Extracting OPE Coefficients of the 3d Ising CFT from the Four-Point Function

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LATTICE 2023 – THEORETICAL DEVELOPMENTS II

Acknowledgements

I would like to thank my collaborators and co-authors

- George T. Fleming, Fermilab

- Richard C. Brower, Boston University
- Venkitesh Ayyar, Boston University
- Evan Owen, Boston University

- Timothy G. Raben, Michigan State University

- Chung-I Tan, Brown University

So far: Development

Evan Owen, Tuesday, 1:30 pm
Richard Brower, Tuesday, 1:50 pm

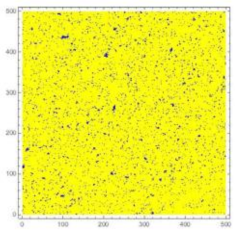
Now: Application

Put the critical Ising model on $\mathbb{R} \times \mathbb{S}^2$ to access quantities that are difficult to calculate on Euclidean lattices

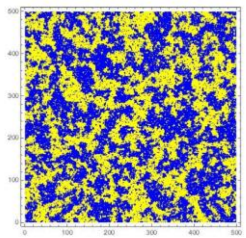
Introduction

The Critical 3d Ising Model

- 3d Ising model: interacting spins $s = \pm 1$



$T \ll T_C$



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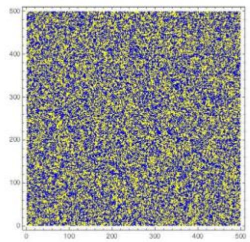

$T \gg T_C$

[Tong 2017]

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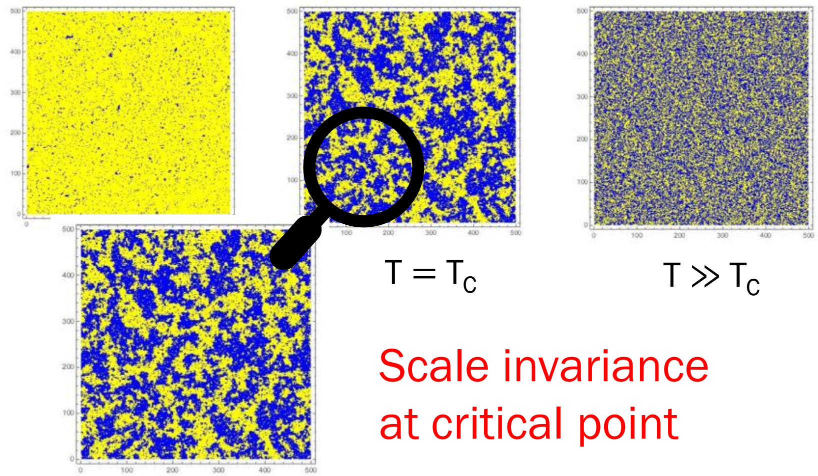

[Tong 2017]

Scale invariance at critical point

- At criticality: physical quantities = power laws

$$C \propto t^{-\alpha} \quad m \propto t^{\beta} \quad \chi \propto t^{-\gamma} \quad \xi \propto t^{-\nu}$$

$$\langle \sigma(0)\sigma(\boldsymbol{r}) \rangle \propto r^{-d+2-\eta} \qquad t = \frac{|T - T_c|}{T_c}$$

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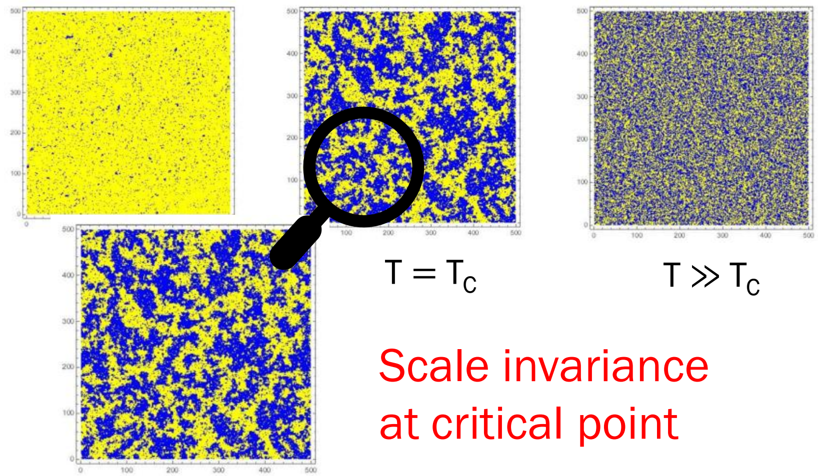

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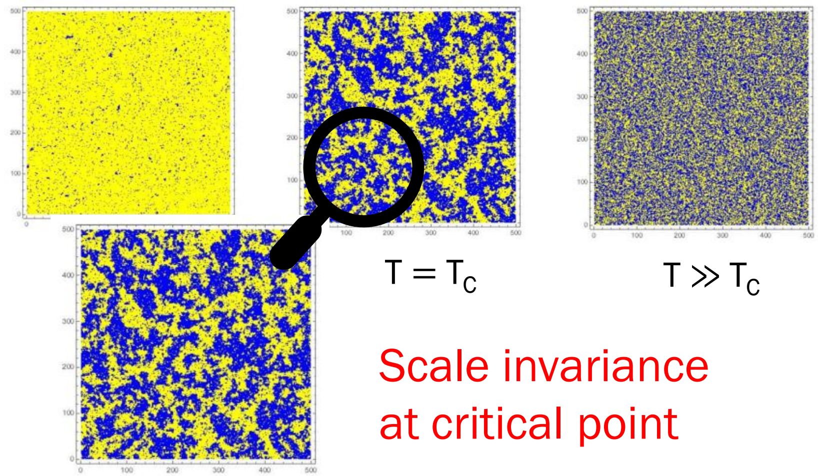

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Determination of critical exponents using Monte Carlo & finite size scaling

[Swendsen, Wang 1987],
[Wolff 1989],
[Pelissetto, Vicari, 2000]

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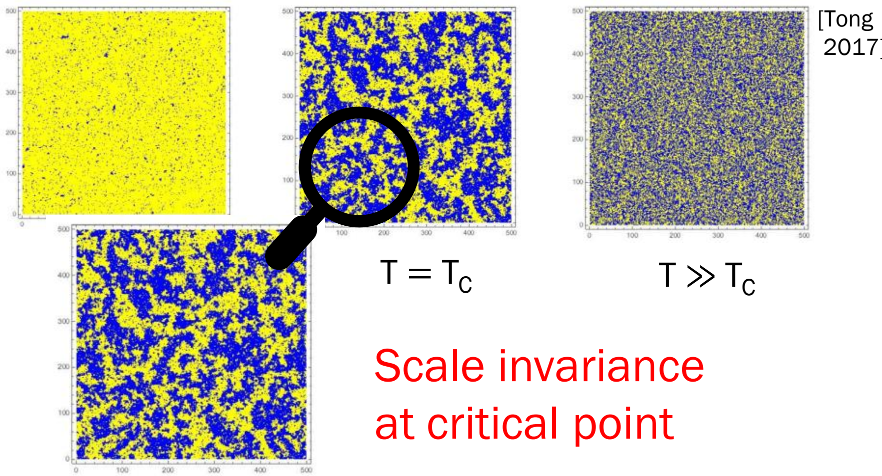

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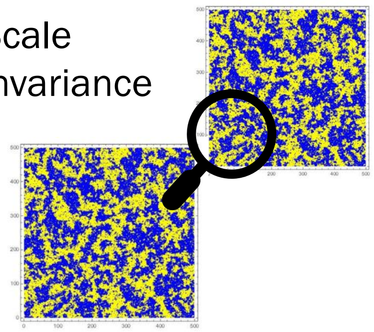
What more is there to learn?

