Hadronic vacuum polarization: comparing lattice QCD and data-driven results in systematically improvable ways

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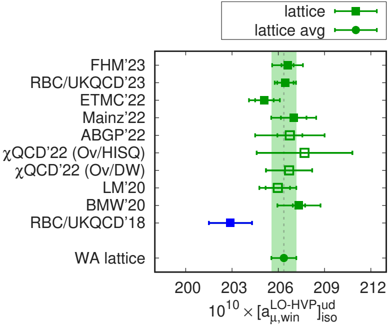
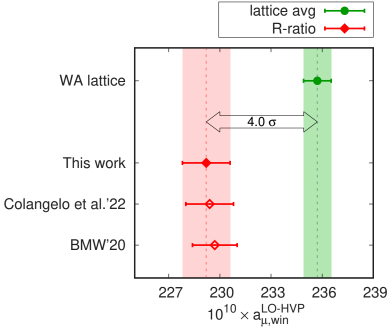
for the BMW and DHMZ collaborations
(in preparation)

Motivation

Significant tensions between lattice and data-driven (DD) results for HVP

- $[\Delta a_\mu^{\text{LO-HVP}}]_{\text{lat-DD}} \sim 2.1\sigma$ [BMW'20, WP'21]
- Simpler $[\Delta a_{\mu,\text{win}}^{\text{LO-HVP}}]_{\text{lat-DD}} \gtrsim 4\sigma$ [Observable proposed in RBC/UKQCD'18]





→ origin of tensions?

→ comparison not trivial

Primary observables

- Lattice: compute with simulations

$$C(t) = \frac{a^3}{3e^2} \sum_{i=1}^{3} \sum_{\vec{x}} \langle J_i(\vec{x}, t) J_i(0) \rangle$$

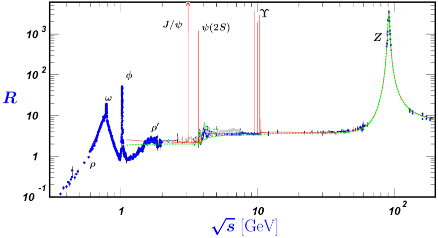
w/ $\frac{J_\mu}{e} = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{2}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c - \frac{1}{3}\bar{b}\gamma_\mu b + \frac{2}{3}\bar{t}\gamma_\mu t$

$a_{\mu(,\text{win})}^{\text{LO-HVP}}$, $\hat{\Pi}(Q^2)$, ... are weigthed sums of $C(t)$ over t

- Data-driven: measure

$$R(s) \equiv \frac{\sigma(e^+e^-(s) \to \text{hadrons}(+\gamma))}{4\pi\alpha^2(s)/(3s)}$$

$a_{\mu(,\text{win})}^{\text{LO-HVP}}$, $\hat{\Pi}(Q^2)$, ... are weigthed integrals of $R(s)$ over s



[PDG compilation]

Lattice $\leftrightarrow$ R-ratio

$$C(t) = \frac{1}{24\pi^2}\int_0^\infty ds\,\sqrt{s}R(s)\,e^{-|t|\sqrt{s}}$$

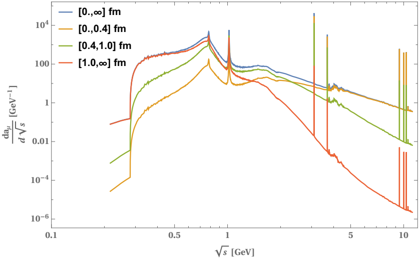
[Bernecker et al '11]

- R-ratio $\longrightarrow$ lattice: "straightforward"

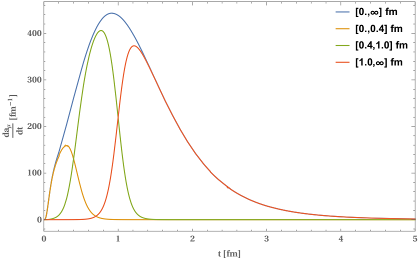
  $\rightarrow$ integrate R-ratio

- Lattice $\longrightarrow$ R-ratio: inverse Laplace transform

  $\rightarrow$ ill-posed problem



$\longleftrightarrow$



Requirements for comparison methodology

- Very few HVP quantities computed on lattice w/:
  - all contributions to $C(t)$: flavors, various contractions, QED and SIB corrections
  - all limits taken: $a \to 0$, $L \to \infty$, $M_\pi \to M_\pi^\phi$, ...
- None w/ correlations among lattice HVP observables
- None w/ uncertainties on these correlations (important for checking stability of inverse problem)

