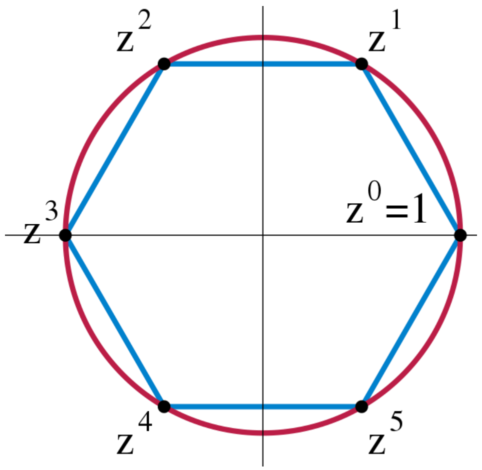
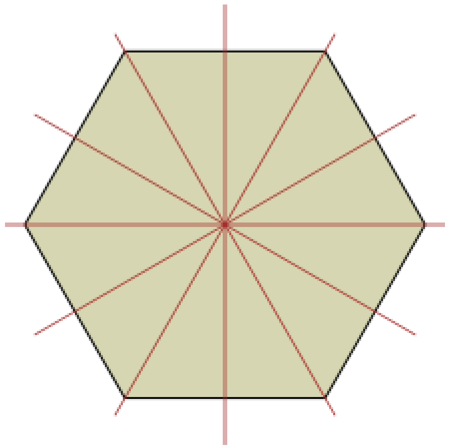
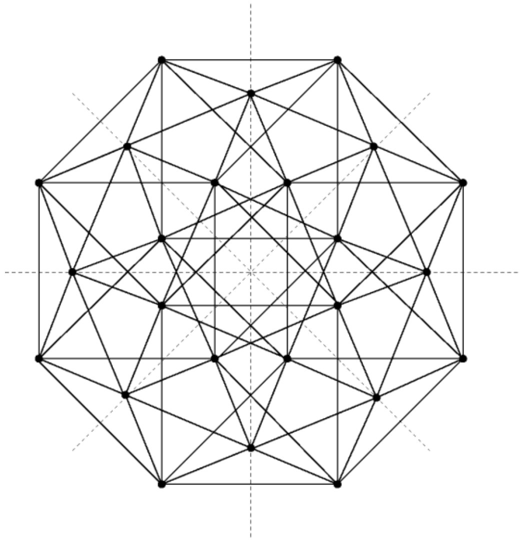
Finite-group Laplacian and the physical Hilbert space of finite-group gauge theories







