Renormalons in Large-Momentum Effective Theory

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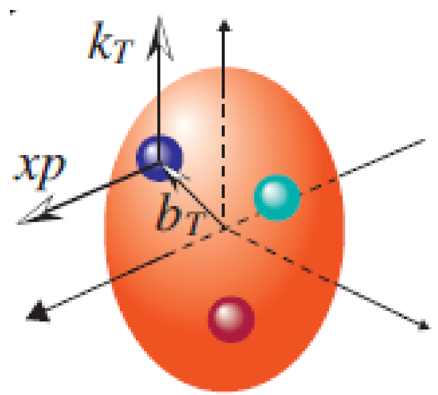
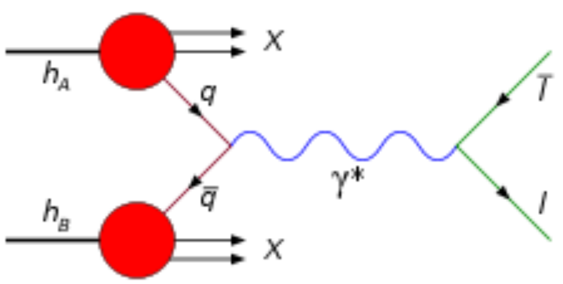
Introduction

- Parton physics plays an important role in mapping out the 3D structure of hadrons and interpreting the experimental data at hadron colliders





Example: Drell-Yan Process

- Factorization

$$\frac{d\sigma}{dQ^2} = \sum_{i,j} \int_0^1 d\xi_a d\xi_b \, f_{i/P_a}(\xi_a) f_{j/P_b}(\xi_b) \frac{d\hat{\sigma}_{ij}(\xi_a, \xi_b)}{dQ^2} \times \left[1 + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{Q}\right)\right] \qquad Q = \sqrt{q^2}$$

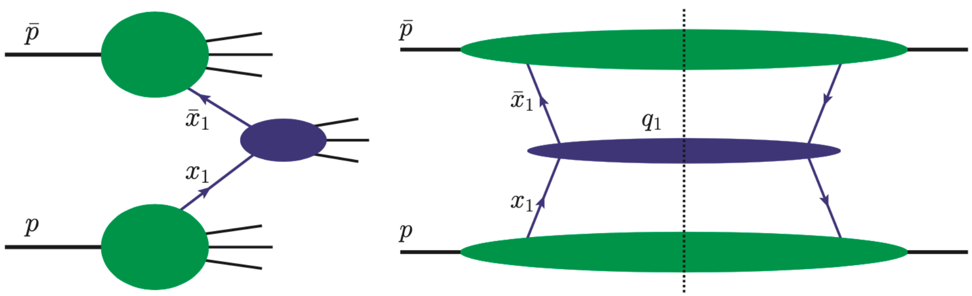
$q_T \ll Q$:

$$\frac{d\sigma}{dQ^2 d^2\mathbf{q_T}} = \sum_{i,j} H_{ij}(Q) \int_0^1 d\xi_a d\xi_b \int d^2\mathbf{b_T} \, e^{i\mathbf{b_T}\cdot\mathbf{q_T}} \times f_{i/P}(\xi_a, \mathbf{b_T}) f_{j/P}(\xi_b, \mathbf{b_T}) \times \left[1 + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{Q}, \frac{q_T}{Q}\right)\right]$$

Introduction

- Tremendous progress has been achieved in calculating the partonic structure of hadrons from Euclidean correlations on the lattice

- Focused on **single parton distributions**



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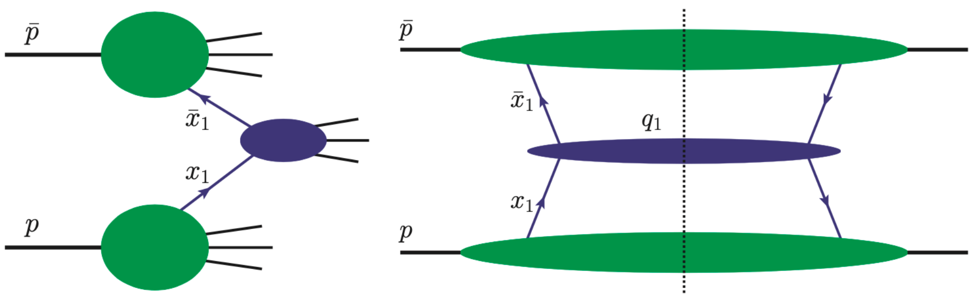

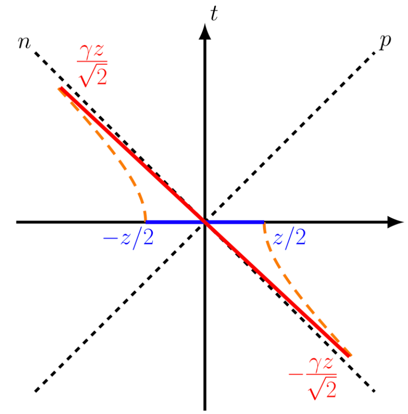

- Collinear PDFs, distribution amplitudes, GPDs, TMDPDFs/wave functions

  Many talks, see Xiang Gao, 9 am Monday

Ji, PRL 13' & SCPMA 14',
Ji, Liu, Liu, JHZ, Zhao, RMP 21'