The 3d Ising CFT

<u>Change POV</u>: Condensed Matter Physics ⇨ Continuum Field Theory

Scale invariance



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Invariance under conformal group $O(d+1,1)$

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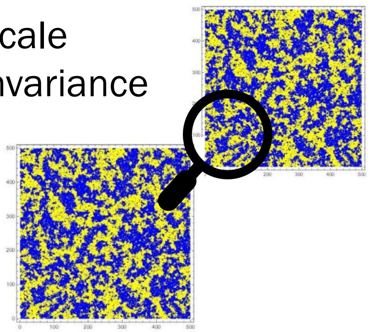

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⇨ Critical Ising Model = Conformal Field Theory (3d Ising CFT)

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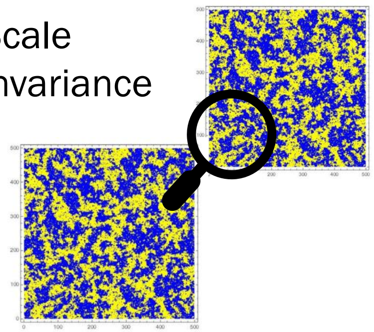

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Primary operators $\mathcal{O}$: $\underset{0^-}{\sigma}, \quad \underset{0^+}{\epsilon}, \quad \underset{0^+}{\epsilon'}, \quad \underset{2^+}{T}, \quad \underset{2^+}{T'}, \quad \underset{4^+}{C}, \ldots$

The 3d Ising CFT

Scaling dimensions Operator Product Expansion (OPE) coefficients

CFT characterized by $\Delta_{\mathcal{O}}$ and $f_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}$

2-point functions

$$\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \frac{1}{|x_1 - x_2|^{2\Delta_{\mathcal{O}}}}$$

(n>2)-point functions

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k f_{\mathcal{O}_i\mathcal{O}_j\mathcal{O}_k} C(x_1 - x_2)\mathcal{O}_k(x_2)$$

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Scaling dimensions known from lattice calculations

$$\Delta_\sigma = \frac{\eta + d - 2}{2}$$
$$\Delta_\epsilon = d - \frac{1}{\nu}$$
$$\Delta_{\epsilon'} = \omega + d$$

[Pelissetto, Vicari, 2000]

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Difficult to determine OPE coefficients with traditional MC methods on Euclidean lattices

Monte Carlo ($f_{\sigma\sigma\epsilon}, f_{\epsilon\epsilon\epsilon}$):
e.g. [Hasenbusch, 2019]

Conformal Bootstrap:
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$\mathbb{R}^3$

conformal map

radial quantization

$\mathbb{R} \times \mathbb{S}^2$

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} radial quantization $\mathbb{R} \times \mathbb{S}^2$

Goal of this work: extract both scaling dimensions AND OPE coefficients from lattice calculations of four-point functions on $\mathbb{R} \times \mathbb{S}^2$

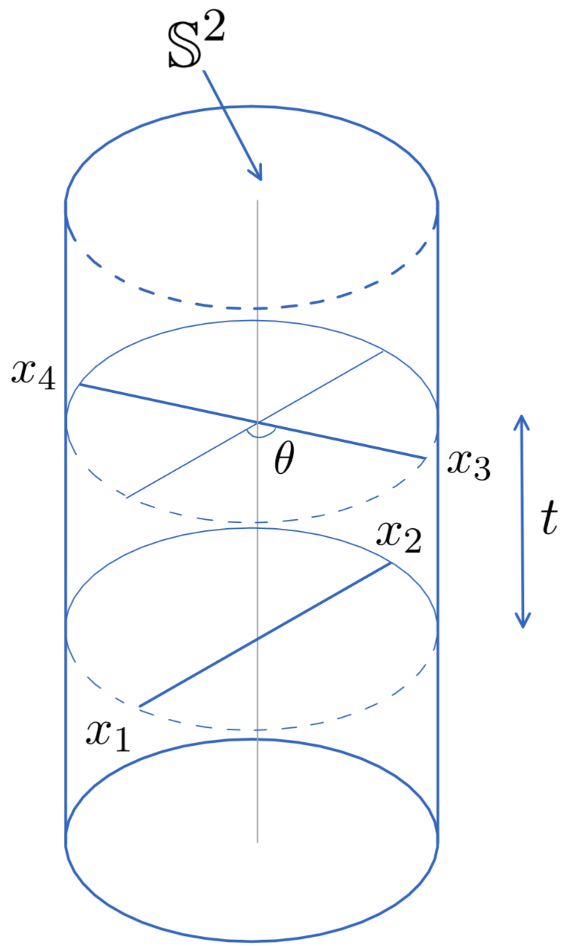
Antipodal 4-point function on $\mathbb{R} \times \mathbb{S}^2 \ni (t, \vec{x})$

4-point function of σ-operators in special frame on the right: [Hogervorst and Rychkov, 2013, Costa et al., 2016]

$$\frac{\langle\sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle}{\langle\sigma(x_1)\sigma(x_2)\rangle\langle\sigma(x_3)\sigma(x_4)\rangle} = 1 + \sum_{j\in 2\mathbb{N}_0} c_j(t) P_j(\cos\theta)$$

$$c_j(t) = \sum_{\mathcal{O}} f^2_{\sigma\sigma\mathcal{O}} \sum_{\substack{n\in 2\mathbb{N}_0 \\ n\geq|j-l|}}^{\infty} B_{n,j}(\Delta_{\mathcal{O}}) e^{-(\Delta_{\mathcal{O}}+n)t}$$

