mathbf{q}}\right]$$
$$d_{11}^{\mathrm{F}} = 1\,\mathrm{tr}\left[S(t_0,t_3)\gamma_5 S(t_3,t_0)\gamma_5\right]\mathrm{tr}\left[S(t_2,t_2)\gamma_\mu e^{-i\mathbf{q}}\right]\mathrm{tr}\left[S(t_1,t_1)\gamma_\nu e^{i\mathbf{q}}\right]$$



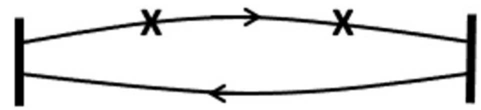
Quark propagator $S_q(t_2,t_1) \equiv \langle q\bar{q}\rangle = \dfrac{1}{\not{D} + m_q}$

Four-point functions Q_{44} for α_E



$4/9 = 2 q_u q_{\bar{d}}$ (current conservation at **q**=0)

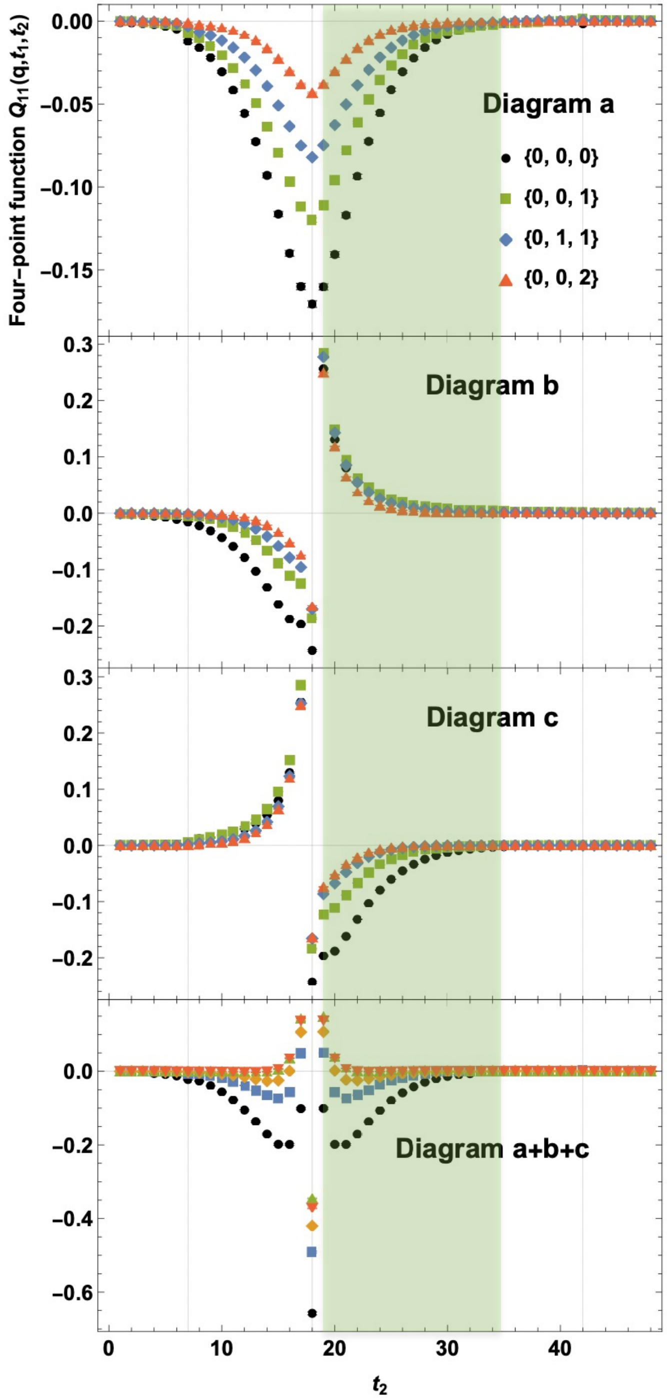
$5/9 = q_u q_u + q_{\bar{d}} q_{\bar{d}}$ (current conservation at **q**=0)

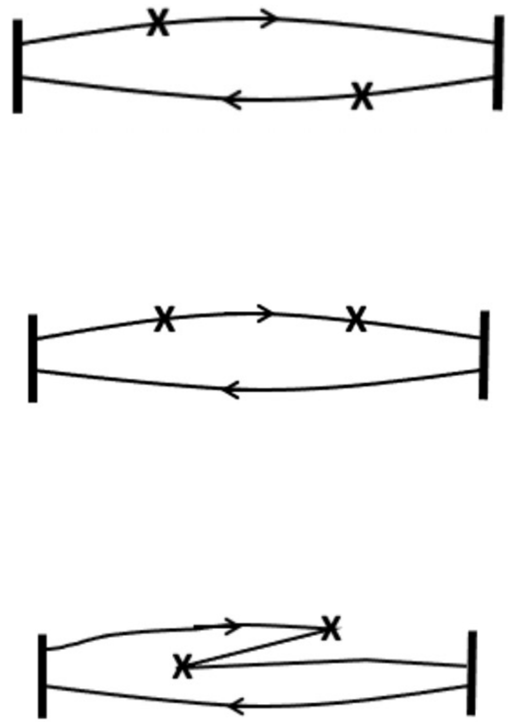


$5/9 = q_u q_u + q_{\bar{d}} q_{\bar{d}}$ (current conservation at **q**=0)