$\rightarrow$ Want approach that:

  - provides useful information w/ limited lattice input
  - can be systematically improved w/ more lattice input
  - can (eventually) incorporate physical constraints
  - includes measure of agreement of lattice & R-ratio results w/ comparison hypothesis
  - accounts for all correlations in lattice and R-ratio observables ...
  - ... including uncertainties on these

- Here use BMW'20: $a_\mu^{\text{LO-HVP}}$, $a_{\mu,\text{win}}^{\text{LO-HVP}}$ & $\delta(\Delta_{\text{had}}^{(5)}\alpha) \equiv \Delta_{\text{had}}^{(5)}\alpha(-1 \to -10\,\text{GeV}^2)$ (preliminary)

Lattice covariances: method

- Uncertainties and correlations critical for comparisons

- Use extension of BMW error method with stat resampling and syst histogram w/ flat and AIC weights [BMW '08, '15, '20, see also Neil et al '23, Pinto et al '23]

  $\rightarrow$ for observables $\{a_j\} = \{a_\mu^{\text{LO-HVP}}, a_{\mu,\text{win}}^{\text{LO-HVP}}, \delta(\Delta_{\text{had}}^{(5)}\alpha), \cdots\}$

$$H(\{a_j\}) = \sum_{\psi^{\text{corr}},\{\psi_j^{\text{aic}},\psi_j^{\text{flat}}\}} \mathcal{N}_{N_\mathcal{O}}[\{a_j\},\{\overline{a_j}\}(\psi^{\text{corr}},\{\psi_j^{\text{aic}},\psi_j^{\text{flat}}\}), C^{\text{stat}}(\psi^{\text{corr}},\{\psi_j^{\text{flat}},\psi_j^{\text{aic}}\})] \times \Pi_j \omega_j(\psi^{\text{corr}},\psi_j^{\text{aic}},\psi_j^{\text{flat}})$$

w/

$$\omega_j(\psi^{\text{corr}},\psi_j^{\text{aic}},\psi_j^{\text{flat}})) = \frac{\text{aic}(\psi^{\text{corr}},\psi_j^{\text{aic}},\psi_j^{\text{flat}})}{\sum_{\psi_j^{\text{aic}}} \text{aic}(\psi^{\text{corr}},\psi_j^{\text{aic}},\psi_j^{\text{flat}})}$$

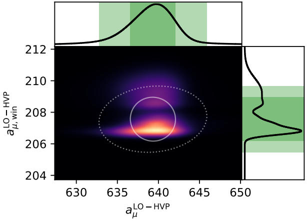
- Build matrix from 1D distributions

- Separate stat. & syst. by solving ($\lambda = 2$)

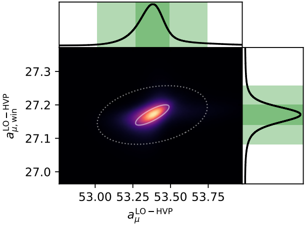
$$\begin{aligned} C &= C^{\text{stat}} + C^{\text{syst}} \\ C_\lambda &= \lambda\, C^{\text{stat}} + C^{\text{syst}} \end{aligned}$$

Lattice covariances: results

- $\delta(\Delta^{(5)}_{\text{had}}\alpha)$ largely uncorrelated w/ other two observables
- Uncertainties and correlations of $a_\mu^{\text{LO-HVP}}$ & $a_{\mu,\text{win}}^{\text{LO-HVP}}$ contributions (units of 10^{-10})



(connected *ud*)



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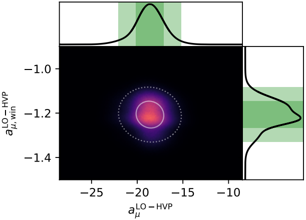


(disconnected)

- Double peak $\rightarrow$ consider 1σ & 2σ intervals

Uncertainties on lattice covariances

- Uncertainties on covariance matrix can compromise the inverse problem
- Stat error estimated from bootstrap on only 48 jackknife samples (sufficient for this study)
- Syst from:
  - For: *ud*, *s*, QED, SIB connected, and disconnected
    $\rightarrow$ get uncertainties from 1 or 2σ quantiles
    $\rightarrow$ 0 or 100% correlations in $a \rightarrow 0$ uncertainties of $T = a_\mu^{\text{LO-HVP}}$ and $W = a_{\mu,\text{win}}^{\text{LO-HVP}}$, w/ $C = T - W$

$$C_{TW} = C_{TW}^{\text{other}} + \begin{bmatrix} (dW)^2 + (dC)^2 & \{0,1\} \times (dW)^2 \\ \{0,1\} \times (dW)^2 & (dW)^2 \end{bmatrix}_{\text{cont}}$$