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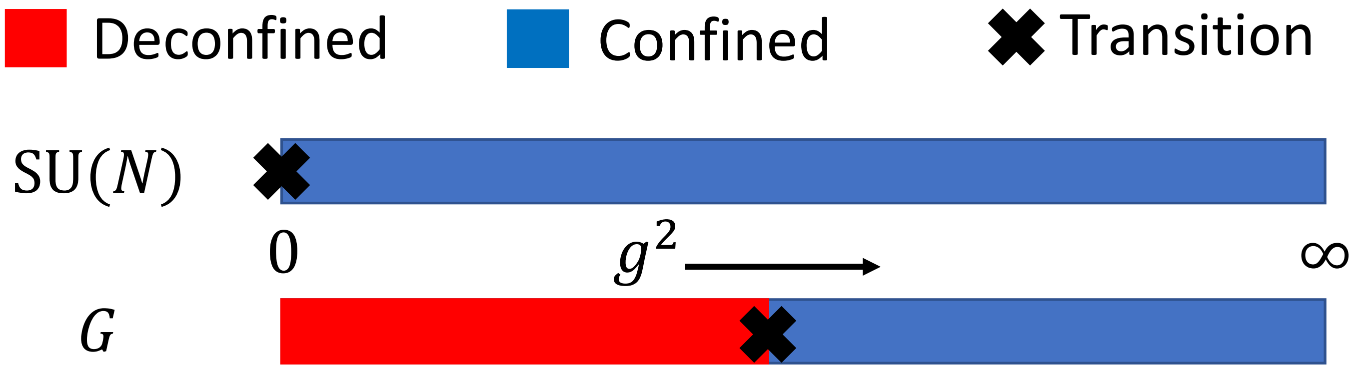
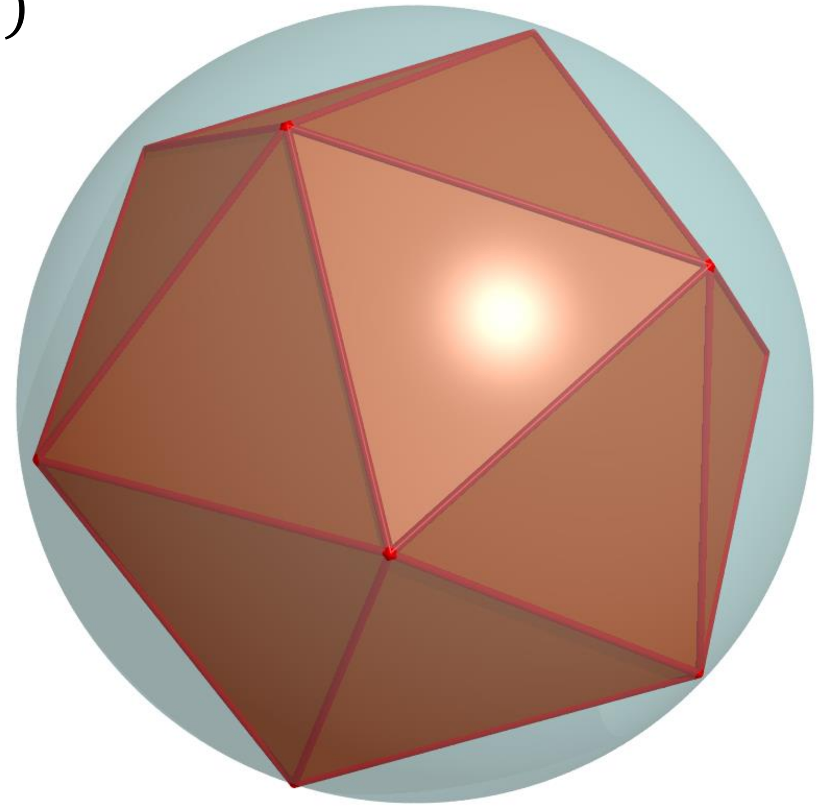
based on *Mariani, Pradhan, Ercolessi, Phys. Rev. D 107, 114513 (2023)*

Quantum simulation of gauge theories: discrete subgroups

Replace Lie group with finite subgroup $G \leq \mathrm{SU}(N)$





For G finite, the transition is only first order

⟶ effective theory

Q: Is there an action/formulation with a *second-order* transition?

[Hasenfratz & Niedermayer ’01]

The Hamiltonian

Transfer matrix from the Wilson action:

[Kogut & Susskind ‘75; Creutz ‘77; Luscher ‘77]

[Harlow & Ooguri ‘18]

Lie group:

$$H = \frac{g^2}{2} \sum_{links} E_l^a E_l^a - \frac{1}{g^2} \sum_{plaq} \mathrm{Re}\,\mathrm{tr}\,\rho(U_\square)$$

E^2 is a Laplacian on the Lie group

Finite group:

$$H = \frac{g^2}{2} \sum_{links} \Delta_l - \frac{1}{g^2} \sum_{plaq} \mathrm{Re}\,\mathrm{tr}\,\rho(U_\square)$$

As we will see, this is a Laplacian on the *finite* group

In both cases H_B involves a choice of representation ρ

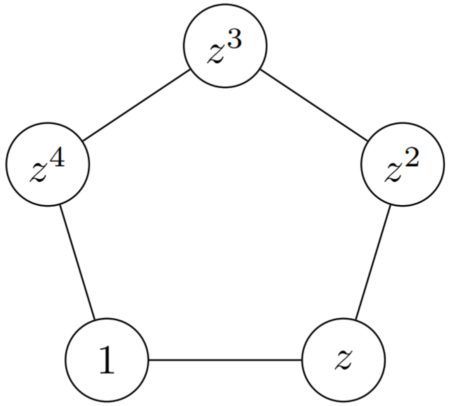
Cayley graph

Geometric structure of G is a **graph**

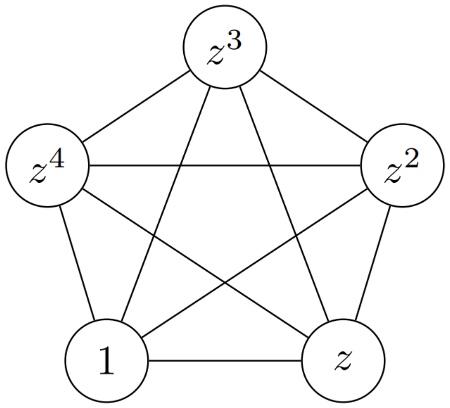
Vertices = group elements

Edges = Pick a subset of **generators** $\Gamma \subset G$. Bond $g \sim h$ if $gh^{-1} \in \Gamma$