- **Main message**: Now reaches the stage of precision control
  - Higher-order perturbative correction, RG resummation, threshold resummation, higher-twist contribution, **renormalon ambiguity**…

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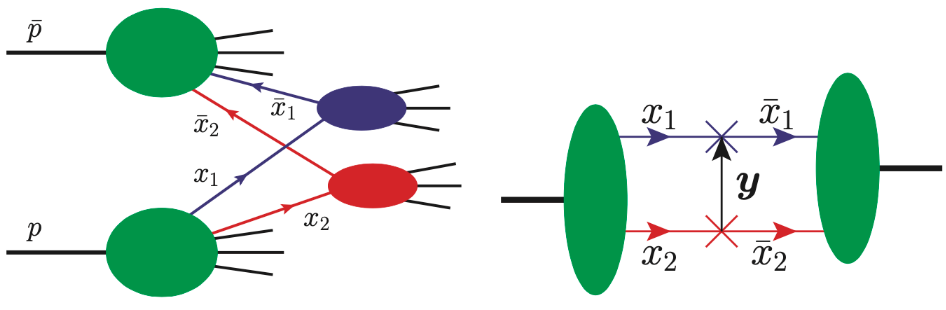
- Tremendous progress has been achieved in calculating the partonic structure of hadrons from Euclidean correlations on the lattice

- New territory: **double/multiple parton distributions**
  [JHZ, 2304.12481, Jaarsma, Rahn, Waalewijn, 2305.09716]



(a) (b)

Two partons from a hadron can have transverse separations

$$\sigma_{DPS} \sim \sum_{ijkl} \int dx_1 dx_2 \int d\bar{x}_1 d\bar{x}_2 \int d^2\boldsymbol{y} f_{ij}(x_1, x_2, \boldsymbol{y}) f_{kl}(\bar{x}_1, \bar{x}_2, \boldsymbol{y}) \hat{\sigma}_{ik} \hat{\sigma}_{jl}.$$

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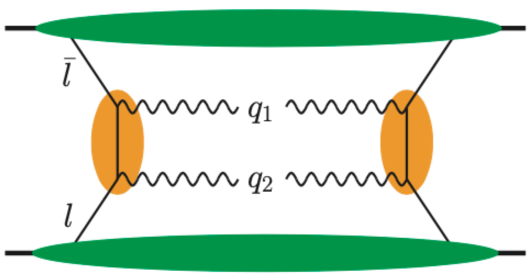
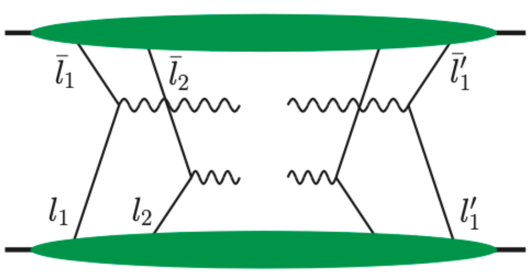
Power counting

	Diagram	$\frac{s\,d\sigma}{\prod_{i=1}^{2} dx_i\, d\bar{x}_i\, d^2\boldsymbol{q}_i}$	$\frac{s\,d\sigma}{\prod_{i=1}^{2} dx_i\, d\bar{x}_i}$
a		$\frac{1}{\Lambda^2 Q^2}$	1
b		$\frac{1}{\Lambda^2 Q^2}$	$\frac{\Lambda^2}{Q^2}$

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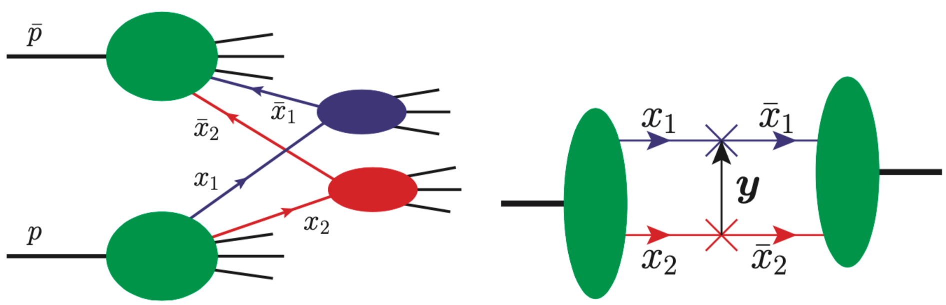

