FINDING VORTICES IN THE BKT TRANSITION OF 2D FERMI GASES

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CODE AVAILABILITY

https://github.com/evanberkowitz/two-dimensional-gasses

https://two-dimensional-gasses.readthedocs.io/
THE BKT TRANSITION
A little about BKT:

Vadim L’Vovich Berezinskii  J. Michael Kosterlitz  David J. Thouless

10.1070/PU1981v024n03ABEH004788
Finite-temperature phase transition in 2D

Happens in the **XY model**, **cold fermi gases**, arrays of Josephson junctions, ...

\[ H = -J \sum_{\langle ij \rangle} s_i \cdot s_j = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \]

Divergent critical exponents (!)

No local order parameters!

**Topological**
\[ H = - J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \simeq H_0 + \frac{J}{2} \int d^2x \left\| \nabla \theta \right\|^2 \]

\[ Vortex: \quad \int d\mathbf{\ell} \cdot \nabla \theta = 2\pi n \]

\[ \nabla \theta \sim \frac{n}{r} \]

\[ Energy: \quad E \sim J\pi n^2 \log(L/r) \]

\[ Entropy: \quad S = \log \Omega \sim \log(L^2/r^2) \]

\[ Free \ Energy: \quad F = E - TS \]

\[ No \ vortices \ at \ all \quad T/J \quad Vortices \ everywhere \]
TURNING AND TURNING IN THE WIDENING GYRE

$T/J = 0.4$ on $250^2$
TURNING AND TURNING IN THE WIDENING GYRE

\[ \frac{T}{J} = 0.4 \text{ on } 250^2 \]

\[ \theta \]

\[ \langle \text{vortex} \rangle \]

\[ \int d\ell \cdot \nabla \theta \rightarrow \sum_i d\hat{x}_i \cdot (s_{i+\hat{x}} - s_i) \]
TURNING AND TURNING IN THE WIDENING GYRE

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\[ \theta \]

- No local order parameters!

- Topological

\[ \int d\vec{\ell} \cdot \nabla \theta \rightarrow \sum_i d\hat{x}_i \cdot (s_{i+\hat{x}} - s_i) \]
FERMI GASSES IN 2D
NONRELATIVISTIC 2D FERMI GASES

- $[\psi] = [L]^{-1}$
- $C_0 M$ is dimensionless
- 2D two-particle scattering amplitude
- Low-energy EFT; 'universal'
- But also: a UV fixed point!

$$H = \int d^2x \frac{(\nabla \psi)^\dagger \cdot (\nabla \psi)}{2M} + C_0(\psi^\dagger \psi)^2$$

$$T = \frac{4/M}{\cot \delta_0(k) - i}$$

$$\cot \delta_0(k) = \frac{2}{\pi} \log ka + \mathcal{O}(k^2)$$

Warning: 2 common conventions
IT'S REAL!

- Trapping!
- Lasers!
- Feshbach Resonances!
- Ions!
PHASE STRUCTURE

- No vortices
- Vortices everywhere

T/T_F

BEC | log k_F a | BCS

T_BKT/T_F
Zhang et al. (2022) PRL 129, 076403

PHASE STRUCTURE

Theory
- BCS mean-field
- Petrov et al. [36]
- Bighin et al. [36]
- Bauer et al. [40]
- Mulkerin et al. [60]

Experiment
- Ries et al. [24]
- Sobirey et al. [34]

This work
- $L=45$, $N_\phi=58$
- $L=\infty$, $N_\phi=\infty$

No vortices

Vortices everywhere

$T_{\text{BKT}}/T_F$ vs. $\log(k_Fa)$

BEC $\rightarrow$ log($k_Fa$) $\rightarrow$ BCS
**DETECTION STRATEGIES**

**Critical velocity**

![Critical velocity graph](image1)

**2-point function**

![2-point function graph](image2)

**$dn_c/dT$**

![$dn_c/dT$ graph](image3)

WHERE ARE THE VORTICES?
A PROBE OF VORTICITY

- Idea: try to find the vortices!

- What are 'the arrows'?

- What tracks the vortices?

- Example

\[ j = -\frac{i}{2M} \left( \psi^\dagger \nabla \psi - \nabla \psi^\dagger \psi \right) \]

\[ \omega = \nabla \times j = -\frac{i}{M} \nabla \psi^\dagger \times \nabla \psi \]

\[ |\chi\rangle = \int d^2 x \, \chi(x) |x\rangle \]

\[ \chi(x) = e^{i\ell \theta} f(r) \]

\[ \langle \chi \mid \omega(x) \mid \chi \rangle \propto \ell \]
LATTICE APPROACH
\[ H = \int d^2 x \frac{(\nabla \psi)^\dagger \cdot (\nabla \psi)}{2M} + C_0(\psi^\dagger \psi)^2 \rightarrow \sum \Delta x^2 \frac{(\nabla \psi)^\dagger \cdot (\nabla \psi)}{2M} + C_0(\psi^\dagger \psi)^2 \]
TOOLS OF LATTICE FIELD THEORY

\[ H = \int d^2x \frac{\nabla \psi \cdot \nabla \psi}{2M} + C_0(\psi^\dagger \psi)^2 \rightarrow \sum \Delta x^2 \frac{\nabla \psi \cdot \nabla \psi}{2M} + C_0(\psi^\dagger \psi)^2 \]