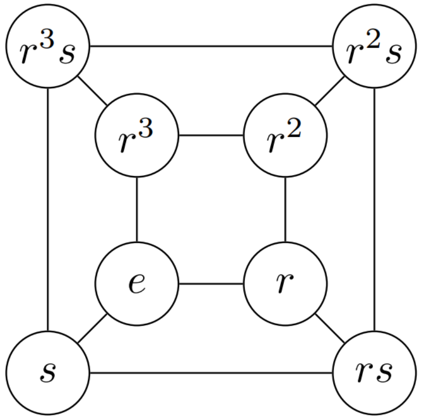
Γ non-unique but must have $\Gamma = \Gamma^{-1}$ and $g\Gamma g^{-1} = \Gamma$.



$\mathbb{Z}_5$
$\Gamma = \{z, z^{-1}\}$



$\mathbb{Z}_5$
$\Gamma = \{z, z^{-1}, z^2, z^{-2}\}$



D_4
$\Gamma = \{r, r^{-1}, s\}$

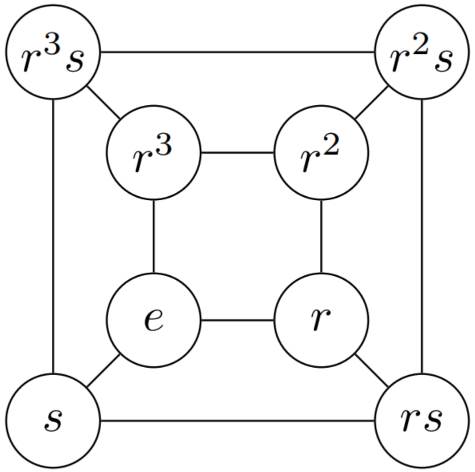
Lorentz symmetry: relates H_B and H_E. Relativistic choice of Γ : connect nearest neighbours according to distance function $d(g,h) = \dim\rho - \mathrm{tr}\,\rho(gh^{-1})$

[see also Caspar et al '16]

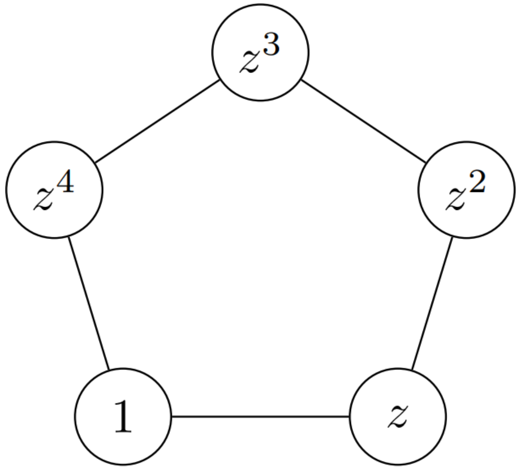
Finite-group Laplacian

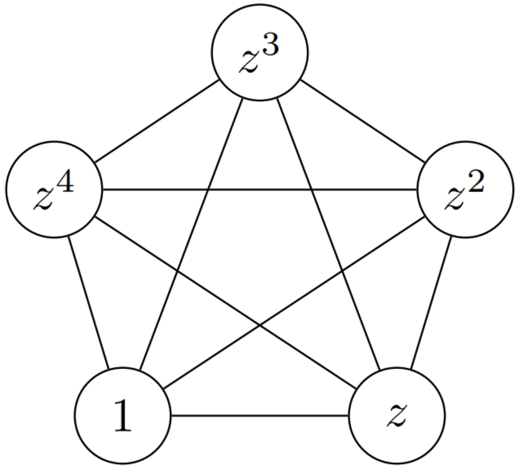
Every graph has a graph Laplacian:

$$\Delta\psi(g) = \sum_{h\sim g}(\psi(g) - \psi(h))$$



Reproduces the transfer-matrix electric Hamiltonian





- Applicable to other cases? subsets, …
- Can import known results about the graph Laplacian