Diagram b and c have unphysical contact interactions (we avoid t_1=t_2)

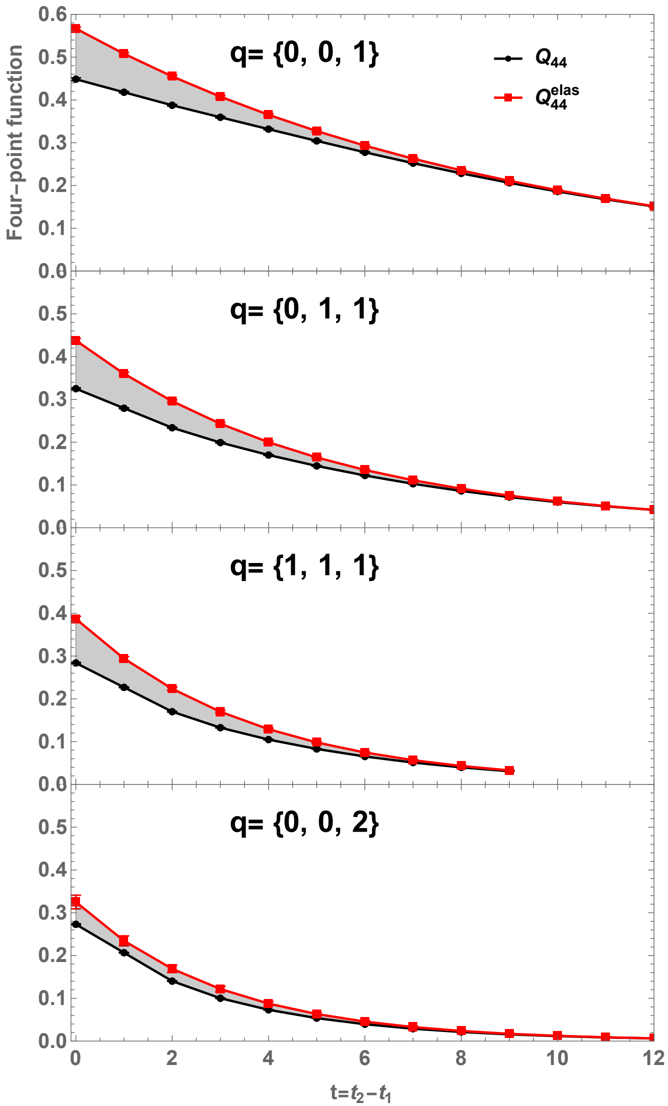
Four-point functions Q_{11} for β_M





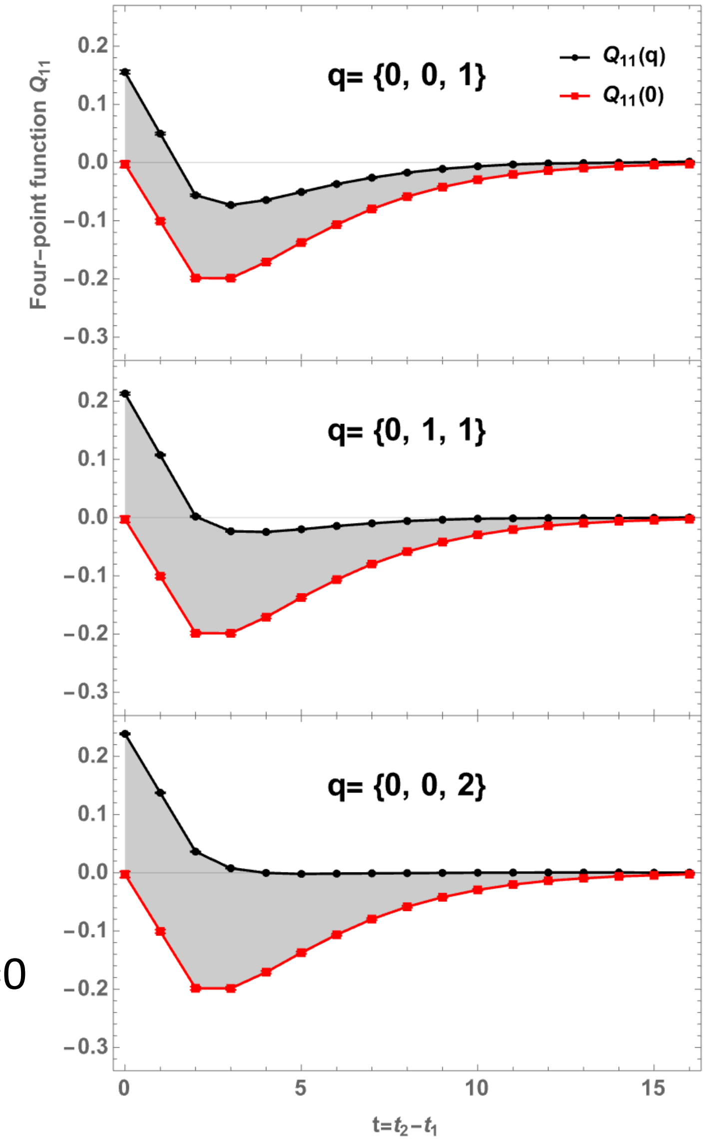
Time integrals

$$\alpha_E^\pi = \frac{\alpha\langle r_E^2\rangle}{3m_\pi} + \frac{2\alpha a}{\boldsymbol{q}^2}\int_0^\infty dt\,[Q_{44}(\boldsymbol{q},t) - Q_{44}^{elas}(\boldsymbol{q},t)]$$