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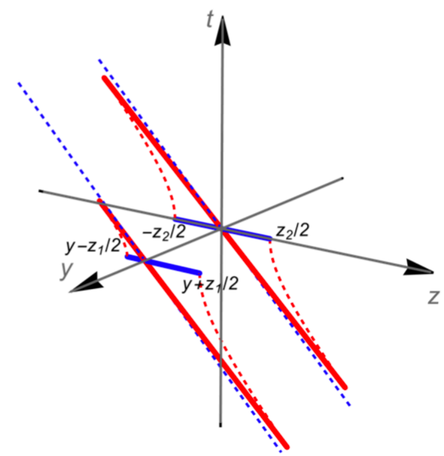

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JHZ, 23'

- Combines features of collinear PDFs and TMDPDFs
- Rapidity divergences can appear already in collinear distributions

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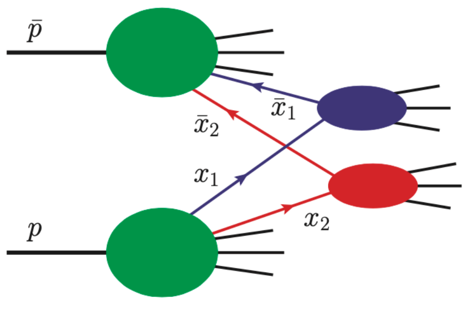

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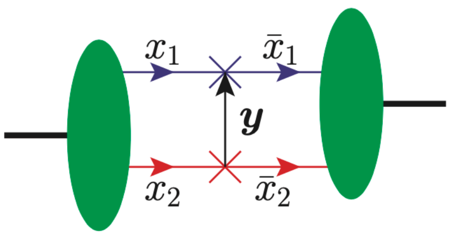

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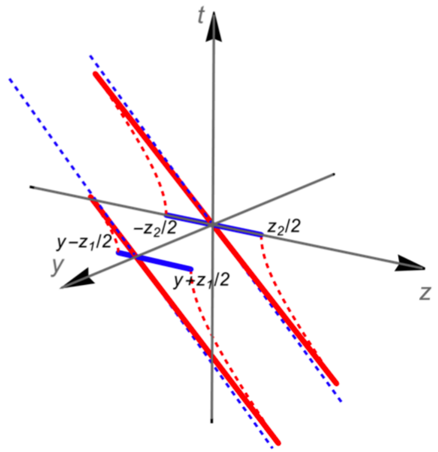

Two partons from a hadron can have transverse separations

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JHZ, 23'

- The same development in single parton distributions (PDFs, TMDs, GPDs…) can be extended to DPDs, and generalized to multiparton distributions

Renormalons

- Renormalons are related to the fact that QCD series Beneke, Phys. Rep., 99'

$$F(\alpha_s) = \sum_n f_n \alpha_s^n$$

is **not convergent** for any $\alpha_s \neq 0$, f_n diverges $\sim a_f^n n!$

- It can still be a useful approximation to $F(\alpha_s)$ if

$$\lim_{\alpha_s \to 0} \alpha_s^{-N} | F(\alpha_s) - \sum_{n=0}^{N} f_n \alpha_s^n | \to 0$$

- For finite α_s, the partial sum usually gives an increasingly better approximation to $F(\alpha_s)$ up to some order $N_0 \sim 1/(|a_f| \alpha_s)$
- Beyond N_0, the approximation does not improve
- Best approximation is reached when truncated at the minimal term which characterizes the truncation error

$$f_{N_0} \alpha_s^{N_0} \sim e^{-1/(|a_f| \alpha_s)}$$

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- In terms of Borel transform

$$B[F](t) = \sum_n f_n \frac{t^n}{n!}$$

the same series expansion as F can be obtained from the Borel integral

$$\int_0^\infty dt\, e^{-t/\alpha_s} B[F](t)$$

if $B[F](t)$ has no singularities for real positive t

- Singularities along integration path correspond to IR renormalons

Numerical evidence of renormalons

- Self-energy of static source Bauer et al, PRL 12', Bali et al, PRD 13', Pineda, 21'

$$\delta m = \frac{1}{a} \sum_{n=0}^{\infty} c_n^{(3,\rho)} \alpha^{n+1}(1/a)$$

- For large n, c_n is universal and equal to the expansion coefficient of heavy quark pole mass r_n/ν up to $\mathcal{O}[\exp(-1/n)]$ terms

$$m_{\rm OS} = m_{\overline{\rm MS}}(\nu) + \sum_{n=0}^{\infty} r_n \alpha_s^{n+1}(\nu),$$

- Large-n behavior

$$c_n^{(3,\rho)} \stackrel{n\to\infty}{=} N_m \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \times \left(1 + \frac{b}{(n+b)} s_1 + \frac{b(b-1)}{(n+b)(n+b-1)} s_2 + \cdots \right).$$

$$\frac{c_n^{(3,\rho)}}{c_{n-1}^{(3,\rho)}} \frac{1}{n} = \frac{\beta_0}{2\pi} \left\{ 1 + \frac{b}{n} - \frac{b s_1}{n^2} + \frac{1}{n^3} [b^2 s_1^2 + b(b-1)(s_1 - 2s_2)] + \mathcal{O}\left(\frac{1}{n^4}\right) \right\}.$$