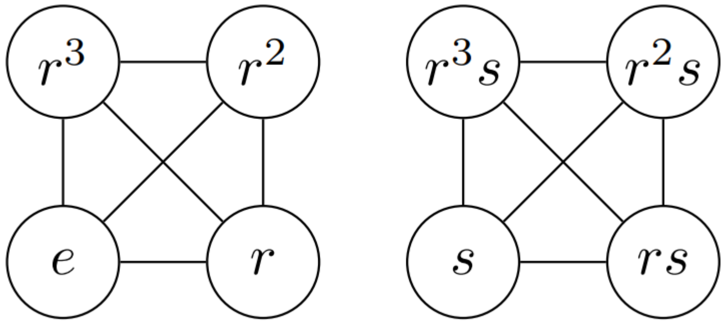
Finite-group Laplacian: electric degeneracy

Degeneracy of lowest eigenvalue of graph Laplacian = Number of connected components of the graph

- Electric degeneracy *on each link*
- Persists in gauge-invariant sector
- Hints to a different phase diagram

D_4

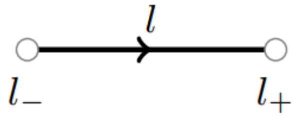
$\Gamma = \{r, r^2, r^3\}$



Two connected components
Twofold degenerate *on each link*

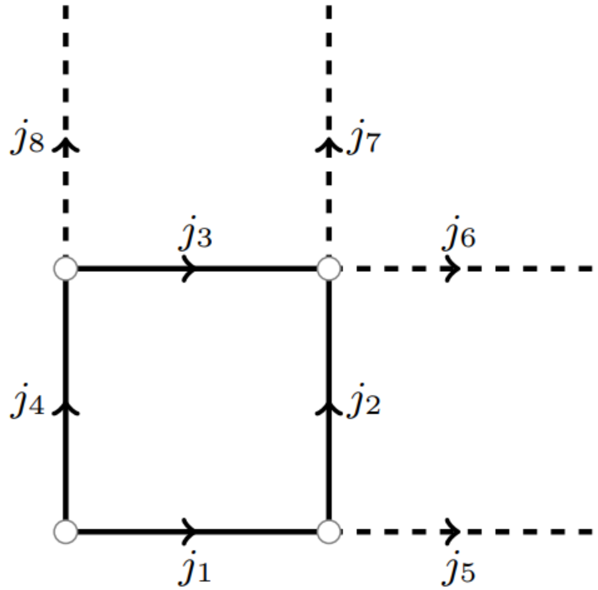
The dimension of the physical subspace

$$\mathcal{H}_{tot} \xrightarrow{\text{Gauss' Law}} \mathcal{H}_{phys}$$



How many resources can be saved in a gauge-invariant simulation scheme?

- Wilson loops **do not** necessarily span $\mathcal{H}_{phys}$ [Durhuus '80]
- Use spin-network states [Baez '94]



$$\mathcal{H}_{phys} = \bigoplus_{\{j\}} \bigotimes_{x} \mathrm{Inv}[\left(\otimes_{l_+=x} V_{j_l}^*\right) \otimes \left(\otimes_{l_-=x} V_{j_l}\right)]$$

Assign:
- an irrep to each link;
- an invariant tensor to each site.

