Extracting OPE Coefficients of the 3d Ising CFT from the Four-Point Function

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• Richard C. Brower, Boston University
• Venkitesh Ayyar, Boston University
• Evan Owen, Boston University
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• Chung-I Tan, Brown University
So far: Development

Evan Owen, Tuesday, 1:30 pm
Richard Brower, Tuesday, 1:50 pm

Now: Application

Put the critical Ising model on $\mathbb{R} \times S^2$ to access quantities that are difficult to calculate on Euclidean lattices
Introduction
The Critical 3d Ising Model

- 3d Ising model: interacting spins $s = \pm 1$
The Critical 3d Ising Model

- 3d Ising model: interacting spins $s = \pm 1$
- At criticality: physical quantities = power laws
  
  $C \propto t^{-\alpha}$  
  $m \propto t^\beta$  
  $\chi \propto t^{-\gamma}$  
  $\xi \propto t^{-\nu}$
  
  $\langle \sigma(0)\sigma(r) \rangle \propto r^{-d+2-\eta}$

  $t = \frac{|T-T_c|}{T_c}$

Scale invariance at critical point
The Critical 3d Ising Model

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  $t = \frac{|T - T_c|}{T_c}$

Scale invariance at critical point

T = $T_c$  $T \gg T_c$

[Unsolved (analytically in 3D)]

[Tong 2017]
The Critical 3d Ising Model

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  Scale invariance at critical point

Determination of critical exponents using Monte Carlo & finite size scaling

[Swendsen, Wang 1987],
[Wolff 1989],
[Pelissetto, Vicari, 2000]

UNSOLVED
(analytically in 3D)
The Critical 3d Ising Model

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  Determination of critical exponents using Monte Carlo & finite size scaling

  [Swendsen, Wang 1987],
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  What more is there to learn?

  **UNSOLVED**

  (analytically in 3D)
The 3d Ising CFT

Change POV: Condensed Matter Physics $\Rightarrow$ Continuum Field Theory

Scale invariance $\quad +$ Poincaré invariance $\Rightarrow$ Invariance under conformal group $O(d+1,1)$
The 3d Ising CFT

Change POV: Condensed Matter Physics → Continuum Field Theory

- Scale invariance
- Invariance under conformal group $O(d + 1, 1)$
- Critical Ising Model = Conformal Field Theory (3d Ising CFT)
The 3d Ising CFT

Change POV: Condensed Matter Physics $\rightarrow$ Continuum Field Theory

Scale invariance + Poincaré invariance $\rightarrow$ Invariance under conformal group $O(d + 1, 1)$

Critical Ising Model = Conformal Field Theory (3d Ising CFT)

Primary operators $\mathcal{O}$: $\sigma, \epsilon, \epsilon^\prime, T, T^\prime, C, ...$

$0^-, 0^+, 0^+, 2^+, 2^+, 4^+$
The 3d Ising CFT

Scaling dimensions  Operator Product Expansion (OPE) coefficients

CFT characterized by $\Delta_{\mathcal{O}}$ and $f_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}$

2-point functions

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2) \rangle = \frac{1}{|x_1 - x_2|^{2\Delta_{\mathcal{O}}}}$$

(n>2)-point functions

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k f_{\mathcal{O}_i\mathcal{O}_j\mathcal{O}_k} C(x_1 - x_2) \mathcal{O}_k(x_2)$$
The 3d Ising CFT

Scaling dimensions and Operator Product Expansion (OPE) coefficients

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\]

Scaling dimensions known from lattice calculations

\[
\Delta_\sigma = \frac{\eta + d - 2}{2}
\]

\[
\Delta_\epsilon = d - \frac{1}{\nu}
\]

\[
\Delta_{\epsilon'} = \omega + d
\]

[Pelissetto, Vicari, 2000]
The 3d Ising CFT

Scaling dimensions \( \Delta_\sigma \) and \( f_{O_1 O_2 O_3} \)

2-point functions

\[
\langle O(x_1)O(x_2) \rangle = \frac{1}{|x_1 - x_2|^{2\Delta_\sigma}}
\]

(n>2)-point functions

\[
O_i(x_1)O_j(x_2) = \sum_k f_{O_i O_j O_k} C(x_1 - x_2) O_k(x_2)
\]

Difficult to determine OPE coefficients with traditional MC methods on Euclidean lattices

Scaling dimensions known from lattice calculations

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\begin{align*}
\Delta_\sigma &= \frac{\eta + d - 2}{2} \\
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\]

Monte Carlo (\( f_{\sigma\sigma\epsilon}, f_{\epsilon\epsilon\epsilon} \)): e.g. [Hasenbusch, 2019]

Conformal Bootstrap:
[El-Showk et al., 2012, 2014]
[Simmons-Duffin, 2016]
[Reehorst, 2021]