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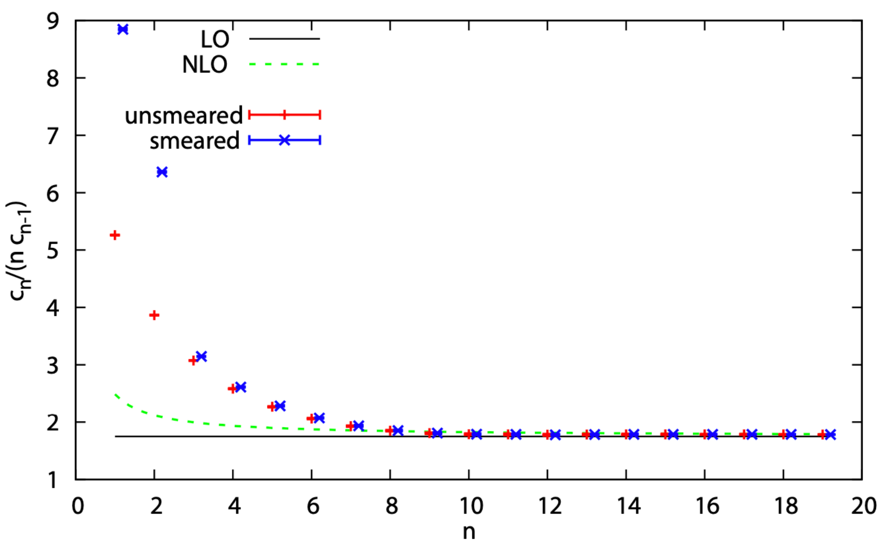

Numerical stochastic perturbation theory calculated up to α_s^{20}

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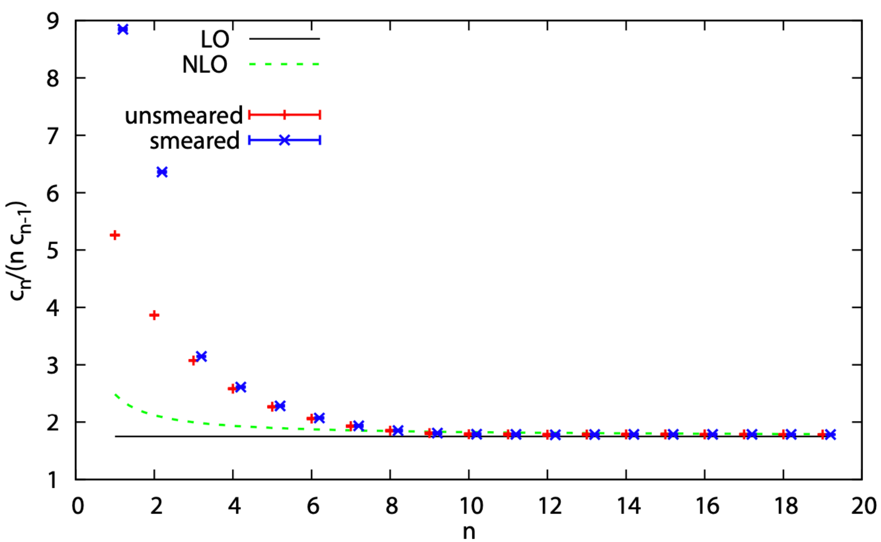

Numerical stochastic perturbation theory calculated up to α_s^{20}

Renormalon-related talks:

Yushan Su, 4:20 pm Tuesday
Andreas Kronfeld, 1:50 pm Thursday
Jack Holligan, 2:10 pm Thursday

Renormalons in LaMET

- Factorization of quasi-PDF

$$\mathcal{Q}(x,p) = \int_{-1}^{1} \frac{dy}{|y|} C_{\mathcal{Q}}\left(\frac{x}{y}, xp, \mu_F\right) q(y,\mu_F) + \frac{1}{p^2}\mathcal{Q}_4(x,p) + \dots$$

$$C_{\mathcal{Q}}(x,p,\mu_F) = \delta(1-x) + c_1\alpha_s + c_2\alpha_s^2 + \dots \; - \frac{\mu_F^2}{p^2} D_{\mathcal{Q}}(x) + \dots,$$

- Logarithmic scale dependence in $c_i(x, \ln p^2/\mu_F^2)$ is canceled by that from the leading-twist PDF $q(x,\mu_F)$
- Power dependence is canceled between leading- and higher-twist contributions Braun, Vladimirov, JHZ, PRD 19'

$$\mathcal{Q}_4(x,p,\mu_F) = \mu_F^2 \int_{-1}^{1} \frac{dy}{|y|} D_{\mathcal{Q}}\left(\frac{x}{y}\right) q(y,\mu_F) + \tilde{\mathcal{Q}}_4(x,p,\mu_F)$$

- In DR, power-like terms do not appear, but the coefficients c_i grow factorially with i
- The sum of perturbative series is only defined to a power accuracy and this ambiguity is compensated by adding a higher-twist contribution