even spin & parity



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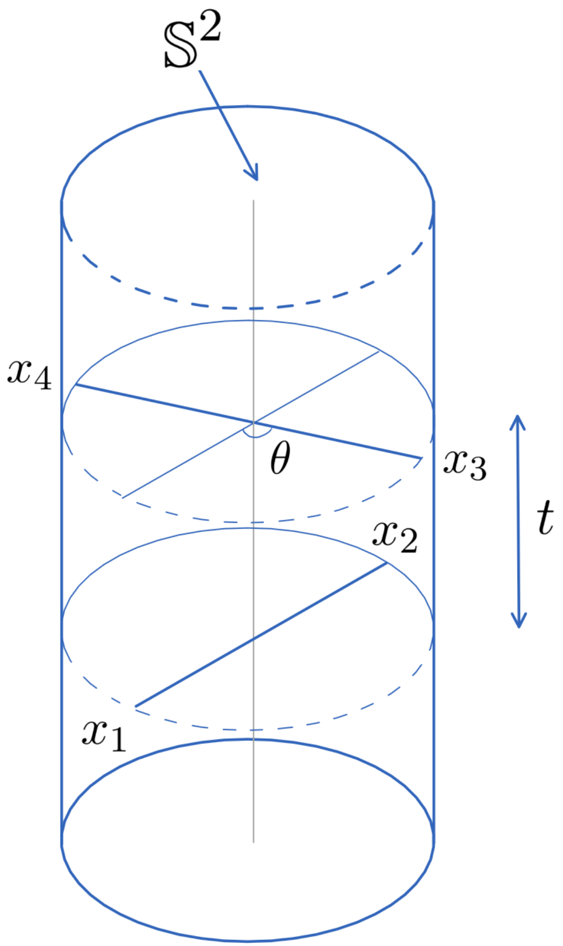
even spin & parity

⇨ Obtain $f_{\sigma\sigma\mathcal{O}}$ and $\Delta_{\mathcal{O}}$ for even-parity & even-spin $\mathcal{O}$ by fits to $c_j(t)$



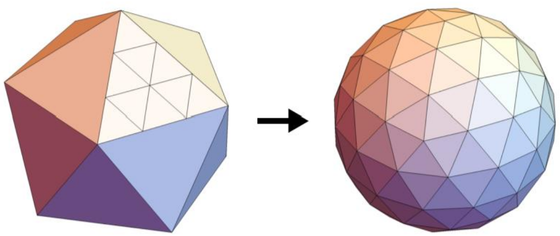
Quantum Finite Elements

[Brower et al., 2018, 2021]

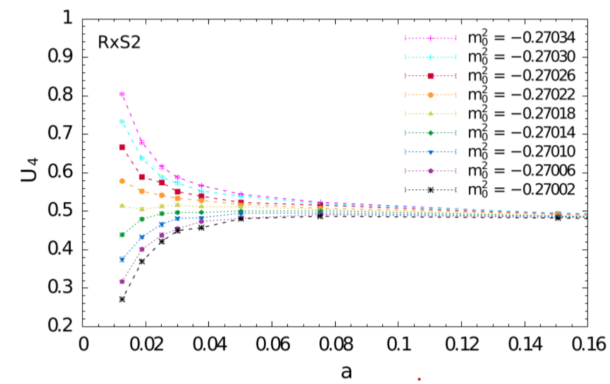
- Discretize φ^4-theory on simplicial lattices approximating $\mathbb{R} \times \mathbb{S}^2$ using Regge Calculus, DEC & FEM

- Introduce perturbative counterterms **Quantum Finite Elements (QFE)** (justified for small coupling)

- Tune mass to critical surface



⇨ convergence to spherically symmetric continuum theory



$$S = \frac{1}{2} \sum_{y \in \langle x,y \rangle} \frac{l^*_{xy}}{l_{xy}} \left(\tilde{\phi}_{t,x} - \tilde{\phi}_{t,y} \right)^2 + \frac{a^2}{4R^2} \sqrt{\tilde{g}_x} \tilde{\phi}^2_{t,x} + \sqrt{\tilde{g}_x} \left[\frac{a^2}{a_t^2} \left(\tilde{\phi}_{t,x} - \tilde{\phi}_{t+1,x} \right)^2 + m_0^2 \tilde{\phi}^2_{t,x} + \lambda_0 \tilde{\phi}^4_{t,x} \right]$$

$$S_{QFE} = S - \sum_{t,x} \sqrt{\tilde{g}_x} \left[6\lambda_0 \delta G_x - 24 \lambda_0^2 \delta G_x^{(3)} \right] \tilde{\phi}^2_{t,x}.$$

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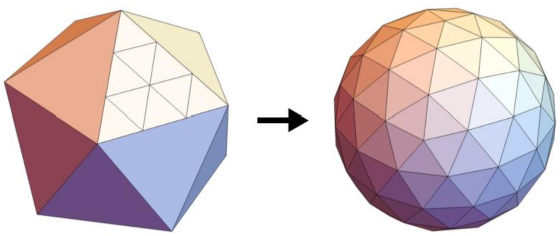

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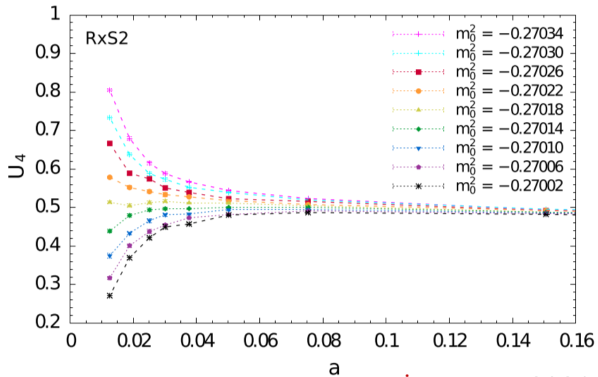

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⇨ Lattice simulations of critical φ^4-theory on lattices approaching $\mathbb{R} \times \mathbb{S}^2$

Numerical Results

MONTE CARLO SIMULATIONS OF CRITICIAL φ^4-THEORY,

PERIODIC BOUNDARY CONDITIONS

Data & Fitting

$$\frac{\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle}{\langle \sigma(x_1)\sigma(x_2)\rangle\langle \sigma(x_3)\sigma(x_4)\rangle} = 1 + \sum_{j\in 2\mathbb{N}_0} c_j(t) P_j(\cos\theta)$$