Fuzzy Sphere:
[Hu, 2023]
The 3d Ising CFT

CFT characterized by $\Delta_{\mathcal{O}}$ and $f_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3}$

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$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle = \frac{1}{|x_1 - x_2|^{2\Delta_{\mathcal{O}}}}$

(n>2)-point functions

$\mathcal{O}_i(x_1) \mathcal{O}_j(x_2) = \sum_{k} f_{\mathcal{O}_i \mathcal{O}_j \mathcal{O}_k} C(x_1 - x_2) \mathcal{O}_k(x_2)$

Difficult to determine OPE coefficients with traditional MC methods on Euclidean lattices

Scaling dimensions known from lattice calculations

$\Delta_{\sigma} = \eta + d - 2$  
$\Delta_{\epsilon} = d - \frac{1}{\nu}$  
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Monte Carlo ($f_{\sigma\sigma\sigma}, f_{\epsilon\epsilon\epsilon}$): e.g. [Hasenbusch, 2019]

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The 3d Ising CFT

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Difficult to determine OPE coefficients with traditional MC methods on Euclidean lattices

Scaling dimensions known from lattice calculations

$\Delta_\sigma = \frac{\eta + d - 2}{2}$

$\Delta_\epsilon = d - \frac{1}{\nu}$

$\Delta_\epsilon' = \omega + d$

Goal of this work: extract both scaling dimensions AND OPE coefficients from lattice calculations of four-point functions on $\mathbb{R} \times \mathbb{S}^2$
Antipodal 4-point function on $\mathbb{R} \times S^2 \ni (t, \vec{x})$

4-point function of $\sigma$-operators in special frame on the right:

$$\frac{\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4) \rangle}{\langle \sigma(x_1)\sigma(x_2) \rangle \langle \sigma(x_3)\sigma(x_4) \rangle} = 1 + \sum_{j \in 2\mathbb{N}_0} c_j(t) P_j(\cos \theta)$$

$$c_j(t) = \sum_{\mathcal{O}} f_{\sigma \sigma \mathcal{O}}^2 \sum_{n \in 2\mathbb{N}_0} B_{n,j}(\Delta \mathcal{O}) e^{-(\Delta \mathcal{O} + n)t}$$

[Hogervorst and Rychkov, 2013, Costa et al., 2016]
Antipodal 4-point function on $\mathbb{R} \times S^2 \ni (t, \vec{x})$

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$$c_j(t) = \sum_{\mathcal{O}} f_{\sigma\sigma\mathcal{O}}^2 \sum_{\substack{n \in 2\mathbb{N}_0 \atop n \geq |j-l|}} B_{n,j}(\Delta_{\mathcal{O}}) e^{-(\Delta_{\mathcal{O}} + n) t}$$

Obtain $f_{\sigma\sigma\mathcal{O}}$ and $\Delta_{\mathcal{O}}$ for even-parity & even-spin $\mathcal{O}$ by fits to $c_j(t)$
Quantum Finite Elements

[Brower et al., 2018, 2021]

- Discretize $\varphi^4$-theory on simplicial lattices approximating $\mathbb{R} \times S^2$ using Regge Calculus, DEC & FEM
- Introduce perturbative counterterms Quantum Finite Elements (QFE) (justified for small coupling)
- Tune mass to critical surface

$\Rightarrow$ convergence to spherically symmetric continuum theory

$S = \frac{1}{2} \sum_{y \in \{x,y\}} l_{xy}^s \left( \tilde{\phi}_{t,x} - \tilde{\phi}_{t,y} \right)^2 + \frac{a^2}{4R^2} \sqrt{g_x} \tilde{\phi}_{t,x}^2$

$+ \sqrt{g_x} \left[ \frac{a^2}{a_t^2} \left( \tilde{\phi}_{t,x} - \tilde{\phi}_{t+1,x} \right)^2 + m_0^2 \tilde{\phi}_{t,x}^2 + \lambda_0 \tilde{\phi}_{t,x}^4 \right]$

$S_{QFE} = S - \sum_{t,x} \sqrt{\tilde{g}_x} \left[ 6\lambda_0 \delta G_x - 24\lambda_0^2 \delta G^{(3)}_x \right] \tilde{\phi}_{t,x}^2$. 
Quantum Finite Elements

- Discretize $\varphi^4$-theory on simplicial lattices approximating $\mathbb{R} \times S^2$ using Regge Calculus, DEC & FEM

- Introduce perturbative counterterms Quantum Finite Elements (QFE) (justified for small coupling)

- Tune mass to critical surface

$S = \frac{1}{2} \sum_{y \in \langle x, y \rangle} \frac{l_{xy}^*}{l_{xy}} \left( \tilde{\phi}_{t,x} - \tilde{\phi}_{t,y} \right)^2 + \frac{a^2}{4R^2} \sqrt{g_x} \tilde{\phi}_{t,x}^2$