Signal is negative of shaded area

$$\beta_E^\pi = -\frac{\alpha\langle r_E^2\rangle}{3m_\pi} + \frac{2\alpha a}{\boldsymbol{q}^2}\int_0^\infty dt\,[Q_{11}(\boldsymbol{q},t) - Q_{11}(\boldsymbol{0},t)]$$



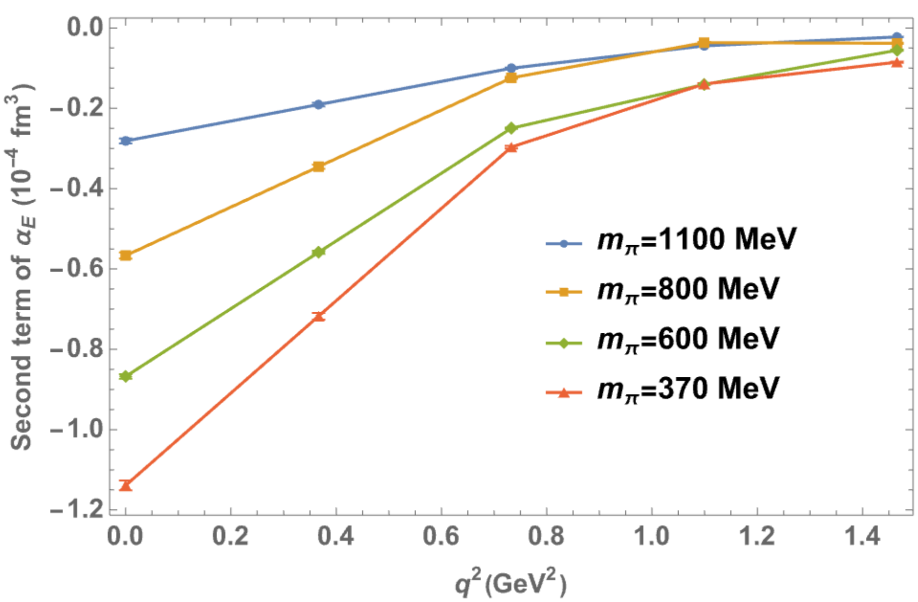
Signal is positive shaded area

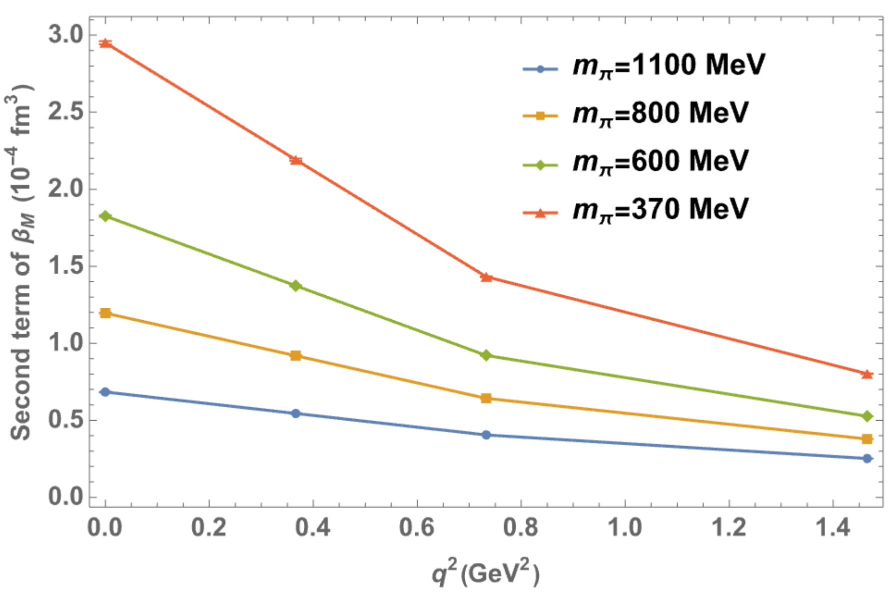
Extrapolation from t=1 to t=0

Extrapolation to $\mathbf{q}^2$=0

$$\alpha_E^\pi = \frac{\alpha\langle r_E^2\rangle}{3m_\pi} + \frac{2\alpha a}{\boldsymbol{q}^2}\int_0^\infty dt\,\left[Q_{44}(\boldsymbol{q},t) - Q_{44}^{elas}(\boldsymbol{q},t)\right]$$

$$\beta_M = -\frac{\alpha\langle r_E^2\rangle}{3m_\pi} + \frac{2\alpha}{\boldsymbol{q}^2}\int_0^\infty dt\left[Q_{11}^{inel}(\boldsymbol{q},t) - Q_{11}^{inel}(\boldsymbol{0},t)\right]$$