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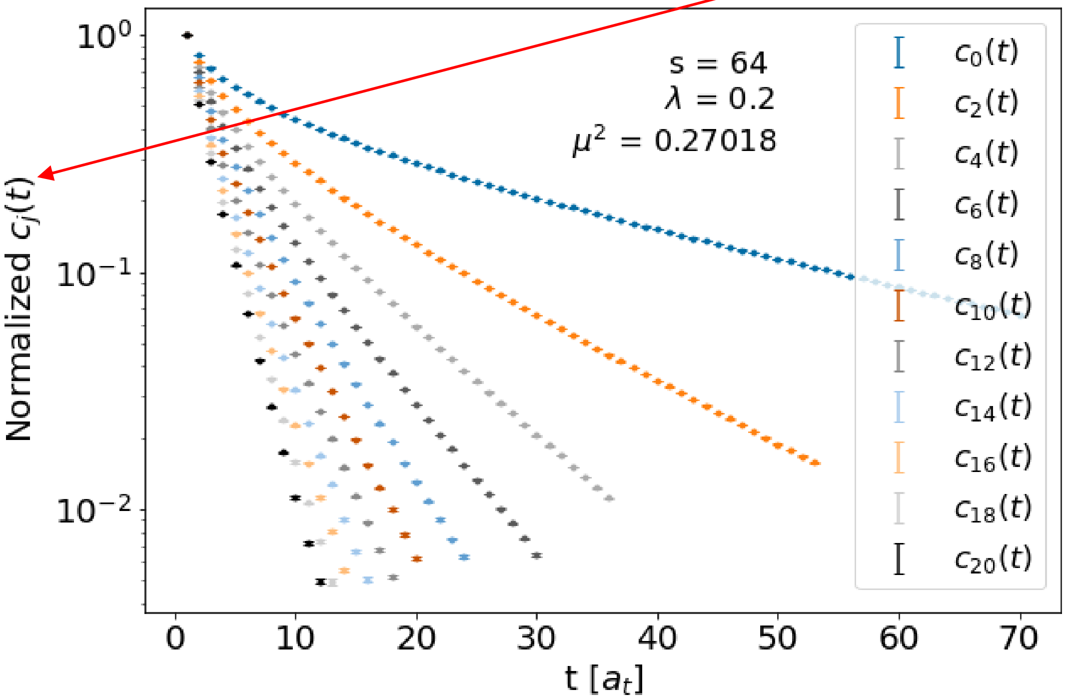
$$+ (t \to N_t - t)$$



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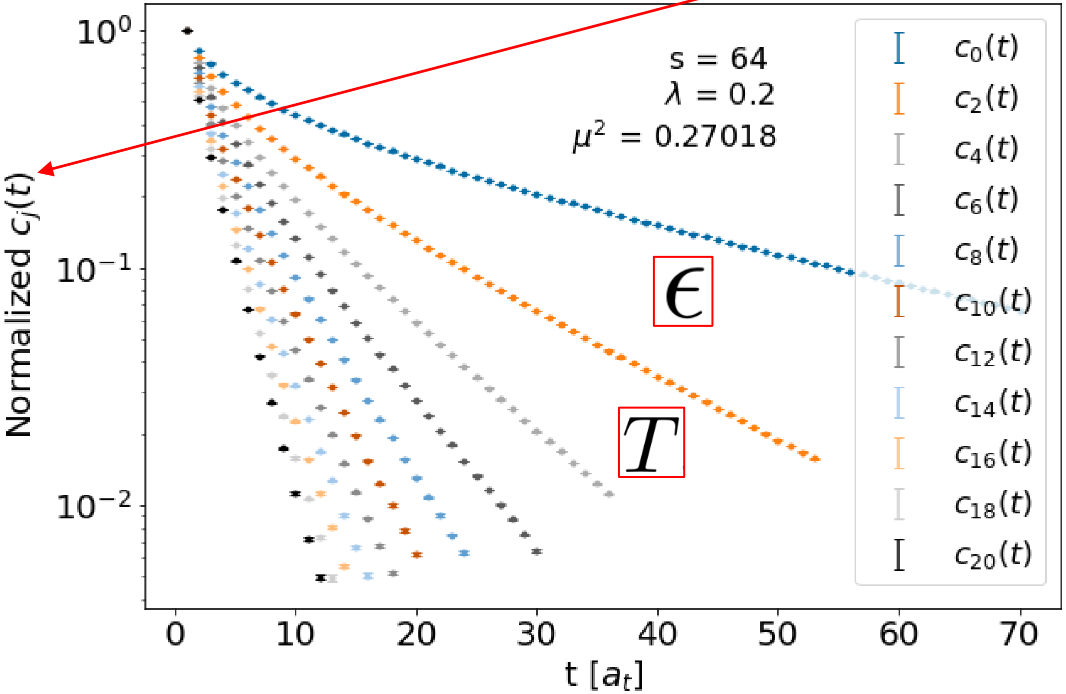


