Renormalons in Large-Momentum Effective Theory

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Parton physics plays an important role in mapping out the 3D structure of hadrons and interpreting the experimental data at hadron colliders.

Example: Drell-Yan Process

\[
\frac{d\sigma}{dQ^2} = \sum_{i,j} \int_0^1 d\xi_a d\xi_b f_{i/P_a}(\xi_a) f_{j/P_b}(\xi_b) \frac{d\hat{\sigma}_{ij}(\xi_a, \xi_b)}{dQ^2} \times \left[ 1 + \mathcal{O}\left( \frac{\Lambda_{QCD}}{Q} \right) \right] \quad Q = \sqrt{q^2}
\]

Factorization

\[
\frac{d\sigma}{dQ^2 d^2 q_T} = \sum_{i,j} H_{ij}(Q) \int_0^1 d\xi_a d\xi_b \int d^2 b_T e^{ib_T \cdot q_T} \times f_{i/P}(\xi_a, b_T) f_{j/P}(\xi_b, b_T) \times \left[ 1 + \mathcal{O}\left( \frac{\Lambda_{QCD}}{Q}, \frac{q_T}{Q} \right) \right]
\]

\[q_T \ll Q\]
Tremendous progress has been achieved in calculating the partonic structure of hadrons from Euclidean correlations on the lattice.

Focused on single parton distributions.
Introduction

Tremendous progress has been achieved in calculating the partonic structure of hadrons from Euclidean correlations on the lattice

Focused on **single parton distributions**

Collinear PDFs, distribution amplitudes, GPDs, TMDPDFs/wave functions

Many talks, see Xiang Gao, 9 am Monday

**Main message:** Now reaches the stage of precision control

- Higher-order perturbative correction, RG resummation, threshold resummation, higher-twist contribution, **renormalon ambiguity**...
Tremendous progress has been achieved in calculating the partonic structure of hadrons from Euclidean correlations on the lattice

New territory: **double/multiple parton distributions**

\[ [JHZ, 2304.12481, Jaarsma, Rahn, Waalewijn, 2305.09716] \]

Two partons from a hadron can have transverse separations

\[
\sigma_{DPS} \sim \sum_{ijkl} \int d\bar{x}_1 d\bar{x}_2 \int d\bar{x}_1 d\bar{x}_2 \int d^2 y f_{ij}(x_1, x_2, y) f_{kl}(\bar{x}_1, \bar{x}_2, y) \hat{\sigma}_{ik} \hat{\sigma}_{jl}.
\]
Introduction

Tremendous progress has been achieved in calculating the partonic structure of hadrons from Euclidean correlations on the lattice

New territory: **double/multiple parton distributions**

\[ [\text{JHZ, 2304.12481, Jaarsma, Rahn, Waalewijn, 2305.09716}] \]

\[
\frac{sd\sigma}{\prod_{i=1}^{2} dx_i d\bar{x}_i d^2 q_i} = \frac{1}{\Lambda^2 Q^2} \quad \text{and} \quad \frac{sd\sigma}{\prod_{i=1}^{2} dx_i d\bar{x}_i} = \frac{\Lambda^2}{Q^2}
\]
Introduction

- Tremendous progress has been achieved in calculating the partonic structure of hadrons from Euclidean correlations on the lattice

- New territory: **double/multiple parton distributions**
  
  [JHZ, 2304.12481, Jaarsma, Rahn, Waalewijn, 2305.09716]

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\[ \sigma_{DPS} \sim \sum_{ijkl} \int dx_1 dx_2 \int d\bar{x}_1 d\bar{x}_2 \int d^2 y f_{ij}(x_1, x_2, y) f_{kl}(\bar{x}_1, \bar{x}_2, y) \hat{\sigma}_{ik} \hat{\sigma}_{jl}. \]

- Combines features of collinear PDFs and TMDPDFs
- Rapidity divergences can appear already in collinear distributions
Tremendous progress has been achieved in calculating the partonic structure of hadrons from Euclidean correlations on the lattice.

New territory: double/multiple parton distributions

The same development in single parton distributions (PDFs, TMDs, GPDs…) can be extended to DPDs, and generalized to multiparton distributions.