Renormalons in LaMET

- Estimate of **twist-4** contribution Braun, Vladimirov, JHZ, PRD 19'

$$\mathcal{Q}_4(x,p,\mu_F) = \mu_F^2 \int_{-1}^{1} \frac{dy}{|y|} D_{\mathcal{Q}}\left(\frac{x}{y}\right) q(y,\mu_F) + \tilde{\mathcal{Q}}_4(x,p,\mu_F)$$

$$\Downarrow$$

$$\mathcal{Q}_4(x,p,\mu_F) = \kappa \Lambda_{\rm QCD}^2 \int_{-1}^{1} \frac{dy}{|y|} D_{\mathcal{Q}}\left(\frac{x}{y}\right) q(y,\mu_F)$$

- Start from the perturbative series of the coefficient function in coordinate space

$$H = \delta(1-\alpha) + \sum_{k=0}^{\infty} h_k a_s^{k+1}, \qquad a_s = \frac{\alpha_s(\mu)}{4\pi}$$

- Borel transform

$$B[H](w) = \sum_{k=0}^{\infty} \frac{h_k}{k!} \left(\frac{w}{\beta_0}\right)^k \qquad H = \delta(1-\alpha) + \frac{1}{\beta_0} \int_0^{\infty} dw e^{-w/(\beta_0 a_s)} B[H](w).$$

- Borel integral has singularities along the integration path
- Ambiguity can be estimated by taking the residue at a given singularity

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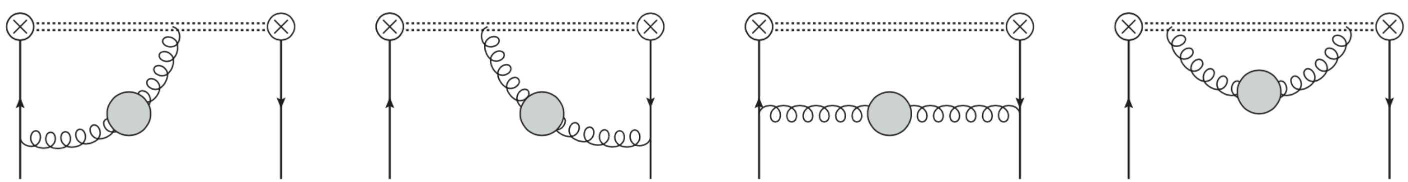
- Singularities of the Borel transform (large β_0 approximation)



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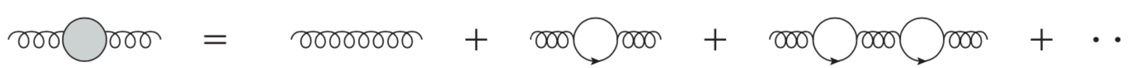

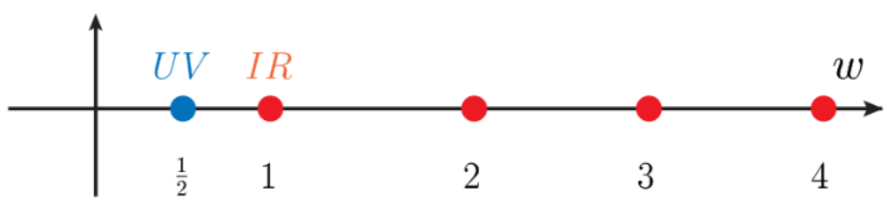

FIG. 2. Singularity structure of the Borel transform.

Renormalons in LaMET

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$$\mathcal{Q}_4(x,p,\mu_F) = \mu_F^2 \int_{-1}^{1} \frac{dy}{|y|} D_{\mathcal{Q}}\left(\frac{x}{y}\right) q(y,\mu_F) + \tilde{\mathcal{Q}}_4(x,p,\mu_F)$$

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$$\mathcal{Q}_4(x,p,\mu_F) = \kappa \Lambda_{\text{QCD}}^2 \int_{-1}^{1} \frac{dy}{|y|} D_{\mathcal{Q}}\left(\frac{x}{y}\right) q(y,\mu_F)$$

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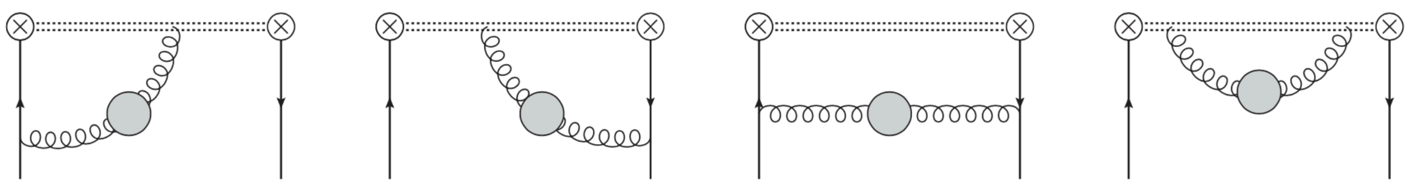