$+ \sqrt{g_x} \left[ \frac{a^2}{a_t^2} \left( \tilde{\phi}_{t,x} - \tilde{\phi}_{t+1,x} \right)^2 + m_0^2 \tilde{\phi}_{t,x}^2 + \lambda_0 \tilde{\phi}_{t,x}^4 \right]$

$S_{QFE} = S - \sum_{t,x} \sqrt{\tilde{g}_x} \left[ 6\lambda_0 \delta G_x - 24\lambda_0^2 \delta G_x^{(3)} \right] \tilde{\phi}_{t,x}^2.$

Lattice simulations of critical $\varphi^4$-theory on lattices approaching $\mathbb{R} \times S^2$
Numerical Results

MONTE CARLO SIMULATIONS OF CRITICAL $\phi^4$-THEORY, PERIODIC BOUNDARY CONDITIONS
Data & Fitting

\[
\frac{\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4) \rangle}{\langle \sigma(x_1)\sigma(x_2) \rangle \langle \sigma(x_3)\sigma(x_4) \rangle} = 1 + \sum_{j \in 2\mathbb{N}_0} c_j(t) P_j(\cos \theta)
\]

\[
c_j(t) = \sum_{\mathcal{O}} f_{\sigma\mathcal{O}}^2 \sum_{n \in 2\mathbb{N}_0 \atop n \geq |j-t|} B_{n,j}(\Delta \mathcal{O}) e^{-(\Delta \mathcal{O} + n)t} g_{\mathcal{O}R} + (t \to N_t - t)
\]

s = 64
\(\lambda = 0.2\)
\(\mu^2 = 0.27018\)
Data & Fitting

\[
\frac{\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4) \rangle}{\langle \sigma(x_1)\sigma(x_2) \rangle \langle \sigma(x_3)\sigma(x_4) \rangle} = 1 + \sum_{j \in 2\mathbb{N}_0} c_j(t) P_j(\cos \theta)
\]

\[
c_j(t) = \sum_{\mathcal{O}} f^2_{\sigma \mathcal{O}} \sum_{n \in 2\mathbb{N}_0, n \geq |j-t|} B_{n,j}(\Delta \mathcal{O}) e^{-(\Delta \mathcal{O} + n) t g_{\mathcal{O}R}} + (t \to N_t - t)
\]

Fitting Procedure

- Simultaneous fits of \(c_0(t)\) and \(c_2(t)\) using primaries \(\epsilon, T, \epsilon', T'\) up to \(n=20\)

- Model averaging [Jay, Neil 2021]

\[
\theta = \sum_{\text{model } i} \theta_{\text{fit } i} p(\text{model } i | \text{data})
\]

Different starting values \(t_{\text{min}}\)
Results - Leading operators

\[ \varepsilon \] - leading for \( l=0 \)

\[ T \] - leading for \( l=2 \)

(a) \[ \Delta' \]

(b) \[ \Delta_T \]

(c) \[ \rho_{00}^2 \]

(d) \[ \rho_{00}^T \]
Results - Leading operators

Extrapolation Procedure

• Fits with

\[ f_{\text{FSS}} = c_1 \left( \frac{a}{R} \right)^{0.83} + c_0 \]
\[ f_{\text{linear}} = c_1 \left( \frac{a}{R} \right) + c_0 \]
\[ f_{\text{quadratic}} = c_2 \left( \frac{a}{R} \right)^2 + c_1 \left( \frac{a}{R} \right) + c_0 \]

• Model averaging [Jay, Neil 2021]

\[ \theta = \sum_{\text{model } i} \theta_{\text{fit } i} \cdot p(\text{model } i|\text{data}) \]

Different fits and maximal values \( a_{\text{max}} \)
Results - Leading operators

<table>
<thead>
<tr>
<th></th>
<th>Bootstrap</th>
<th>Fit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta_{\epsilon}$</td>
<td>1.41265(36)</td>
<td>1.3904(75)</td>
</tr>
<tr>
<td>$f_{\sigma\epsilon}^2$</td>
<td>1.10639(26)</td>
<td>1.160(14)</td>
</tr>
<tr>
<td>$\Delta_T$</td>
<td>3</td>
<td>3.002(14)</td>
</tr>
<tr>
<td>$f_{\sigma\sigma T}^2$</td>
<td>0.10636583(30)</td>
<td>0.11280(70)</td>
</tr>
</tbody>
</table>
Results - Ratio of OPE coefficients

\[
\left( \frac{f_{\sigma\sigma\epsilon}}{f_{\sigma\sigma T}} \right)^{\text{fit}} = 10.30(16) \\
\left( \frac{f_{\sigma\sigma\epsilon}}{f_{\sigma\sigma T}} \right)^{\text{bootstrap}} = 10.4017(24)
\]
Conclusion and outlook