Two partons from a hadron can have transverse separations

$$\sigma_{DPS} \sim \sum_{ijkl} \int dx_1 dx_2 \int d\bar{x}_1 d\bar{x}_2 \int d^2 y f_{ij}(x_1, x_2, y) f_{kl}(\bar{x}_1, \bar{x}_2, y) \hat{\sigma}_{ik} \hat{\sigma}_{jl}.$$  

The same development in single parton distributions (PDFs, TMDs, GPDs…) can be extended to DPDs, and generalized to multiparton distributions.
Renormalons

Renormalons are related to the fact that QCD series \textbf{Beneke, Phys. Rep., 99'}
\[
F(\alpha_s) = \sum_n f_n \alpha_s^n
\]
is not convergent for any \(\alpha_s \neq 0\), \(f_n\) diverges \(\sim a_f^n n!\!

- It can still be a useful approximation to \(F(\alpha_s)\) if
\[
\lim_{\alpha_s \to 0} \frac{1}{N_0} f_n \alpha_s^n \to 0
\]

- For finite \(\alpha_s\), the partial sum usually gives an increasingly better approximation to \(F(\alpha_s)\) up to some order \(N_0 \sim 1/(|a_f| \alpha_s)\)

- Beyond \(N_0\), the approximation does not improve

Best approximation is reached when truncated at the minimal term which characterizes the truncation error

\[
f_{N_0} \alpha_s^{N_0} \sim e^{-1/(|a_f| \alpha_s)}
\]
Renormalons are related to the fact that QCD series [Beneke, Phys. Rep., 99']

\[ F(\alpha_s) = \sum_{n} f_n \alpha_s^n \]

is not convergent for any \( \alpha_s \neq 0 \), \( f_n \) diverges \( \sim a_f^n n! \).

It can still be a useful approximation to \( F(\alpha_s) \) if

\[ \lim_{\alpha_s \to 0} \alpha_s^{-N} | F(\alpha_s) - \sum_{n=0}^{N} f_n \alpha_s^n | \to 0 \]

In terms of Borel transform

\[ B[F](t) = \sum_{n} f_n \frac{t^n}{n!} \]

the same series expansion as \( F \) can be obtained from the Borel integral

\[ \int_{0}^{\infty} dt \, e^{-t/\alpha_s} B[F](t) \]

if \( B[F](t) \) has no singularities for real positive \( t \)

Singularities along integration path correspond to IR renormalons
Numerical evidence of renormalons


\[
\delta m = \frac{1}{a} \sum_{n=0}^{\infty} c_{n}^{(3,\rho)} \alpha^{n+1}(1/a)
\]

- For large \( n \), \( c_n \) is universal and equal to the expansion coefficient of heavy quark pole mass \( r_n/\nu \) up to \( \mathcal{O}[\exp(-1/n)] \) terms

\[
m_{\text{OS}} = m_{\overline{\text{MS}}}(\nu) + \sum_{n=0}^{\infty} r_n \alpha_s^{n+1}(\nu),
\]

- Large-\( n \) behavior

\[
c_{n}^{(3,\rho)} \sim N_m \left( \frac{\beta_0}{2\pi} \right)^n \frac{\Gamma(n + 1 + b)}{\Gamma(1 + b)} \times \left( 1 + \frac{b}{n + b} s_1 + \frac{b(b - 1)}{(n + b)(n + b - 1)} s_2 + \cdots \right).
\]

\[
\frac{c_n^{(3,\rho)}}{c_{n-1}^{(3,\rho)}} \sim \frac{\beta_0}{2\pi} \left( 1 + \frac{b}{n} - \frac{b s_1}{n^2} + \frac{1}{n^3} [b^2 s_1^2 + b(b - 1)(s_1 - 2s_2)] + \mathcal{O} \left( \frac{1}{n^4} \right) \right).
\]

Numerical evidence of renormalons
Self-energy of static source \( \text{Bauer et al, PRL 12', Bali et al, PRD 13', Pineda, 21'} \)

\[
\delta m = \frac{1}{\alpha} \sum_{n=0}^{\infty} c_{n}^{(3,\rho)} \alpha^{n+1}(1/\alpha)
\]