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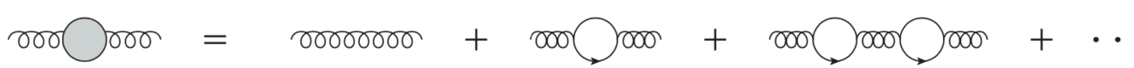

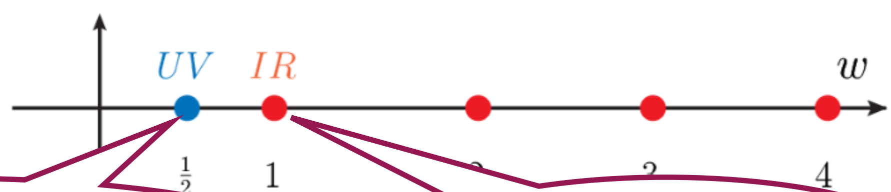

From Wilson line self-energy, removed by renormalization

Leading power correction

Renormalons in LaMET

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Quasi-PDF

$$\mathcal{Q}(x,p) = q(x)\left\{1 + \mathcal{O}\left(\frac{\Lambda^2}{p^2}\cdot\frac{1}{x^2(1-x)}\right)\right\}$$

Pseudo-PDF

$$\mathcal{P}(x,z) = q(x)\{1 + \mathcal{O}(z^2\Lambda^2(1-x))\}$$

- Zero-momentum matrix element helps to suppress the power correction at $x \to 1$
- Confirmed in Liu, Chen, PRD 21'

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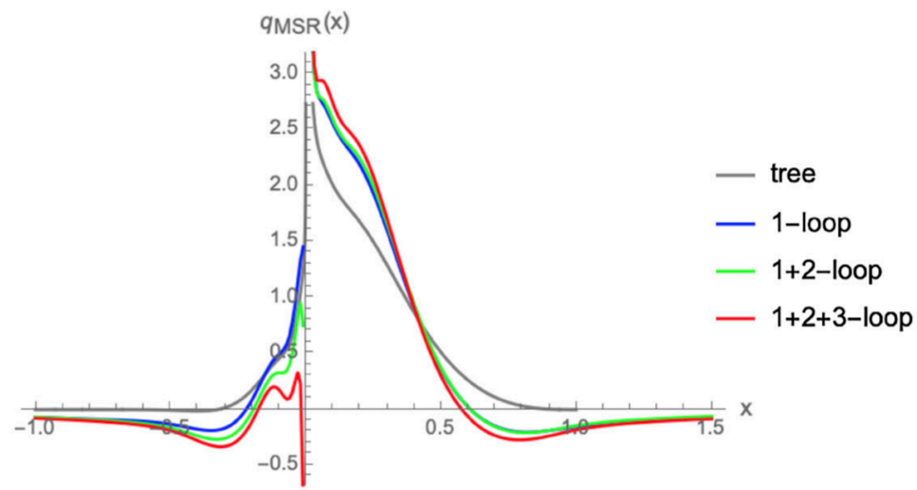
- Remove the leading renormalon ambiguity Liu, Chen, PRD 21'

R-scheme

Hoang et al, PRD 10'



$$\tilde{Q}_R(x,P_z,P'_z,\Lambda') = \frac{P_z^2\tilde{Q}(x,P_z,\Lambda') - P_z'^2\tilde{Q}(x,P'_z,\Lambda')}{P_z^2 - P_z'^2}$$

$$= \int_{-1}^{1}\frac{dy}{|y|}\left[\frac{P_z^2 Z(\frac{x}{y},yP_z,\Lambda',\mu) - P_z'^2 Z(\frac{x}{y},yP'_z,\Lambda',\mu)}{P_z^2 - P_z'^2}\right] Q(y,\mu) + \mathcal{O}\left(\frac{\alpha_s \ln(P_z/P'_z)}{P_z^2 - P_z'^2}, \frac{1}{P_z^2 P_z'^2}\right)$$

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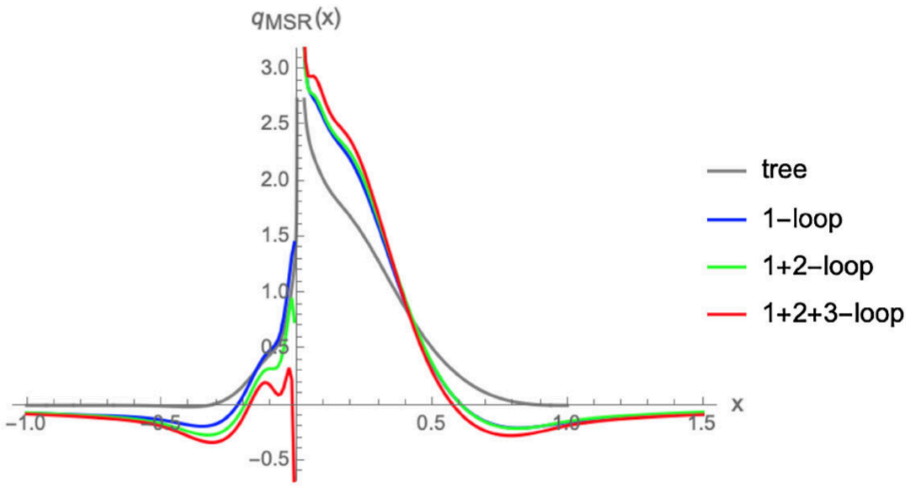