  - Similarly for *c*

$\Rightarrow$ in units of 10^{-20}:

$$C_{\text{lat}}^{1\sigma,0\%} = \begin{bmatrix} 30.13(4.88) & -0.05(0.03) \\ -0.05(0.03) & 1.95(0.47) \end{bmatrix} \qquad C_{\text{lat}}^{2\sigma,0\%} = \begin{bmatrix} 34.04(16.80) & 0.32(0.05) \\ 0.32(0.05) & 1.12(0.07) \end{bmatrix}$$

$$C_{\text{lat}}^{1\sigma,100\%} = \begin{bmatrix} 30.13(4.88) & 1.56(0.03) \\ 1.56(0.03) & 1.95(0.47) \end{bmatrix} \qquad C_{\text{lat}}^{2\sigma,100\%} = \begin{bmatrix} 34.04(16.80) & 1.94(0.05) \\ 1.94(0.05) & 1.12(0.07) \end{bmatrix}$$

Testing lattice

- 1-by-1 comparisons

Observable	lattice [BMW '20]	data-driven	diff.	% diff.	σ	p-value [%]
$a_\mu^{\text{LO-HVP}} \times 10^{10}$	707.5(5.5)	694.0(4.0)	13.5(6.8)	1.9(1.0)	2.0	4.7
$a_{\mu,\text{win}}^{\text{LO-HVP}} \times 10^{10}$	236.7(1.4)	229.2(1.4)	7.5(2.0)	3.2(0.8)	3.8	0.01
$\delta(\Delta_{\text{had}}^{(5)}\alpha) \times 10^4$	48.67(0.32)	48.02(0.32)	0.65(0.45)	1.3(0.9)	1.4	15

$\Rightarrow$ excess in $[\Delta(a_\mu^{\text{LO-HVP}} - a_{\mu,\text{win}}^{\text{LO-HVP}})]_{\text{lat-DD}} \sim 6.0(7.9) \times 10^{-10}$

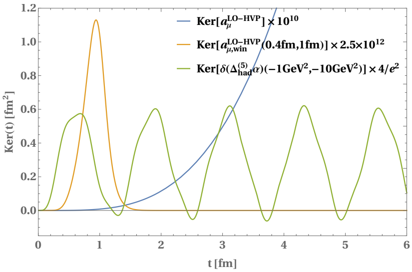
- Simultaneous comparisons w/ correlations

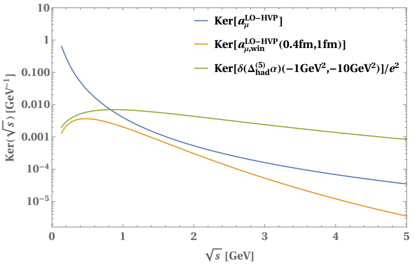
$$\chi^2(a_j) = \sum_{j,k} \left[a_j^{\text{lat}} - a_j\right] [C_{\text{lat}}^{-1}]_{jk} \left[a_k^{\text{lat}} - a_k\right] + \sum_{j,k} \left[a_j^{\text{R}} - a_j\right] [C_{\text{R}}^{-1}]_{jk} \left[a_k^{\text{R}} - a_k\right]$$

# observ.	χ^2/dof	p-value [%]
2	$14.4/2 - 18.8/2$	$0.002 - 0.017$
3	$14.4/3 - 18.8/3$	$0.009 - 0.63$

- Some dilution compared to $a_{\mu,\text{win}}^{\text{LO-HVP}}$ alone, but still significant tension

Consequences for lattice $C(t)$





$\Rightarrow$ SD:ID:LD windows: [using KNT'18 compilation]

- 10%:33%:57% for $a_\mu^{\text{LO-HVP}}$
- 70%:29%:1% for $\delta(\Delta_{\text{had}}^{(5)}\alpha)$

$+$ tensions and agreements above

$\Rightarrow$ excess in $C(t)$ for $t \sim [0.4, 1.5]\,\text{fm}$

$\Rightarrow$ probably for $t \gtrsim 1.5\,\text{fm}$

$\Rightarrow$ possible suppression for $t \lesssim 0.4\,\text{fm}$ (mainly based on preliminary $\delta(\Delta_{\text{had}}^{(5)}\alpha)$)