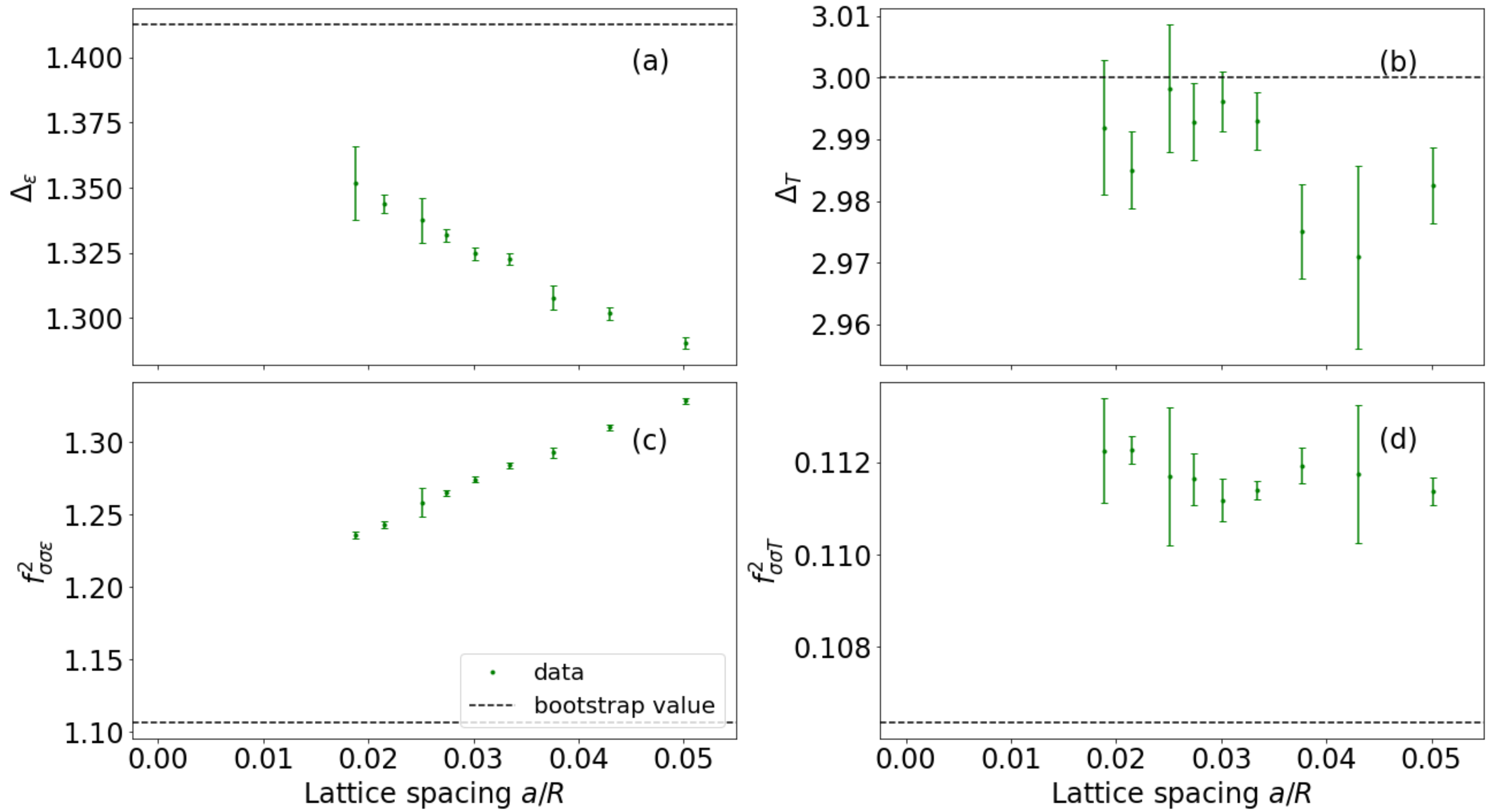
Fitting Procedure

- Simultaneous fits of $c_0(t)$ and $c_2(t)$ using primaries $\epsilon, \quad T, \quad \epsilon', \quad T'$ up to n=20
- Model averaging [Jay, Neil 2021]

$$\theta = \sum_{\text{model } i} \theta_{\text{fit}\,i}\; p(\text{model } i|\text{data})$$

Different starting values t_{min}

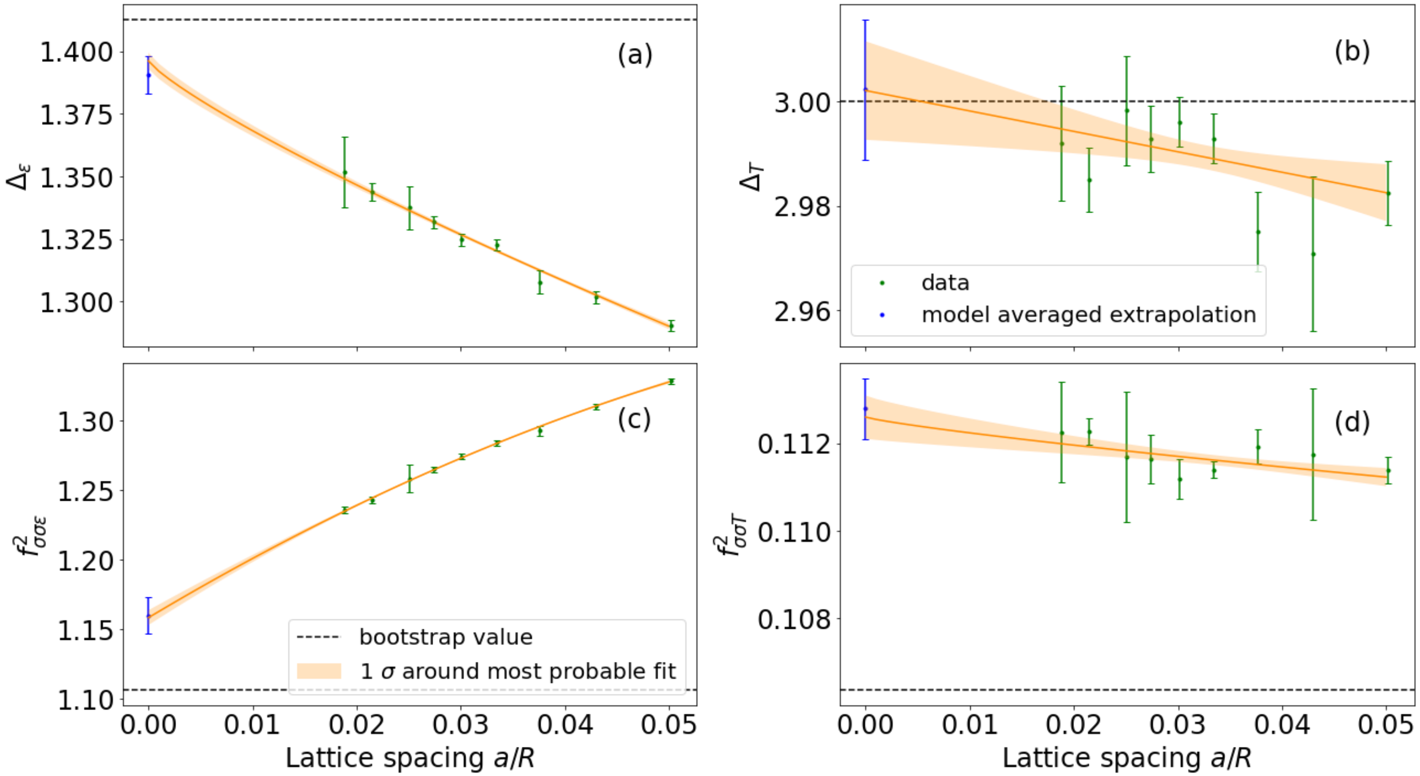
Results - Leading operators



ϵ - leading for l=0

T - leading for l=2

Results - Leading operators



Extrapolation Procedure

- Fits with

$$f_{\mathrm{FSS}} = c_1(\frac{a}{R})^{0.83} + c_0$$

$$f_{\mathrm{linear}} = c_1(\frac{a}{R}) + c_0$$

$$f_{\mathrm{quadratic}} = c_2(\frac{a}{R})^2 + c_1(\frac{a}{R}) + c_0$$

- Model averaging [Jay, Neil 2021]