- Work in the grand-canonical ensemble
- Trotterize + control time discretization
- Eliminate fermions for auxiliary field path integral
- Markov-Chain Monte Carlo / HMC

\[ Z = \text{tr} \left[ e^{-\beta(H-\mu N)} \right] = Z(\Delta t) + \mathcal{O}(\Delta t^2) \]

\[ Z(\Delta t) = \int \mathcal{D} \phi \ e^{-S(\phi)} \]

A real equal sign!
Finite volume 2-body energies \iff\ \cot \delta_0(k) = \frac{2}{\pi} \log ka + \mathcal{O}(k^2)

\begin{align*}
\cot \delta &- \frac{2}{\pi} \log \sqrt{x} = \frac{1}{\pi^2} \\
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\end{align*}

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\end{align*}

Energies converge with $\Delta x^2$
EXAMPLE CONTINUUM BEHAVIOR
SANITY CHECK: TAN'S CONTACT

\[ \hat{C} = 2\pi M \frac{dH}{d \log a} \]
A PROBE OF VORTICITY

- What tracks the vortices?
- $\omega(x)$ is local

Need $\langle \omega(x) \omega(y) \rangle$ correlations

Caution: OPE analysis shows it diverges as $\partial^2 \delta(x-y)$

$$B_n = \int d^2 x \ |x|^n \langle \omega(x) \omega(0) \rangle$$

intensive

$n \geq 2$ well-behaved in the continuum

$$\omega = \nabla \times j = -\frac{i}{M} \nabla \psi^\dagger \times \nabla \psi$$

$$\sum_x \Delta x^2 \omega(x) = 0 \quad \text{with periodic boundary conditions}$$

$$\omega(x) \omega(0)$$
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$\omega = \nabla \times j = -\frac{i}{M} \nabla \psi^\dagger \times \nabla \psi$

$\sum_x \Delta x^2 \omega(x) = 0$ with periodic boundary conditions
LIMITS OF VORTICITY PROBES

Free theory, $\mu=0$, $\beta/ML^2 = 0.061$

Spatial continuum

Free theory, $\mu=0$

Fixed lattice spacing
Infinite volume

$nx_0 = 7$
$\beta/ML^2 = 0.061$
FIRST RESULTS!
\[ x^2 \langle \omega(x) \omega(0) \rangle \]

\[ \log k_F a = 3.3 \quad T/T_F = 1.2 \quad N = 4. \]

Each took about 1 hour on 1 NVIDIA A100 GPU

Same parameters as the contact check

N = 9, scaled L by 1.5
PHASE STRUCTURE

$T/T_F$

BEC $\log k_F a$ BCS

No vortices $T_{\text{BKT}}/T_F$ Vortices everywhere

our test calculation $T/T_F = 1.2$
CURRENT CAMPAIGN

Berkowitz + Warrington 2309.????

No vortices

T/T_F

BEC

log k_F a = 0

log k_F a

MB_2/k_F^4

BCS

T/T_F with fixed log k_F a

BEC

Vortices everywhere

T_{BKT}/T_F

Vortices everywhere

T/T_F
BACKUP SLIDES
CODE AVAILABILITY

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**BUT KNOWN THEOREM?**

- Mermin-Wagner(-Berezinskii-Coleman): continuous symmetries don't spontaneously break in 2D

- Therefore, correlation functions don't go to a finite constant at long distance.

- BKT: OK, but if they decay with a power law?

\[ H = -J \sum_{\langle ij \rangle} s_i \cdot s_j \quad \text{O(2)} \]

\[ = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \]
$H = - J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \simeq H_0 + \frac{J}{2} \int d^2x \left| \nabla \theta \right|^2$
EXPECTED VALUES AND SUSCEPTIBILITIES

$|\omega|^2$ vs $T/BKT/J$ and susceptibility vs $T/BKT/J$
A PROBE OF VORTICITY

- What tracks the vortices?
- \( \omega(x) \) is local

- Need \( \langle \omega(x) \omega(y) \rangle \) correlations
  Caution: OPE analysis shows it diverges as \( \delta^2 \delta(x-y) \)

- \( B_n = \int d^2 x \ |x|^n \langle \omega(x) \omega(0) \rangle \)
  intensive
  \( n \geq 2 \) well-behaved in the continuum

\[
\begin{align*}
\omega &= \nabla \times j = - \frac{i}{M} \nabla \psi^\dagger \times \nabla \psi \\
\sum_x \Delta x^2 \omega(x) &= 0 \quad \text{with periodic boundary conditions}
\end{align*}
\]
A PROBE OF VORTICITY

- What tracks the vortices?
- $\omega(x)$ is local
- Need $\langle \omega(x) \, \omega(y) \rangle$ correlations
  Caution: OPE analysis shows it diverges as $\partial^2 \delta(x-y)$
- $B_n = \int d^2x \, |x|^n \langle \omega(x) \, \omega(0) \rangle$
  intensive
  $n \geq 2$ well-behaved in the continuum

$$\omega = \nabla \times j = - \frac{i}{M} \nabla \psi^\dagger \times \nabla \psi$$

$$\sum_x \Delta x^2 \omega(x) = 0 \quad \text{with periodic boundary conditions}$$

$$\Omega(k) = \int d^2x \, e^{-ik \cdot x} \langle \omega(x) \omega(0) \rangle$$
ANTICIPATED BEHAVIOR

No vortices $T_{BKT}/T_F$ with $fi$ fixed

$\log k_F a$

$T/T_F$ with fixed $\log k_F a$

$MB_2/k_F^4$