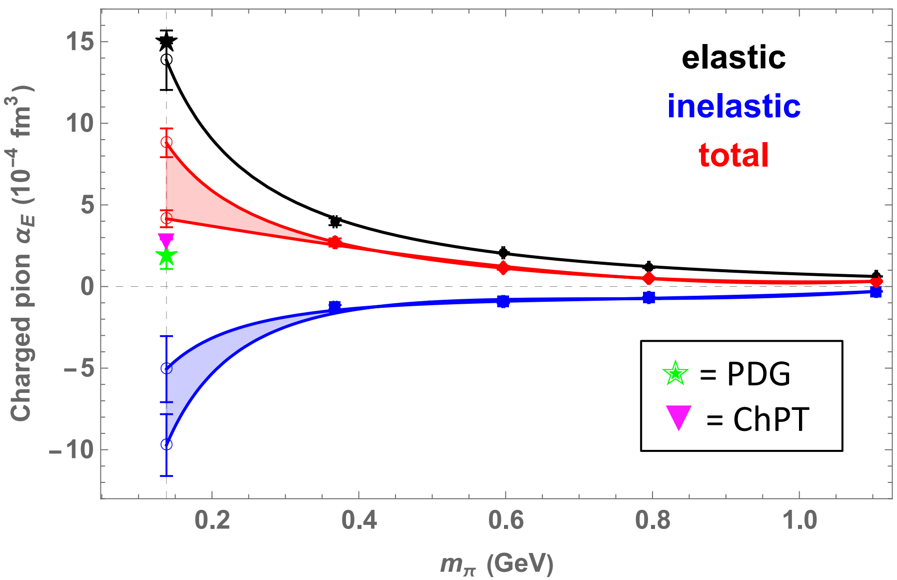


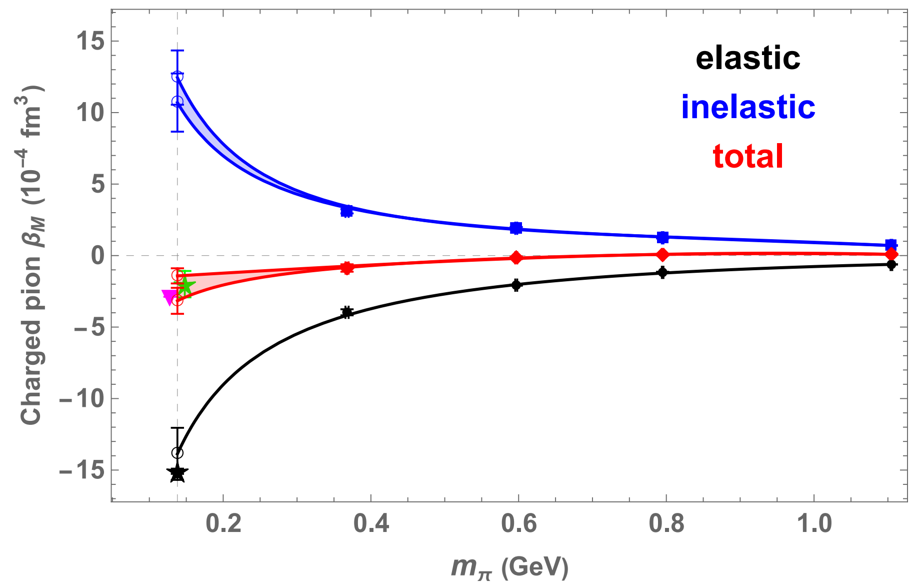


Chiral extrapolation

$$\alpha_E^\pi = \frac{\alpha \langle r_E^2 \rangle}{3m_\pi} + \frac{2\alpha a}{\boldsymbol{q}^2} \int_0^\infty dt \left[Q_{44}(\boldsymbol{q},t) - Q_{44}^{elas}(\boldsymbol{q},t) \right]$$

$$\beta_M = -\frac{\alpha \langle r_E^2 \rangle}{3m_\pi} + \frac{2\alpha}{\boldsymbol{q}^2} \int_0^\infty dt \left[Q_{11}^{inel}(\boldsymbol{q},t) - Q_{11}^{inel}(\boldsymbol{0},t) \right]$$