For large \( n \), \( c_n \) is universal and equal to the expansion coefficient of heavy quark pole mass \( r_n/\nu \) up to \( \mathcal{O}[\exp(-1/n)] \) terms

\[
m_{OS} = m_{\overline{MS}}(\nu) + \sum_{n=0}^{\infty} r_n \alpha_s^{n+1}(\nu),
\]

Large-\( n \) behavior

Numerical evidence of renormalons

Numerical stochastic perturbation theory calculated up to \( \alpha_s^{20} \)
Self-energy of static source *Bauer et al, PRL 12’, Bali et al, PRD 13’, Pineda, 21’*

$$\delta m = \frac{1}{a} \sum_{n=0}^{\infty} c_n^{(3,\rho)} \alpha^{n+1}(1/a)$$

For large $n$, $c_n$ is universal and equal to the expansion coefficient of heavy quark pole mass $r_n/\nu$ up to $\mathcal{O}[\exp(-1/n)]$ terms

$$m_{\text{OS}} = m_{\text{MS}}(\nu) + \sum_{n=0}^{\infty} r_n \alpha_s^{n+1}(\nu),$$

**Large-$n$ behavior**

**Numerical evidence of renormalons**

Bauer et al, PRL 12’, Bali et al, PRD 13’, Pineda, 21’

Numerical stochastic perturbation theory calculated up to $\alpha_s^{20}$

Renormalon-related talks:

Yushan Su, 4:20 pm Tuesday
Andreas Kronfeld, 1:50 pm Thursday
Jack Holligan, 2:10 pm Thursday
Renormalons in LaMET

- Factorization of quasi-PDF

\[ Q(x, p) = \int_{-1}^{1} \frac{dy}{|y|} C_Q\left(\frac{x}{y}, xp, \mu_F\right) q(y, \mu_F) + \frac{1}{p^2} Q_4(x, p) + \ldots \]

\[ C_Q(x, p, \mu_F) = \delta(1-x) + c_1 \alpha_s + c_2 \alpha_s^2 + \ldots - \frac{\mu_F^2}{p^2} D_Q(x) + \ldots, \]

- Logarithmic scale dependence in \( c_i(x, \ln p^2/\mu_F^2) \) is canceled by that from the leading-twist PDF \( q(x, \mu_F) \)

- Power dependence is canceled between leading- and higher-twist contributions [Braun, Vladimirov, JHZ, PRD 19']

\[ Q_4(x, p, \mu_F) = \mu_F^2 \int_{-1}^{1} \frac{dy}{|y|} D_Q\left(\frac{x}{y}\right) q(y, \mu_F) + \tilde{Q}_4(x, p, \mu_F) \]

- In DR, power-like terms do not appear, but the coefficients \( c_i \) grow factorially with \( i \)

- The sum of perturbative series is only defined to a power accuracy and this ambiguity is compensated by adding a higher-twist contribution
Renormalons in LaMET

- Estimate of **twist-4** contribution \(Braun, Vladimirov, JHZ, PRD 19'\)

\[
Q_4(x, p, \mu_F) = \mu_F^2 \int_{y}^{1} \frac{dy}{|y|} D_Q \left( \frac{x}{y} \right) q(y, \mu_F) + \tilde{Q}_4(x, p, \mu_F)
\]

\[
\downarrow
\]

\[
Q_4(x, p, \mu_F) = \kappa \Lambda_{QCD}^2 \int_{y}^{1} \frac{dy}{|y|} D_Q \left( \frac{x}{y} \right) q(y, \mu_F)
\]

- Start from the perturbative series of the coefficient function in coordinate space

\[
H = \delta(1 - \alpha) + \sum_{k=0}^{\infty} h_k a_s^{k+1}, \quad a_s = \frac{\alpha_s(\mu)}{4\pi}
\]