• Proof of concept:
  Can extract CFT quantities, including OPE coefficients from four-point function with this method

• Good agreeance with conformal bootstrap for scaling dimensions & ratios of OPE coefficients,

• Possible systematic errors:
  1. Excited state contamination (subleading & higher spin operators)
  2. Wraparound effects
  3. Tuning of the critical mass
  4. Perturbative counterterms $\Rightarrow$ QFE only valid for $\lambda \to 0$

• Outlook: Implement with lattice methods currently under development
Thank you for your attention!
Sources


Sources


Sources


Back-up Slides
Subleading operator

\[ \Delta_{\epsilon'}^{\text{fit}} = 3.78(25) \]

\[ \Delta_{\epsilon'}^{\text{bootstrap}} = 3.82968(23) \]
Central charge

\[ C_T = \frac{\Delta^2_{\sigma} \Delta^2_{T}}{16 f^2_{\sigma\sigma T}} \]

\[ C_T^{\text{fit}} / C_T^{\text{free}} = 0.896(13) \]
\[ C_T^{\text{bootstrap}} / C_T^{\text{free}} = 0.946543(42) \]

\[ \left( \frac{C_T / C_T^{\text{free}} \cdot f_{\phi\phi}^2 / \Delta^2_\epsilon}{C_T / C_T^{\text{free}} \cdot f_{\phi\phi}^2 / \Delta^2_\epsilon} \right)^{\text{bootstrap}} = 0.52478(13) \]

\[ \left( \frac{C_T / C_T^{\text{free}} \cdot f_{\phi\phi}^2 / \Delta^2_\epsilon}{C_T / C_T^{\text{free}} \cdot f_{\phi\phi}^2 / \Delta^2_\epsilon} \right)^{\text{fit}} = 0.528(16) \]
The Free Case
Simulation Details

- Tamayo-Brower cluster algorithm combined with Metropolis and overrelaxation
- Lattice refinements $s \in \{24, 28, 36, 40, 44, 48, 56, 64\}$ on sphere $\Rightarrow 10s^2 + 2$ lattice sites
- $N_t = 16s$ “timesteps” and periodic boundary conditions along $\mathbb{R}$
- Multiple runs for each $s$:
  - $s = 24 \rightarrow N = 8000$
  - $28 \leq s \leq 56 \rightarrow N = 1600$
  - $s = 64 \rightarrow N = 800$
- Measure the 4-point function in each sweep and project on Legendre Polynomials
Fitting Details

- Fit functions:

\[
c_0^{\text{fit}}(t) = \sum_{n=0}^{n_{\text{max}}} f_{\sigma \epsilon}^2 B_{n,0}(\Delta_{\epsilon}) e^{-(\Delta_{\epsilon} + n) t a_t / R} \\
+ \sum_{n=0}^{n_{\text{max}} - 2} f_{\sigma \epsilon}^2 B_{n,0}(\Delta_{\epsilon}) e^{-(\Delta_{\epsilon} + n) t a_t / R} \\
+ \sum_{n=2}^{n_{\text{max}} - 4} f_{\sigma \epsilon}^2 B_{n,0}(\Delta_{\epsilon}) e^{-(\Delta_{\epsilon} + n) t a_t / R} \\
+ (t \to N_t - t)
\]

\[
c_2^{\text{fit}}(t) = \sum_{n=0}^{n_{\text{max}} - 2} f_{\sigma \epsilon}^2 B_{n,2}(\Delta_{\epsilon}) e^{-(\Delta_{\epsilon} + n) t a_t / R} \\
+ \sum_{n=0}^{n_{\text{max}} - 4} f_{\sigma \epsilon}^2 B_{n,2}(\Delta_{\epsilon}) e^{-(\Delta_{\epsilon} + n) t a_t / R} \\
+ \sum_{n=2}^{n_{\text{max}} - 4} f_{\sigma \epsilon}^2 B_{n,2}(\Delta_{\epsilon}) e^{-(\Delta_{\epsilon} + n) t a_t / R} \\
+ (t \to N_t - t)
\]

- Least Squares Minimization using the L-BFGS-B algorithm implemented in SciPy
- Perform fits for fixed \((t_0^{\text{max}}, t_2^{\text{max}})\) s.t. the error of the effective mass calculated from the \(c_j(t)\) doesn’t exceed 12.5% but varying \((t_0^{\text{min}}, t_2^{\text{min}})\)
- Calculate the model probability for each fit [Jay, Neil 2021]
- Eliminate 1) fits with unphysical or unconstrained fit parameters, 2) fits with model probability < 0.01 or 3) fits for which \((t_0^{\text{min}}, t_2^{\text{min}})\) was far from the optimal tuple
- Renormalize model probability and model average