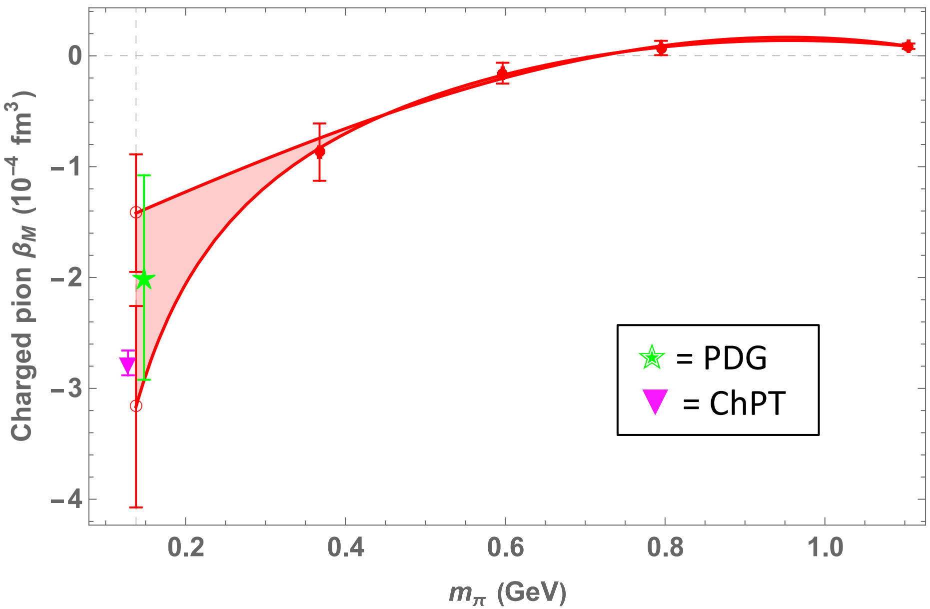
$$a + b\, m_\pi + c\, m_\pi^3$$

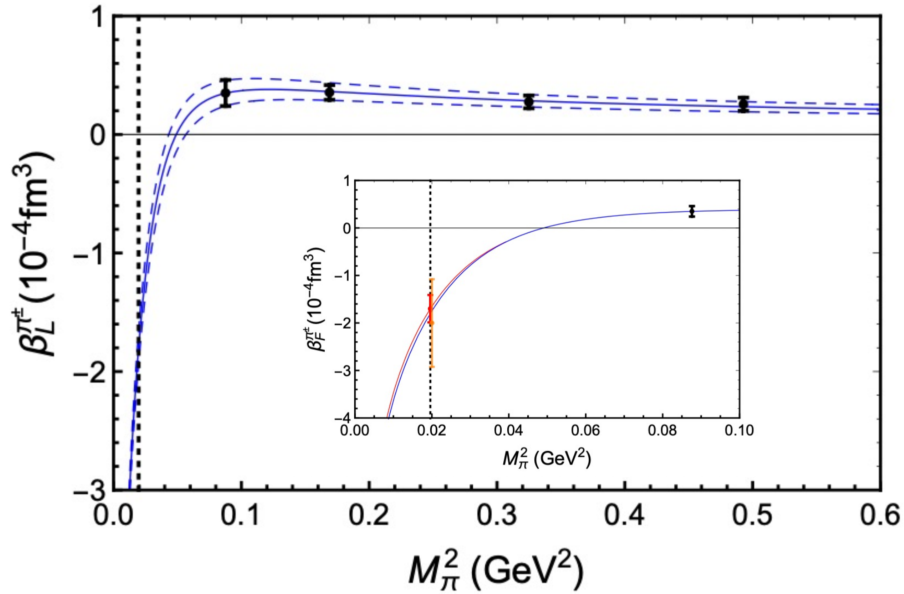
$$\frac{a}{m_\pi} + b\, m_\pi + c\, m_\pi^3$$

Four-point function vs. Background field

arXiv:2307.08620 (2023),
Lee, Wilcox , Alexandru, Culver

PRD104, 054506 (2021),
He, Leinweber, Thomas, Wang





Quenched Wilson ensembles on 24^3x48 at m_π= 0.37, 0.6, 0.8, 1.1 GeV.

PACS-CS 2+1 ensembles on 32^3x64 at m_π= 0.296 , 0.411 , 0.572, 0.702 GeV.

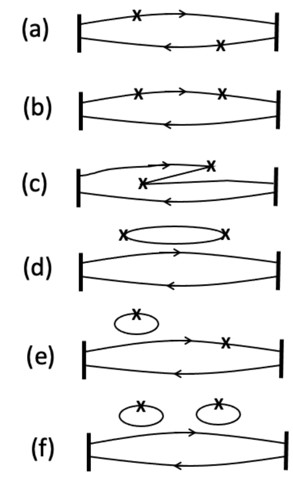
$$a + b\, m_\pi + c\, m_\pi^3$$

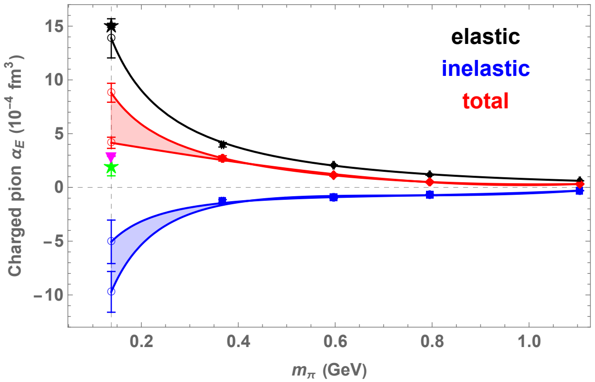
$$\frac{a}{m_\pi} + b\, m_\pi + c\, m_\pi^3$$

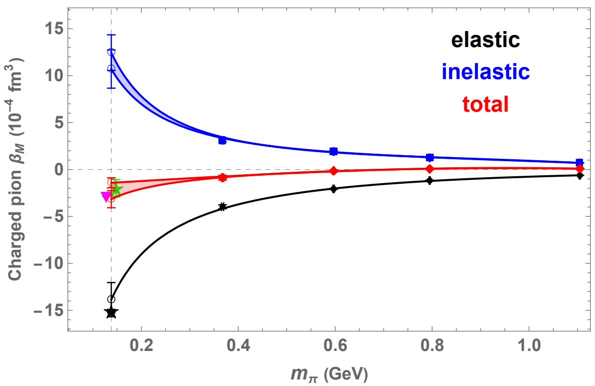
$$\beta_L^{\pi^\pm} = \beta_0 \frac{1 + c_1 M_\pi^2 + c_3 M_\pi^4}{1 + c_4 M_\pi^4}\,, \quad \text{(Padé)}$$