The dimension of the physical subspace

$$\mathcal{H}_{tot} \xrightarrow{\text{Gauss' Law}} \mathcal{H}_{phys}$$

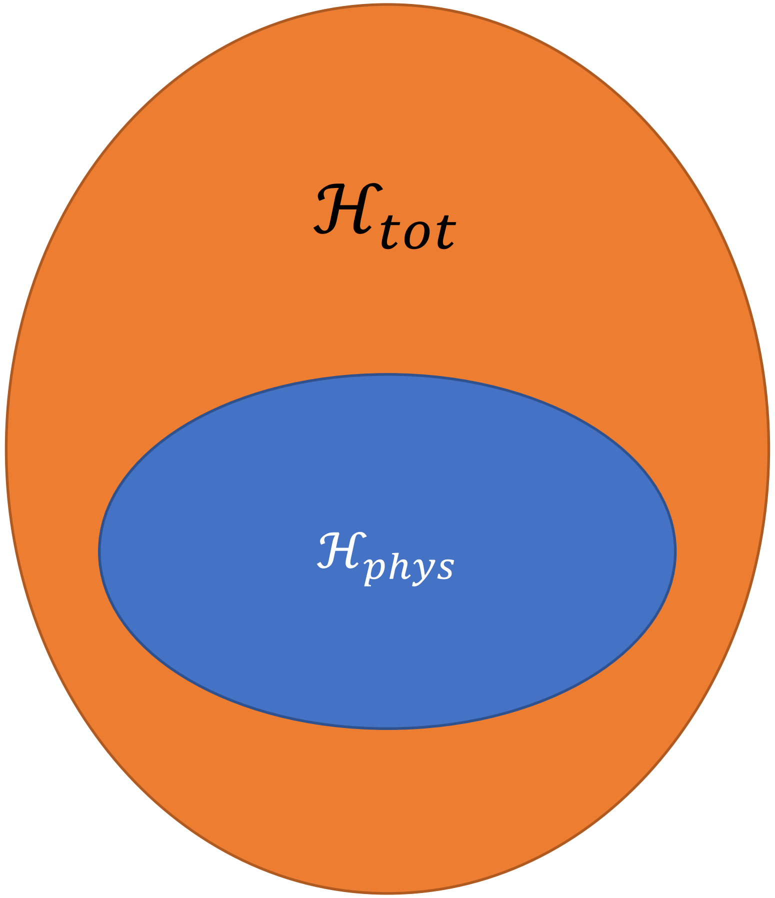
For a **pure gauge theory** with arbitrary finite group G on an arbitrary lattice with V sites and E links:

$$\dim \mathcal{H}_{tot} = |G|^E$$

$$\boxed{\dim \mathcal{H}_{phys} = \sum_C \left(\frac{|G|}{|C|}\right)^{E-V}}$$

C are the conjugacy classes of G.

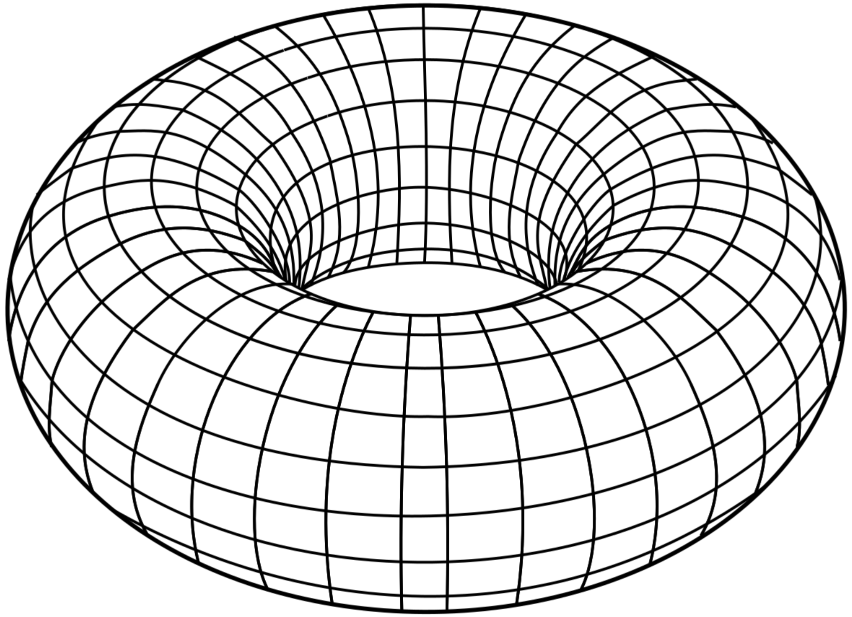
Roughly speaking: $$\frac{\dim \mathcal{H}_{phys}}{\dim \mathcal{H}_{tot}} \approx \frac{1}{|G|^V}$$



What about matter fields?

If matter Hilbert space is local, $\mathcal{H}_{matt} = \bigotimes_x \mathcal{H}_M$ then ok:

$$\dim \mathcal{H}_{phys} = \sum_C \left(\frac{|G|}{|C|}\right)^{E-V} \chi_M(C)^V$$



Example: no charged states on a torus.

$\mathbb{Z}_N$ theory. Place $q = 1$ charge on each site.

$$\dim \mathcal{H}_{phys} = N^{E-V} \sum_{k=0}^{N-1} e^{2\pi i k \frac{V}{N}}$$

which is zero unless $Q_{tot} = V \equiv 0 \pmod N$.

But general case more complicated.

Conclusions

The electric Hamiltonian is a natural Laplace operator on the finite group, with consequences for its degeneracy.