- Borel transform

\[
B[H](w) = \sum_{k=0}^{\infty} \frac{h_k}{k!} \left( \frac{w}{\beta_0} \right)^k, \quad H = \delta(1 - \alpha) + \frac{1}{\beta_0} \int_{0}^{\infty} dwe^{-w/(\beta_0 a_s)} B[H](w).
\]

- Borel integral has singularities along the integration path

- Ambiguity can be estimated by taking the residue at a given singularity
Estimate of twist-4 contribution \( \text{Braun, Vladimirov, JHZ, PRD 19'} \)

\[
Q_4(x, p, \mu_F) = \mu_F^2 \int_{-1}^{1} \frac{dy}{|y|} D_Q \left( \frac{x}{y} \right) q(y, \mu_F) + \tilde{Q}_4(x, p, \mu_F)
\]

\[\uparrow\]

\[
Q_4(x, p, \mu_F) = \kappa \Lambda_{\text{QCD}}^2 \int_{-1}^{1} \frac{dy}{|y|} D_Q \left( \frac{x}{y} \right) q(y, \mu_F)
\]

Singularities of the Borel transform (large \( \beta_0 \) approximation)

\[
\begin{align*}
\text{(a)} & \quad = \quad \text{UV} \\
\text{(b)} & \quad + \quad \text{IR} \\
\text{(c)} & \quad + \quad \text{IR} \\
\text{(d)} & \quad + \quad \cdots
\end{align*}
\]

FIG. 2. Singularity structure of the Borel transform.
Renormalons in LaMET

- Estimate of **twist-4** contribution by Braun, Vladimirov, JHZ, PRD 19′

\[
Q_4(x, p, \mu_F) = \mu_F^2 \int_{-1}^{1} \frac{dy}{|y|} D_Q \left( \frac{x}{y} \right) q(y, \mu_F) + \tilde{Q}_4(x, p, \mu_F)
\]

\[\Downarrow\]

\[
Q_4(x, p, \mu_F) = \kappa \Lambda_{QCD}^2 \int_{-1}^{1} \frac{dy}{|y|} D_Q \left( \frac{x}{y} \right) q(y, \mu_F)
\]

- Singularities of the Borel transform (large \(\beta_0\) approximation)

From Wilson line self-energy, removed by renormalization

Leading power correction
Renormalons in LaMET

- Estimate of **twist-4** contribution: \( \text{Braun, Vladimirov, JHZ, PRD 19}' \)
  \[ Q_4(x, p, \mu_F) = \mu_F^2 \int_{-1}^{1} \frac{dy}{|y|} D_Q\left(\frac{x}{y}\right) q(y, \mu_F) + \tilde{Q}_4(x, p, \mu_F) \]

\[ \Downarrow \]

\[ Q_4(x, p, \mu_F) = \kappa \Lambda_{QCD}^2 \int_{-1}^{1} \frac{dy}{|y|} D_Q\left(\frac{x}{y}\right) q(y, \mu_F) \]

- **Quasi-PDF**

\[ Q(x, p) = q(x) \left\{ 1 + \mathcal{O}\left(\frac{\Lambda^2}{p^2} \cdot \frac{1}{x^2(1-x)}\right) \right\} \]

- **Pseudo-PDF**

\[ P(x, z) = q(x)\{1 + \mathcal{O}(z^2\Lambda^2(1-x))\} \]

- Zero-momentum matrix element helps to suppress the power correction at \( x \rightarrow 1 \)

- Confirmed in **Liu, Chen, PRD 21’**
Estimate of **twist-4** contribution: \( \text{Braun, Vladimirov, JHZ, PRD 19'} \)

\[
\mathcal{Q}_4(x, p, \mu_F) = \mu_F^2 \int_{-1}^{1} \frac{d\gamma}{|y|} D_Q \left( \frac{x}{y} \right) q(y, \mu_F) + \tilde{\mathcal{Q}}_4(x, p, \mu_F)
\]

\[
\tilde{\mathcal{Q}}_4(x, p, \mu_F) = \kappa \Lambda_{\text{QCD}}^2 \int_{-1}^{1} \frac{d\gamma}{|y|} D_Q \left( \frac{x}{y} \right) q(y, \mu_F)
\]