Conclusion

- Proof-of-concept simulations for charged pion show promise of four-point function methodology.
  - Clear pictures for α_E and β_M
  - Requires 2pt and 4pt (but not 3pt) functions
- Open issues
  - Extrapolation to t=0 (contact term)
  - Extrapolation to $\mathbf{q}^2$=0 (static limit)
  - Chiral extrapolation
  - Quenched approximation
  - Only connected contributions so far
  - Discrepancy with background field method for β_M
- Outlook
  - Dynamical ensembles (two-flavor nHYP-clover, 315 and 227 MeV, elongated geometries for volume study and smaller $\mathbf{q}^2$)
  - Disconnected contributions
  - Next target: proton and neutron







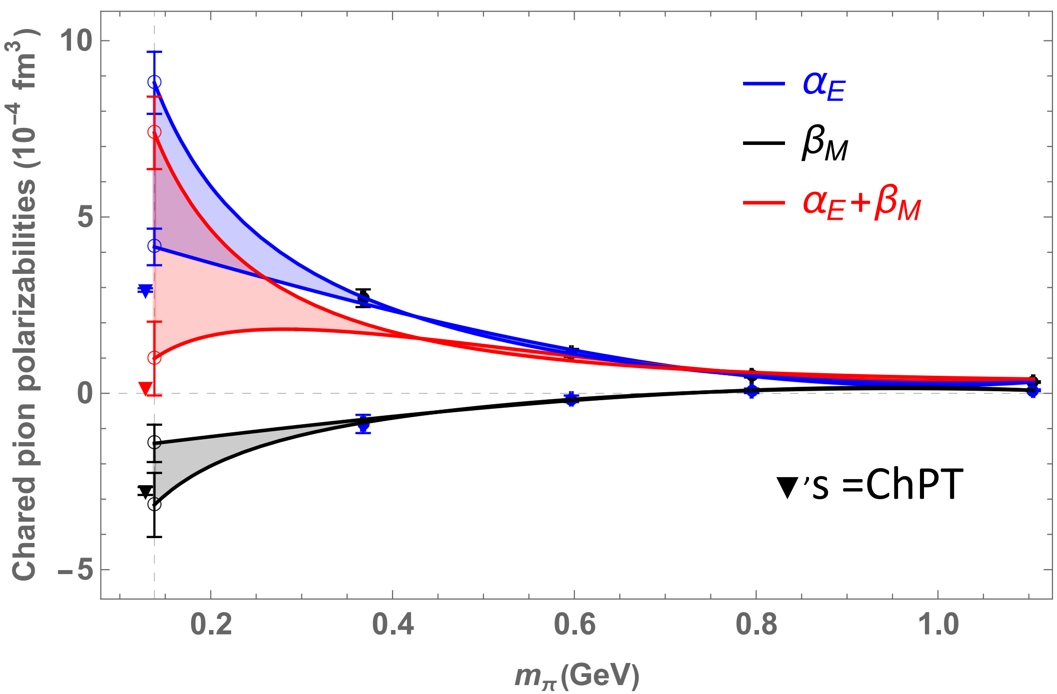
Reserve

$\alpha_E + \beta_M$

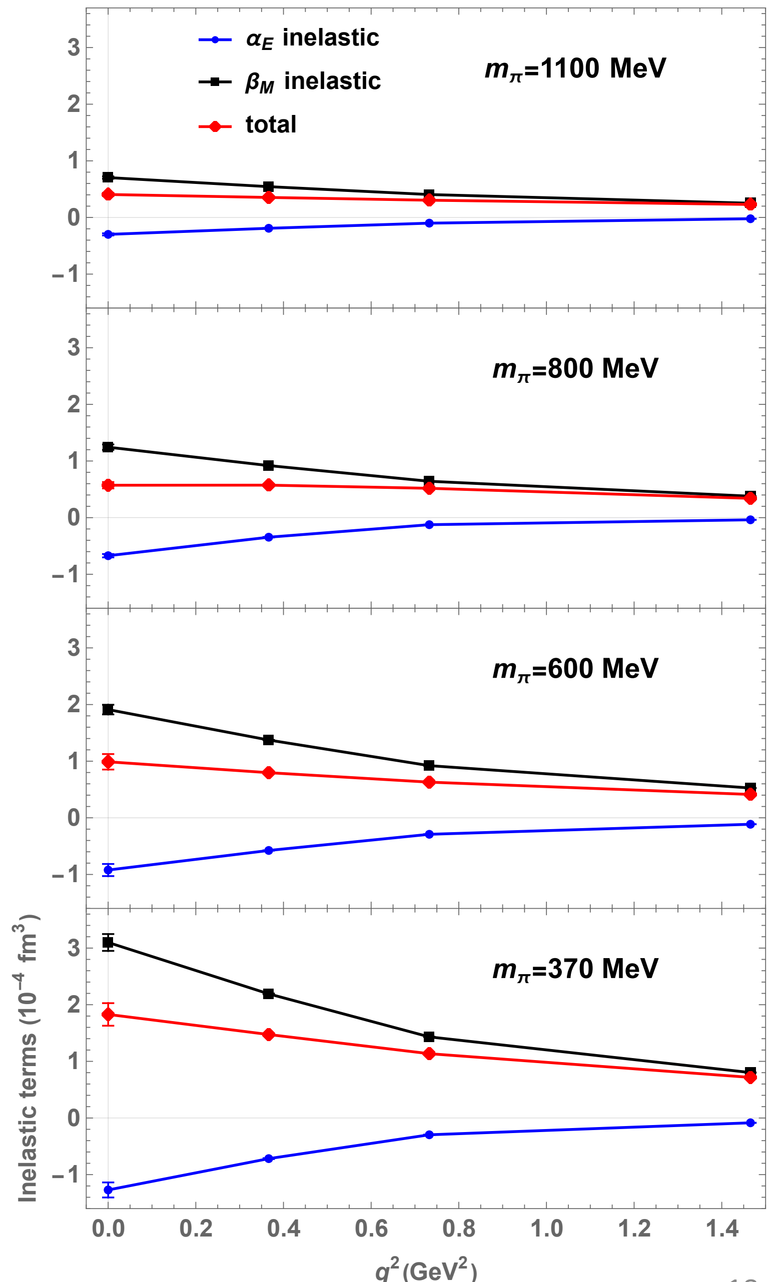
$$\alpha_E^\pi = \frac{\alpha\langle r_E^2\rangle}{3m_\pi} + \frac{2\alpha a}{\boldsymbol{q}^2}\int_0^\infty dt\,[Q_{44}(\boldsymbol{q},t) - Q_{44}^{elas}(\boldsymbol{q},t)]$$