- Good convergence in RI/MOM scheme matrix element observed

Renormalons in LaMET

- There can be **twist-3** contribution Zhang, Holligan, Ji, Su, PLB 23'

$$\mathcal{Q}(x,p) = \int_{-1}^{1} \frac{dy}{|y|} C_{\mathcal{Q}}\left(\frac{x}{y}, xp, \mu_F\right) q(y,\mu_F) + \frac{1}{p^2}\mathcal{Q}_4(x,p) + \ldots$$

See Yushan Su, 4:20 pm Tuesday

$$\mathcal{O}\left(\frac{\Lambda_{\mathbf{QCD}}}{\mathbf{x\,p}}\right)$$

- Renormalization of linear divergence from Wilson line self-energy

$$h^R(z, P_z) = h^B(z, P_z) e^{(\delta m - m_0)z}$$

introduces an intrinsic ambiguity of $\mathcal{O}(z\Lambda_{QCD})$

- Modified OPE

$$\begin{aligned} &h^R(z, P_z, \mu, \tau) \\ &= \left(1 - m_0(\tau) z\right) \sum_{k=0}^{\infty} C_k\left(\alpha_s(\mu), \mu^2 z^2\right) \lambda^k a_{k+1}(\mu) + \mathcal{O}(z^2) \\ &= \sum_{k=0}^{\infty} \left[C_k\left(\alpha_s(\mu), \mu^2 z^2\right) - z m_0(\tau)\right] \lambda^k a_{k+1}(\mu) + \mathcal{O}(z\alpha_s, z^2), \end{aligned}$$

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$\mathscr{O}\left(\frac{\Lambda_{\text{QCD}}}{\text{x p}}\right)$

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$$h^R(z, P_z, \mu, \tau)$$
$$= \left(1 - m_0(\tau) z\right) \sum_{k=0}^{\infty} C_k\left(\alpha_s(\mu), \mu^2 z^2\right) \lambda^k a_{k+1}(\mu) + \mathcal{O}(z^2)$$
$$= \sum_{k=0}^{\infty} \left[C_k\left(\alpha_s(\mu), \mu^2 z^2\right) - z m_0(\tau)\right] \lambda^k a_{k+1}(\mu) + \mathcal{O}(z\alpha_s, z^2),$$

$m_0(\tau)$ can be determined by fitting to zero-momentum matrix element

Renormalons in LaMET

- There can be **twist-3** contribution Zhang, Holligan, Ji, Su, PLB 23'

$$\mathcal{Q}(x,p) = \int_{-1}^{1} \frac{dy}{|y|} C_{\mathcal{Q}}\left(\frac{x}{y}, xp, \mu_F\right) q(y,\mu_F) + \frac{1}{p^2}\mathcal{Q}_4(x,p) + \ldots$$

See Yushan Su, 4:20 pm Tuesday

$\mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{\mathbf{x\,p}}\right)$

- Renormalization of linear divergence from Wilson line self-energy

$$h^R(z,P_z) = h^B(z,P_z)e^{(\delta m - m_0)z}$$

introduces an intrinsic ambiguity of $\mathcal{O}(z\Lambda_{QCD})$

- Modified OPE

$$\begin{aligned} &h^R(z,P_z,\mu,\tau) \\ &= \left(1 - m_0(\tau)z\right)\sum_{k=0}^{\infty} C_k\left(\alpha_s(\mu),\mu^2 z^2\right)\lambda^k a_{k+1}(\mu) + \mathcal{O}(z^2) \\ &= \sum_{k=0}^{\infty}\left[C_k\left(\alpha_s(\mu),\mu^2 z^2\right) - z m_0(\tau)\right]\lambda^k a_{k+1}(\mu) + \mathcal{O}(z\alpha_s, z^2), \end{aligned}$$

$m_0(\tau)$ can be determined by fitting to zero-momentum matrix element

Renormalons in LaMET

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See Yushan Su, 4:20 pm Tuesday

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introduces an intrinsic ambiguity of $\mathcal{O}(z\Lambda_{QCD})$

- Leading renormalon contribution

$$C_k(\alpha_s(z^{-1}), 1)_{\mathrm{PV}} = N_m \frac{4\pi}{\beta_0} \int_{0,\mathrm{PV}}^{\infty} du \times e^{-\frac{4\pi u}{\alpha_s(z^{-1})\beta_0}} \frac{1}{(1-2u)^{1+b}} (1 + c_1(1-2u) + \ldots),$$