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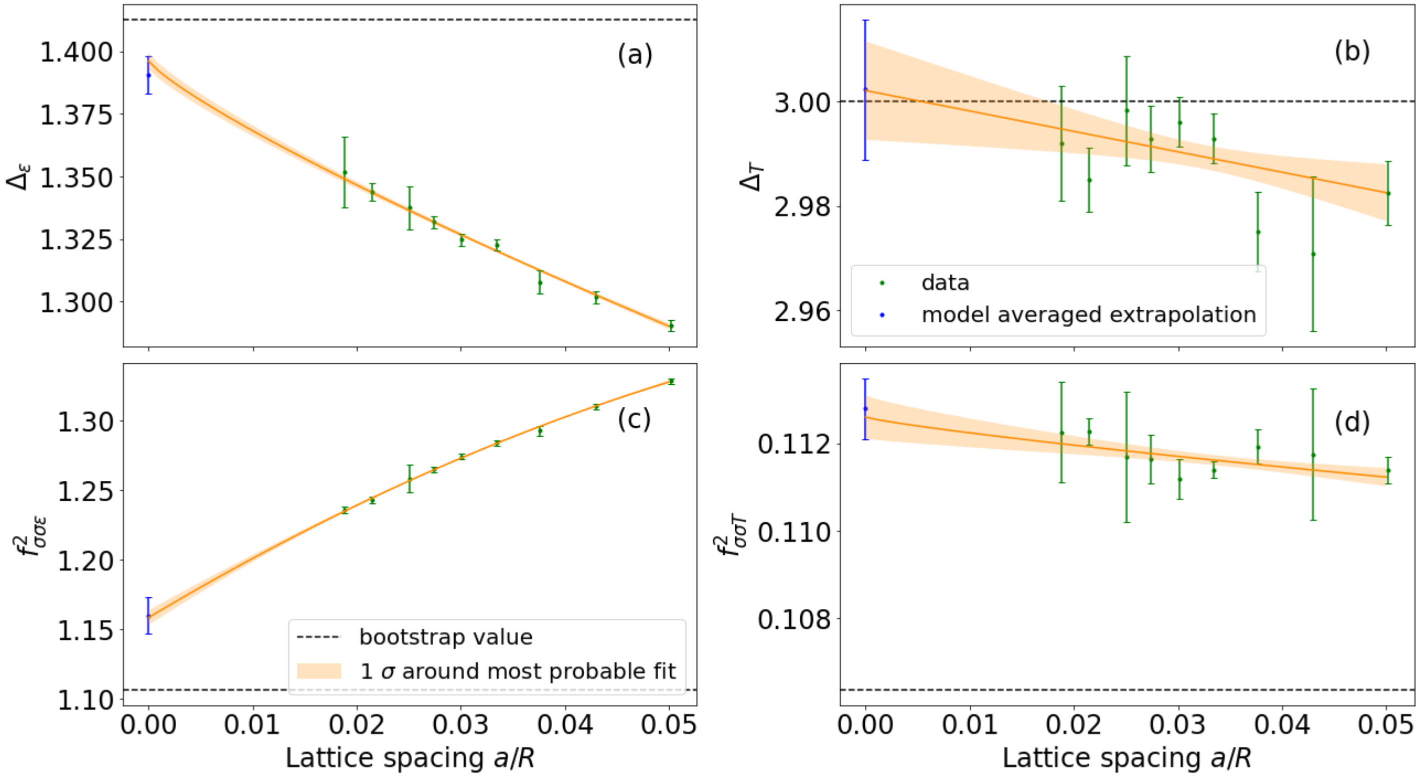
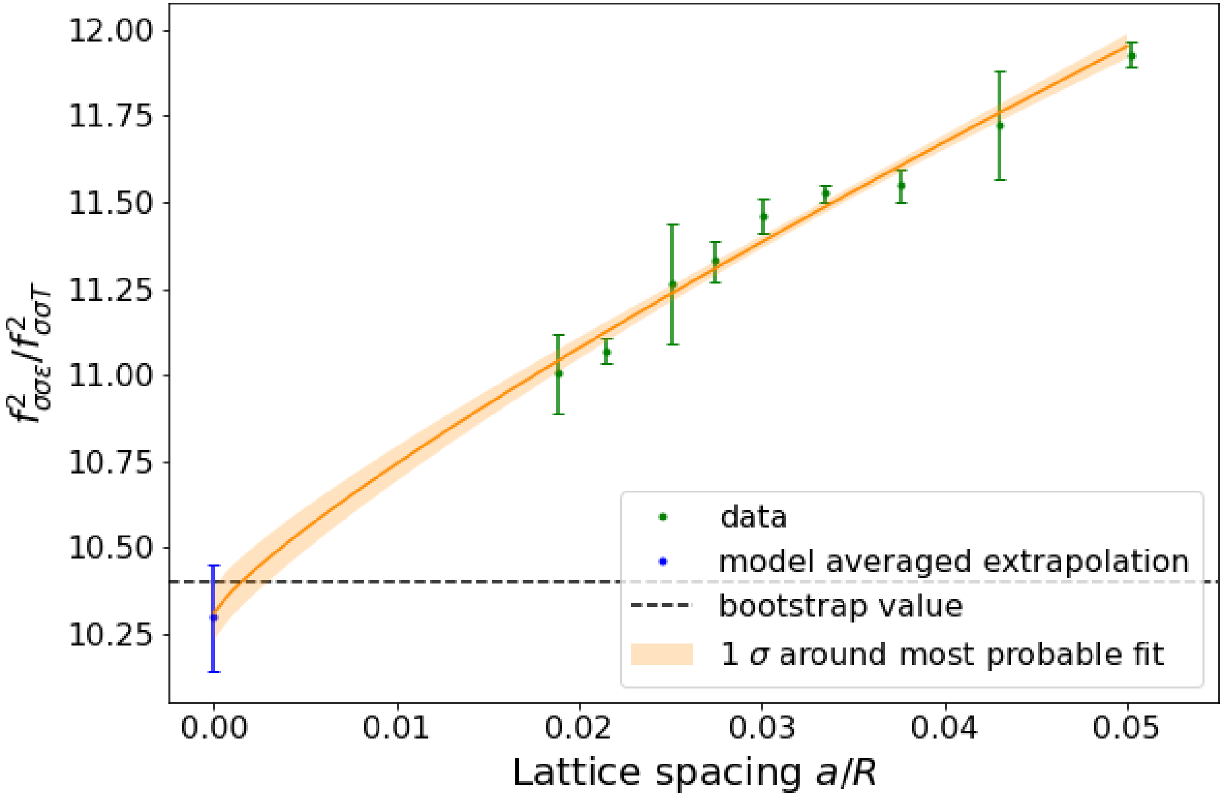
Different fits and maximal values a_{max}

Results - Leading operators



	Bootstrap	Fit
Δ_ϵ	1.41265(36)	1.3904(75)
$f^2_{\sigma\sigma\epsilon}$	1.10639(26)	1.160(14)
Δ_T	3	3.002(14)
$f^2_{\sigma\sigma T}$	0.10636583(30)	0.11280(70)

Results - Ratio of OPE coefficients



$$\left(f^2_{\sigma\sigma\epsilon}/f^2_{\sigma\sigma T}\right)^{fit} = 10.30(16)$$

$$\left(f^2_{\sigma\sigma\epsilon}/f^2_{\sigma\sigma T}\right)^{bootstrap} = 10.4017(24)$$

Conclusion and outlook

- Proof of concept:
  Can extract CFT quantities, including OPE coefficients from four-point function with this method

- Good agreeance with conformal bootstrap for scaling dimensions & ratios of OPE coefficients,

- Possible systematic errors:
  1. Excited state contamination (subleading & higher spin operators)
  2. Wraparound effects
  3. Tuning of the critical mass
  4. Perturbative counterterms ➡ QFE only valid for $\lambda \to 0$

- Outlook: Implement with lattice methods currently under development

Thank you for your attention!