$$\beta_E^\pi = -\frac{\alpha\langle r_E^2\rangle}{3m_\pi} + \frac{2\alpha a}{\boldsymbol{q}^2}\int_0^\infty dt\,[Q_{11}(\boldsymbol{q},t) - Q_{11}(\boldsymbol{0},t)]$$

Pion mass dependence



Momentum dependence



“Pion electric polarizabilities from lattice QCD”

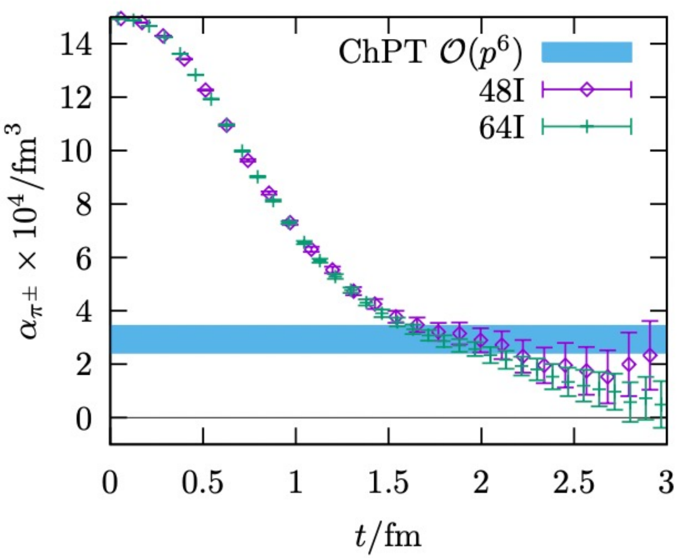
X. Feng, T. Izubuchi, L. Jin, M. Golterman

arXiv:2201.01396 (Lattice 2021)

Domain-wall ensembles
at physical pion mass

	Volume	a^{-1} (GeV)	L (fm)	M_π (MeV)	$t_{\rm sep}$ (a)
48I	$48^3 \times 96$	1.730(4)	5.5	135	12
64I	$64^3 \times 128$	2.359(7)	5.4	135	18
24D	$24^3 \times 64$	1.0158(40)	4.7	142	8
32D	$32^3 \times 64$	1.0158(40)	6.2	142	8

$$\alpha_\pi(t) = -\int_{-t<t_x<t} \int_{\vec{x}} \frac{t_x^2}{24\pi} \frac{1}{2M_\pi} \langle \pi | T\vec{J}(t_x,\vec{x}) \cdot \vec{J}(0,\vec{0}) | \pi \rangle - \alpha_\pi^{\rm Born}$$