- **Remove the leading renormalon ambiguity**: Liu, Chen, PRD 21’

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**R-scheme**

Hoang et al, PRD 10’

\[
\tilde{Q}_R(x, P_z, P_z', \Lambda') = \frac{P_z^2 \tilde{Q}(x, P_z, \Lambda') - P_z'^2 \tilde{Q}(x, P_z', \Lambda')}{P_z^2 - P_z'^2}
\]

\[
= \int_{-1}^{1} \frac{d\gamma}{|y|} \left[ \frac{P_z^2 Z(\frac{x}{y}, yP_z, \Lambda', \mu) - P_z'^2 Z(\frac{x}{y}, yP_z', \Lambda', \mu)}{P_z^2 - P_z'^2} \right] Q(y, \mu) + \mathcal{O} \left( \frac{\alpha_s \ln (P_z / P_z')}{P_z^2 - P_z'^2}, \frac{1}{P_z^2 P_z'^2} \right)
\]
Renormalons in LaMET

- Estimate of **twist-4** contribution Braun, Vladimirov, JHZ, PRD 19’

\[ Q_4(x, p, \mu_F) = \mu_F^2 \int_{-1}^{1} \frac{dy}{|y|} D_Q \left( \frac{x}{y} \right) q(y, \mu_F) + \tilde{Q}_4(x, p, \mu_F) \]

\[ \Rightarrow Q_4(x, p, \mu_F) = \kappa \Lambda_{\text{QCD}}^2 \int_{-1}^{1} \frac{dy}{|y|} D_Q \left( \frac{x}{y} \right) q(y, \mu_F) \]

- Remove the leading renormalon ambiguity Liu, Chen, PRD 21’

- **R-scheme**
  Hoang et al, PRD 10’

- Good convergence in RI/MOM scheme matrix element observed
There can be \textbf{twist-3} contribution \cite{Zhang, Holligan, Ji, Su, PLB 23'}

\[ Q(x, p) = \int_{-1}^{1} \frac{dy}{|y|} C_Q \left( \frac{x}{y}, xp, \mu_F \right) q(y, \mu_F) + \frac{1}{p^2} Q_4(x, p) + \ldots \]

See Yushan Su, 4:20 pm Tuesday

Renormalization of linear divergence from Wilson line self-energy

\[ h^R(z, P_z) = h^B(z, P_z) e^{(\delta m - m_0)z} \]

introduces an intrinsic ambiguity of \( \mathcal{O}(z \Lambda_{QCD}) \)

Modified OPE

\[
\begin{align*}
  h^R(z, P_z, \mu, \tau) & = \left(1 - m_0(\tau)z\right) \sum_{k=0}^{\infty} C_k \left( \alpha_s(\mu), \mu^2 z^2 \right) \lambda^k a_{k+1}(\mu) + \mathcal{O}(z^2) \\
  & = \sum_{k=0}^{\infty} \left[ C_k \left( \alpha_s(\mu), \mu^2 z^2 \right) - zm_0(\tau) \right] \lambda^k a_{k+1}(\mu) + \mathcal{O}(z\alpha_s, z^2),
\end{align*}
\]
Renormalons in LaMET

- There can be **twist-3** contribution (Zhang, Holligan, Ji, Su, PLB 23')
  \[ Q(x, p) = \int_{-1}^{1} \frac{dy}{|y|} C_Q \left( \frac{x}{y}, xp, \mu_F \right) q(y, \mu_F) + \frac{1}{p^2} Q_4(x, p) + \ldots \]

  See Yushan Su, 4:20 pm Tuesday

- Renormalization of linear divergence from Wilson line self-energy
  \[
  h^R(z, P_z) = h^B(z, P_z) e^{(\delta m_0 - m_0)z} \\
  \mathcal{O}(z \Lambda_{QCD})
  \]

  introduces an intrinsic ambiguity of \( \mathcal{O}(z \Lambda_{QCD}) \)