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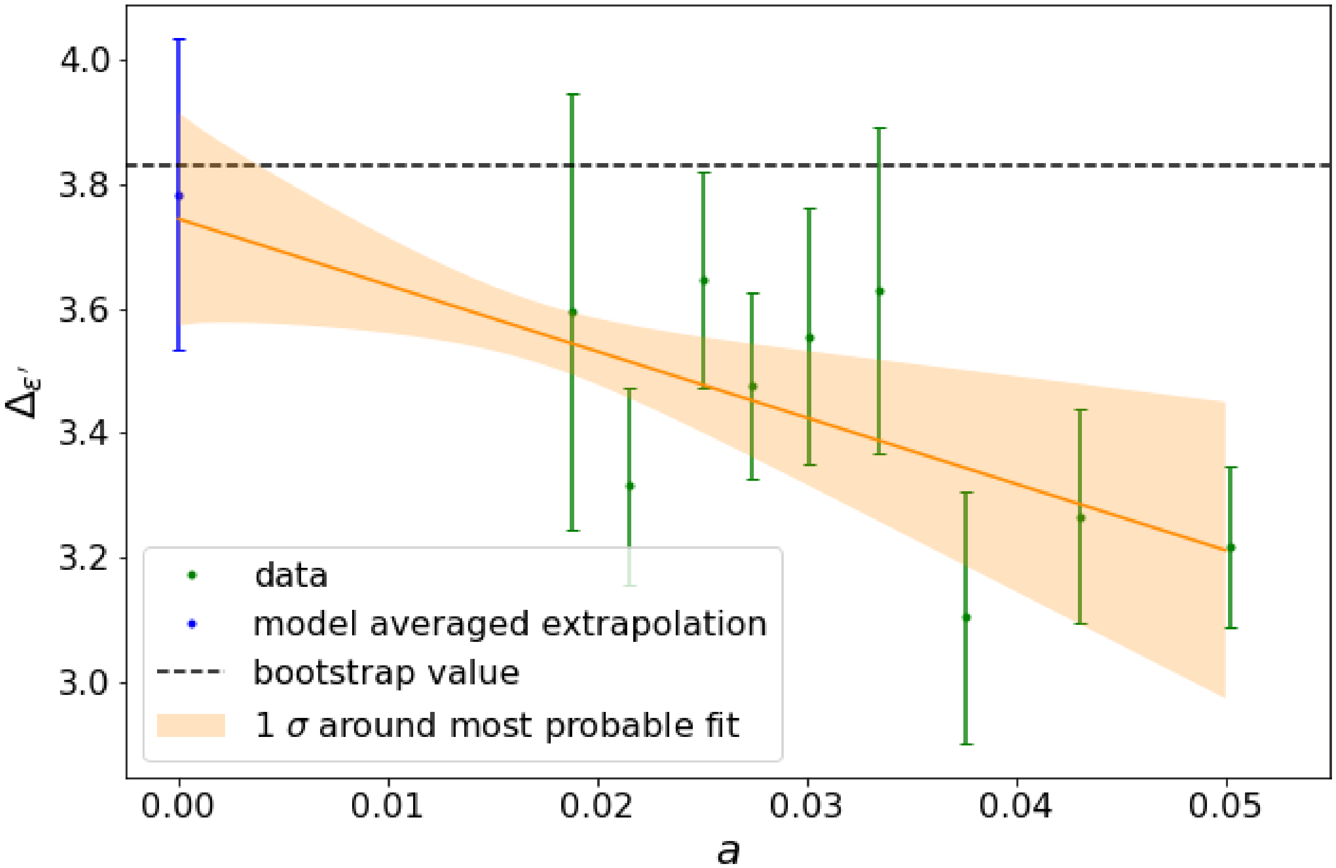
Back-up Slides