- Matching revised accordingly

$$C^{\mathrm{LRR}}(\alpha_s(z^{-1}), 1) = C_k(\alpha_s(z^{-1}), 1) + \left[C_k(\alpha_s(z^{-1}), 1)_{\mathrm{PV}} - \sum_i r_i \alpha_s^{i+1}(z^{-1})\right]$$

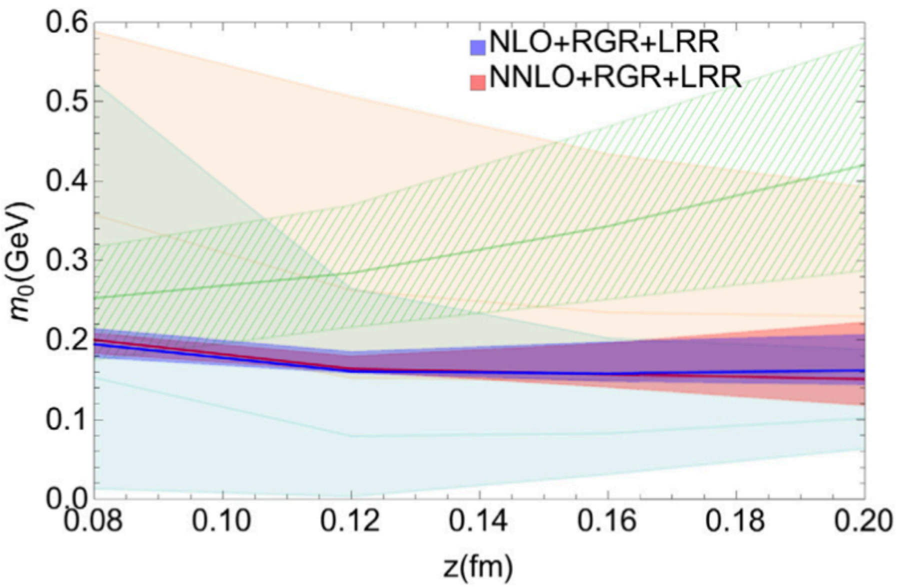
Renormalons in LaMET

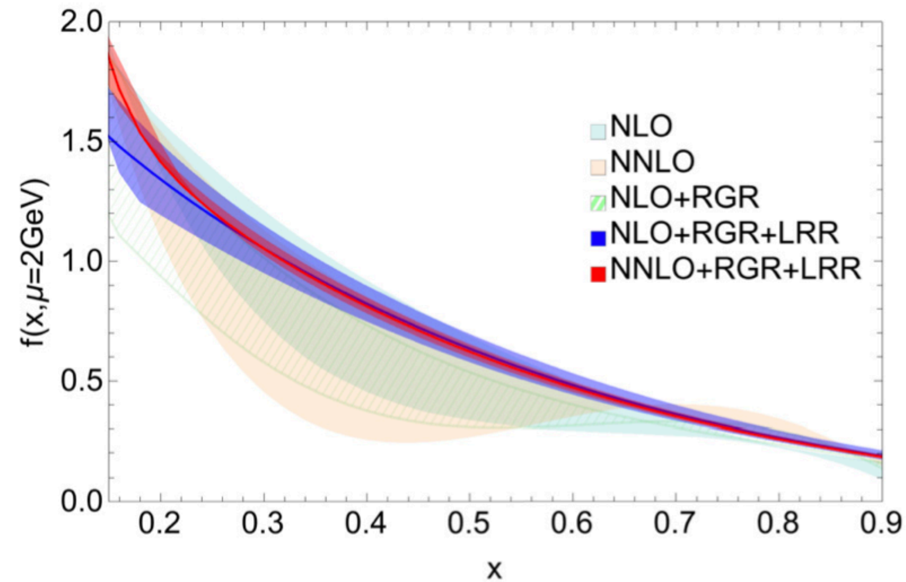
- There can be **twist-3** contribution Zhang, Holligan, Ji, Su, PLB 23'

$$\mathcal{Q}(x,p) = \int_{-1}^{1} \frac{dy}{|y|} C_{\mathcal{Q}}\left(\frac{x}{y}, xp, \mu_F\right) q(y,\mu_F) + \frac{1}{p^2}\mathcal{Q}_4(x,p) + \ldots$$

$$\mathcal{O}\left(\frac{\Lambda_{\mathbf{QCD}}}{\mathbf{x\,p}}\right)$$

See Yushan Su, 4:20 pm Tuesday





Summary and outlook

- A lot of progress has been achieved in calculating the partonic structure of hadrons from lattice
  - Collinear PDFs, DAs, GPDs
  - TMDPDFs/wave functions

- For single parton distributions, we have reached the stage of precision control
  - Higher-order perturbative correction
  - RG/threshold resummation
  - Higher-twist contribution/renormalons
  - Mainly to collinear PDFs, extended to GPDs, TMDs?

- Double/multiple parton distributions can be studied following similar spirit and remain to be explored