- Modified OPE
  \[
  h^R(z, P_z, \mu, \tau) \\
  = \left( 1 - m_0(\tau)z \right) \sum_{k=0}^{\infty} C_k \left( \alpha_s(\mu), \mu^2 z^2 \right) \lambda^k a_{k+1}(\mu) + \mathcal{O}(z^2) \\
  = \sum_{k=0}^{\infty} \left[ C_k \left( \alpha_s(\mu), \mu^2 z^2 \right) - zm_0(\tau) \right] \lambda^k a_{k+1}(\mu) + \mathcal{O}(z \alpha_s, z^2),
  \]

  \( m_0(\tau) \) can be determined by fitting to zero-momentum matrix element
There can be **twist-3** contribution 

\[ Q(x, p) = \int_{-1}^{1} \frac{dy}{|y|} C_Q \left( \frac{x}{y}, xp, \mu_F \right) q(y, \mu_F) + \frac{1}{p^2} Q_4(x, p) + \ldots \]

See Yushan Su, 4:20 pm Tuesday

**Renormalization of linear divergence from Wilson line self-energy**

\[ h^R(z, P_z) = h^B(z, P_z)e^{(\delta m - m_0)z} \]

introduces an intrinsic ambiguity of \( \mathcal{O}(z \Lambda_{QCD}) \)

**Modified OPE**

\[
\begin{align*}
    h^R(z, P_z, \mu, \tau) &= \left(1 - m_0(\tau)z\right) \sum_{k=0}^{\infty} C_k \left( \alpha_s(\mu), \mu^2 z^2 \right) \lambda^k a_{k+1}(\mu) + \mathcal{O}(z^2) \\
    &= \sum_{k=0}^{\infty} \left[ \alpha_s(\mu), \mu^2 z^2 \right] \lambda^k a_{k+1}(\mu) + \mathcal{O}(z^2),
\end{align*}
\]

\( m_0(\tau) \) can be determined by fitting to zero-momentum matrix element
There can be **twist-3** contribution

\[ Q(x, p) = \int_{-1}^{1} \frac{dy}{|y|} C_Q \left( \frac{x}{y}, xp, \mu_F \right) q(y, \mu_F) + \frac{1}{p^2} Q_4(x, p) + \ldots \]

Renormalization of linear divergence from Wilson line self-energy introduces an intrinsic ambiguity of \( \mathcal{O}(z \Lambda_{QCD}) \)

Leading renormalon contribution

\[ h^R(z, P_z) = h^B(z, P_z) e^{(\delta m - m_0)z} \]

Matching revised accordingly

\[ C_{k}(\alpha_s(z^{-1}), 1)_{\text{PV}} = N_m \frac{4\pi}{\beta_0} \int_{0, \text{PV}}^{\infty} du \times e^{-\frac{4\pi u}{\alpha_s(z^{-1})\beta_0}} \frac{1}{(1 - 2u)^{1+b}} (1 + c_1(1 - 2u) + \ldots) \]

\[ C^{\text{LRR}}(\alpha_s(z^{-1}), 1) = C_k(\alpha_s(z^{-1}), 1) + \left[ C_k(\alpha_s(z^{-1}), 1)_{\text{PV}} - \sum_i r_i \alpha_s^{i+1}(z^{-1}) \right] \]

See Yushan Su, 4:20 pm Tuesday
There can be **twist-3** contribution \[ Q(x, p) = \int_{-1}^{1} \frac{dy}{|y|} C_Q \left( \frac{x}{y}, xp, \mu_F \right) q(y, \mu_F) + \frac{1}{p^2} Q_4(x, p) + \ldots \]

\( \mathcal{O} \left( \frac{\Lambda_{\text{QCD}}}{x \cdot p} \right) \)

See Yushan Su, 4:20 pm Tuesday
Summary and outlook

- A lot of progress has been achieved in calculating the partonic structure of hadrons from lattice
  - Collinear PDFs, DAs, GPDs
  - TMDPDFs/wave functions

- For single parton distributions, we have reached the stage of precision control
  - Higher-order perturbative correction
  - RG/threshold resummation
  - Higher-twist contribution/renormalons
    Mainly to collinear PDFs, extended to GPDs, TMDs?

- Double/multiple parton distributions can be studied following similar spirit and remain to be explored