We gave an exact formula for the dimension of the physical subspace of pure gauge theories.

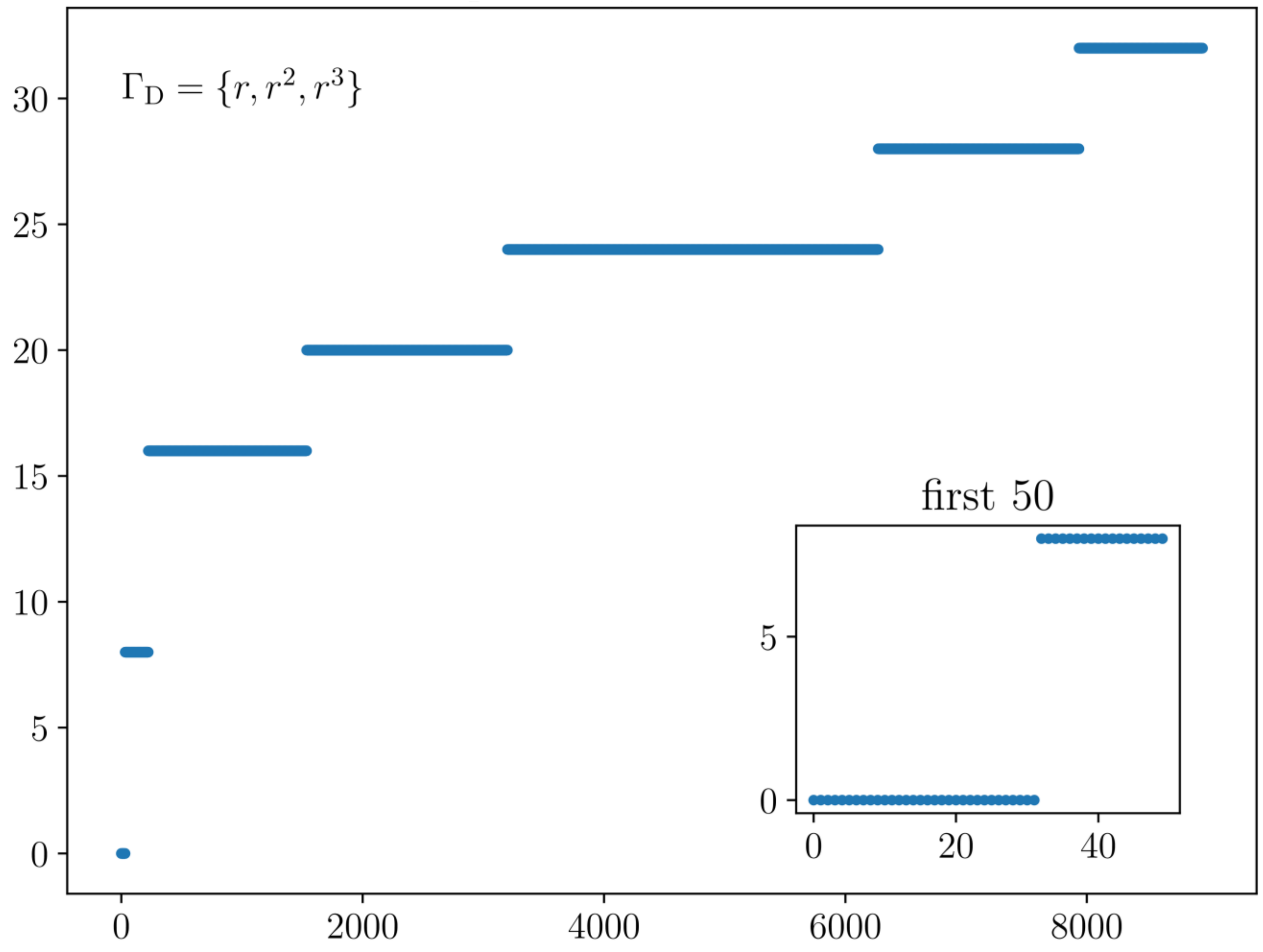
Further questions:
Applications of graph Laplacian to other cases.

Exploration of theories with electric degeneracy.

Dimension of physical Hilbert space with matter fields.

Backup slides

Degeneracy in the physical Hilbert space



Spectrum of H_E in the gauge-invariant basis.

Choose $\Gamma = \{r, r^2, r^3\}$

D_4 pure gauge theory on a 2×2 periodic lattice.