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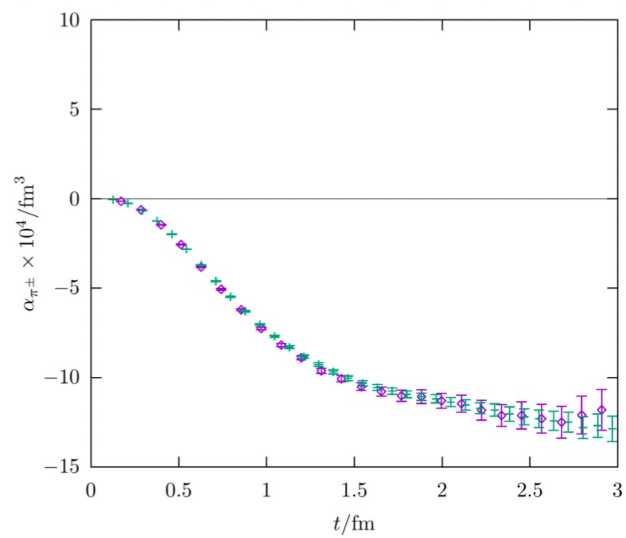
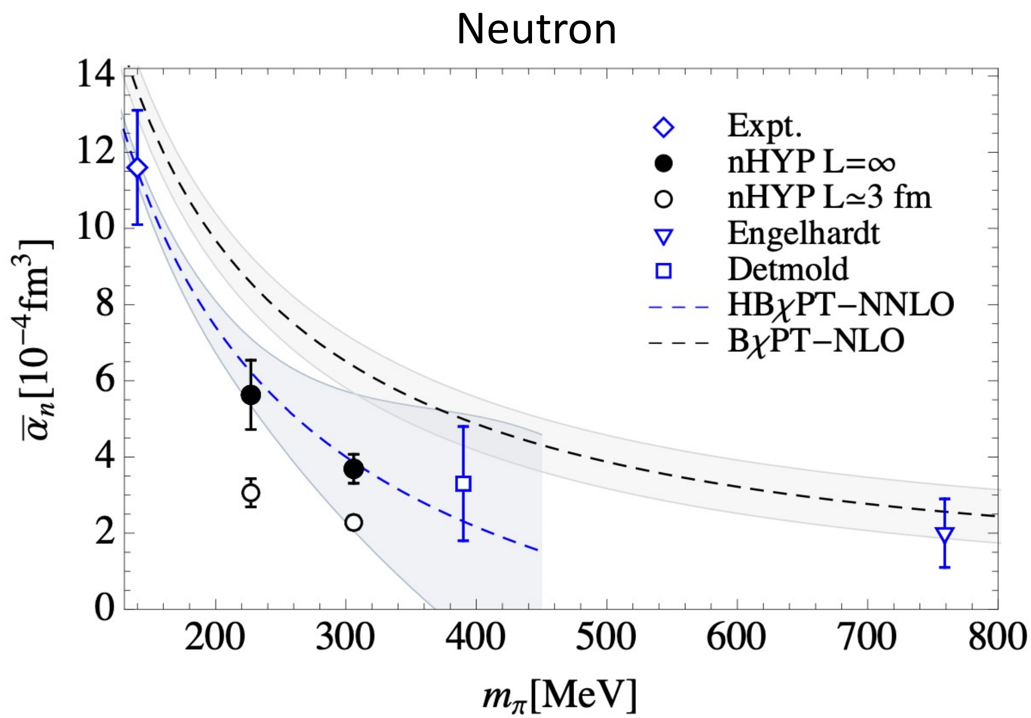


$$+ \frac{\alpha \langle r_E^2 \rangle}{3m_\pi}$$

$$\alpha_E^\pi = \frac{\alpha\langle r_E^2\rangle}{3m_\pi} + \frac{2\alpha a}{\boldsymbol{q}^2} \int_0^\infty dt \left[Q_{44}(\boldsymbol{q},t) - Q_{44}^{elas}(\boldsymbol{q},t) \right]$$

Examples from background field method



π^0: $\alpha_E \simeq -0.5$
K^0: $\alpha_E = 0.356(74)$

π^0: $\alpha_E = -0.69(7)(4) = -\beta_M$ (PDG)
K^0: $\alpha_E = 0.58 = -\beta_M$ (ChPT)

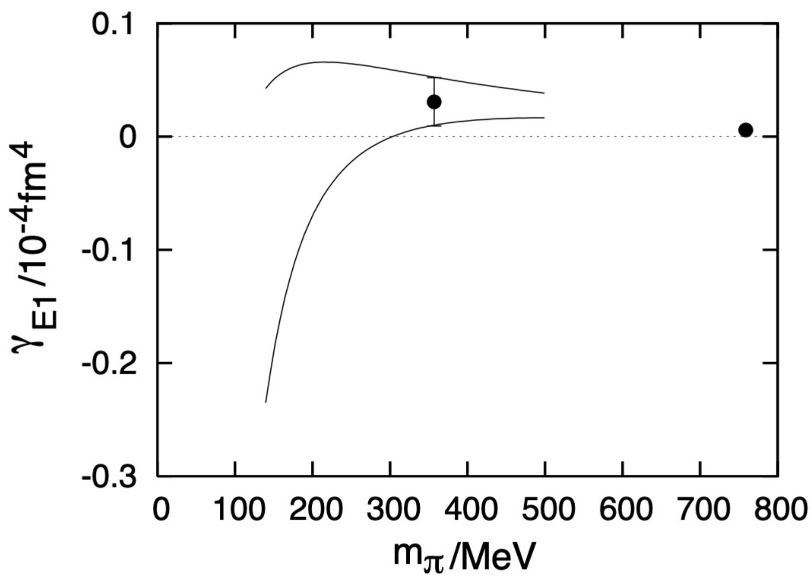
PRD94 (2016), Lujan, Alexandru, Freeman, Lee

Background field + 4pt function method

Perturbative expansion in the background field at the action level leads to the same diagrammatic structure in 4pt method.

Neutron electric polarizability: $\alpha_E = -2.0(0.9)$ PRD76 (2007), Engelhardt

Neutron spin polarizability



arXiv1111.2686 (Lattice2011), Engelhardt