Subleading operator

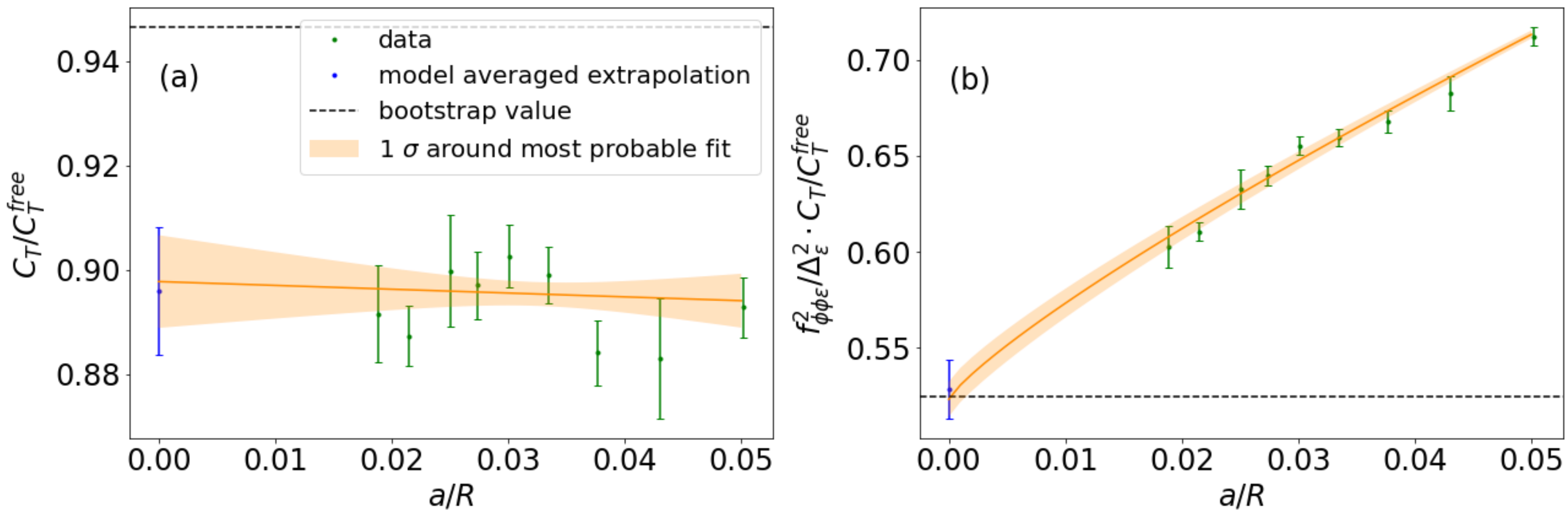


$$\Delta_{\epsilon'}^{fit} = 3.78(25)$$

$$\Delta_{\epsilon'}^{bootstrap} = 3.82968(23)$$

Central charge

$$C_T = \frac{\Delta_\sigma^2 \Delta_T^2}{16 f_{\sigma\sigma T}^2}$$



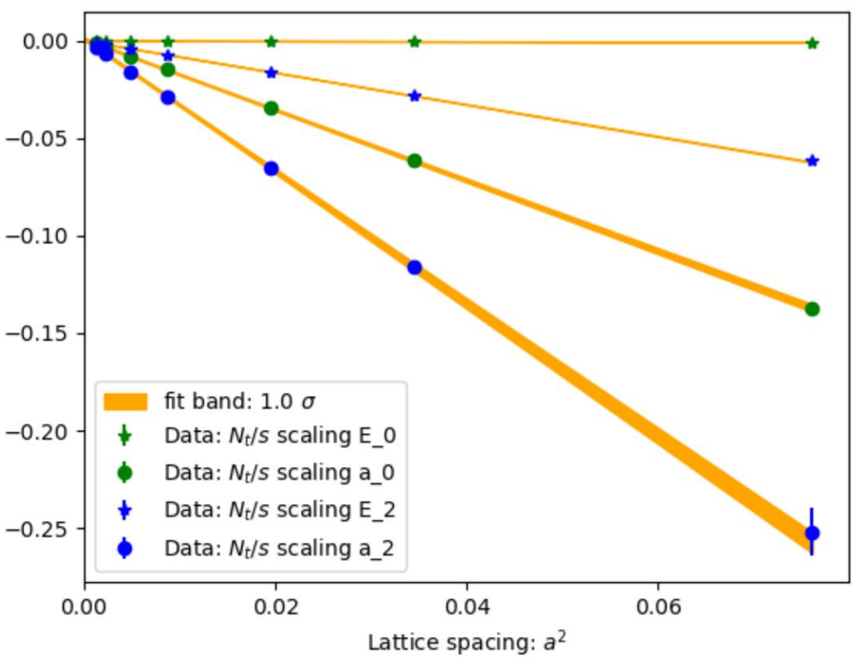
$$C_T^{fit}/C_T^{free} = 0.896(13)$$

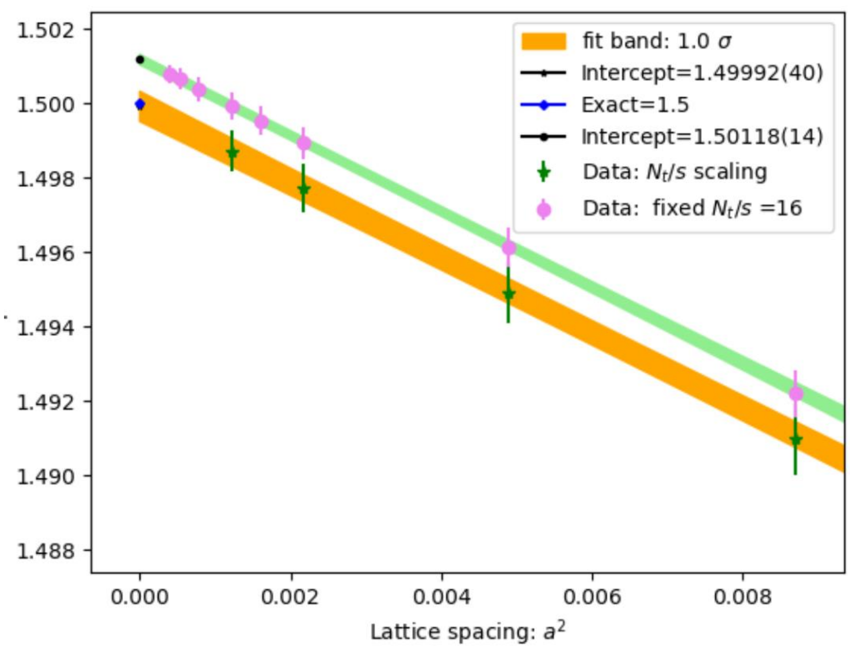
$$C_T^{bootstrap}/C_T^{free} = 0.946543(42)$$

$$\left(C_T/C_T^{free} \cdot f_{\phi\phi\epsilon}^2/\Delta_\epsilon^2\right)^{fit} = 0.528(16)$$

$$\left(C_T/C_T^{free} \cdot f_{\phi\phi\epsilon}^2/\Delta_\epsilon^2\right)^{bootstrap} = 0.52478(13)$$

The Free Case





Simulation Details

- Tamayo-Brower cluster algorithm combined with Metropolis and overrelaxation
- Lattice refinements $s \in \{24, 28, 36, 40, 44, 48, 56, 64\}$ on sphere ➡ $10s^2 + 2$ lattice sites
- $N_t = 16s$ “timesteps” and periodic boundary conditions along $\mathbb{R}$
- Multiple runs for each s :

$$s = 24 \to N = 8000$$
$$28 \leq s \leq 56 \to N = 1600$$
$$s = 64 \to N = 800$$

- Measure the 4-point function in each sweep and project on Legendre Polynomials

Fitting Details

- Fit functions:

$$c_0^{fit}(t) = \sum_{n=0}^{n_{max}} f_{\sigma\sigma\epsilon}^2 B_{n,0}(\Delta_\epsilon) e^{-(\Delta_\epsilon+n)ta_t/R} + \sum_{n=0}^{n_{max}-2} f_{\sigma\sigma\epsilon'}^2 B_{n,0}(\Delta_{\epsilon'}) e^{-(\Delta_{\epsilon'}+n)ta_t/R} + \sum_{n=2}^{n_{max}-4} f_{\sigma\sigma T'}^2 B_{n,0}(\Delta_{T'}) e^{-(\Delta_{T'}+n)ta_t/R} + (t \to N_t - t)$$

$$c_2^{fit}(t) = \sum_{n=0}^{n_{max}-2} f_{\sigma\sigma T}^2 B_{n,2}(\Delta_T) e^{-(\Delta_T+n)ta_t/R} + \sum_{n=0}^{n_{max}-4} f_{\sigma\sigma T'}^2 B_{n,2}(\Delta_{T'}) e^{-(\Delta_{T'}+n)ta_t/R} + \sum_{n=2}^{n_{max}} f_{\sigma\sigma\epsilon}^2 B_{n,2}(\Delta_\epsilon) e^{-(\Delta_\epsilon+n)ta_t/R} + \sum_{n=2}^{n_{max}-2} f_{\sigma\sigma\epsilon'}^2 B_{n,2}(\Delta_{\epsilon'}) e^{-(\Delta_{\epsilon'}+n)ta_t/R} + (t \to N_t - t)$$

- Least Squares Minimization using the L-BFGS-B algorithm implemented in SciPy
- Perform fits for fixed (t_0^{max}, t_2^{max}) s.t. the error of the effective mass calculated from the $c_j(t)$ doesn't exceed 12.5% but varying (t_0^{min}, t_2^{min})
- Calculate the model probability for each fit [Jay, Neil 2021]
- Eliminate 1) fits with unphysical or unconstrained fit parameters, 2) fits with model probability < 0.01 or 3) fits for which (t_0^{min}, t_2^{min}) was far from the optimal tuple
- Renormalize model probability and model average