Testing R-ratio: methodology

- Chop $a_j^{\rm R}$ into contributions $a_{jb}^{\rm R}$ from same $\sqrt{s}$-intervals I_b for all j

$$a_j^{\rm R} = \sum_b a_{jb}^{\rm R}$$

- To accommodate lattice results $a_j^{\rm lat}$, allow common rescaling of $a_{jb}^{\rm R}$, for all j, in certain I_b

$$a_j^{\rm lat} = \sum_b (1+\delta_b) a_{jb}^{\rm R}$$

  $\rightarrow$ simplest interpretation: R-ratio rescaled in I_b

  $\rightarrow$ however, constrains shape of R-ratio modification in limited way

  $\rightarrow$ Φ deformation may be allowed

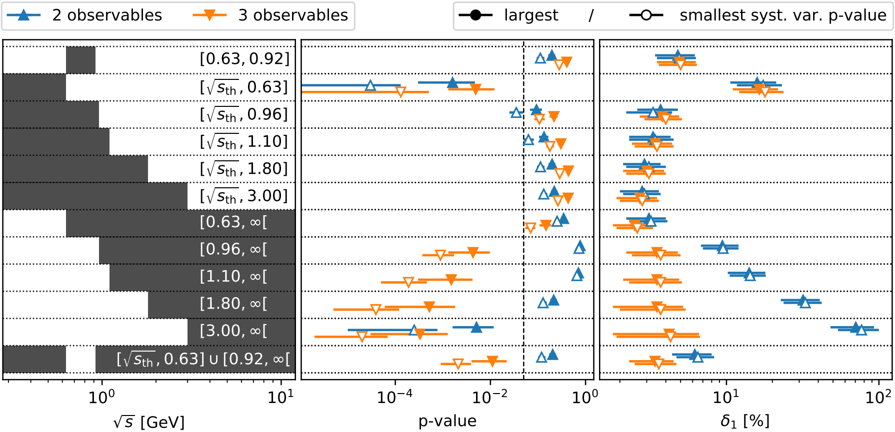
- If $N_j \geq N_b$, system (over-)constrained

- Solve via weighted avg or χ^2 minimization
  $\rightarrow$ compatible results

- None of these rescalings allowed by measured R-ratio

Testing R-ratio: results

Consider $a_j = a_\mu^{\text{LO-HVP}}$, $a_{\mu,\text{win}}^{\text{LO-HVP}}$ (2 observables) w/ $a_j = \delta(\Delta_{\text{had}}^{(5)}\alpha)$ (3 observables)



- Stat and syst uncertainties on lattice covariance matrices do not change overall picture
- Summary of modifications of R-ratio allowed by lattice results . . .

Testing R-ratio: results summary

Modifications to measured R-ratio that could explain lattice results are:

- possible in ρ-peak interval $[0.63,\ 0.92]\,\mathrm{GeV}$ for 2 & 3 observables

  $\rightarrow$ requires rescaling of observables in that interval by $\sim (5.0 \pm 1.5)\%$

- disfavored in interval below ρ-peak, $[\sqrt{s_{\rm th}},\ 0.63\,\mathrm{GeV}]$

- but possible in $[\sqrt{s_{\rm th}},\ \sqrt{s_{\rm max}}]$ w/ $\sqrt{s_{\rm max}}:\ 0.96 \rightarrow 3.0\,\mathrm{GeV}$ that include ρ-peak, for 2 & 3 observables

  $\rightarrow$ rescalings $\sim (4 \pm 1)\% \rightarrow (3 \pm 1)\%$ for $\sqrt{s_{\rm max}} \nearrow$

- possible in $[\sqrt{s_{\rm min}},\ \infty[$ w/ $\sqrt{s_{\rm min}}:\ 0.63 \rightarrow 1.8\,\mathrm{GeV}$, for 2 observables

  $\rightarrow$ rescalings $\sim (3 \pm 1)\% \rightarrow (32 \pm 9)\%$ for $\sqrt{s_{\rm min}} \nearrow$

- but disfavored in $[3.0\,\mathrm{GeV},\ \infty[$, for 2 & 3 observables

- and adding $\delta(\Delta^{(5)}_{\rm had}\alpha)$ constraint eliminates the possibility of rescalings in $[\sqrt{s_{\rm min}},\ \infty[$ w/ $\sqrt{s_{\rm min}}:\ 0.96 \rightarrow 3.0\,\mathrm{GeV}$ that do not include ρ-peak

Conclusions

- Presented flexible method for comparing lattice QCD and data-driven HVP results
- Find that discrepancies/agreements between lattice and data-driven results for $a_\mu^{\text{LO-HVP}}$, $a_{\mu,\text{win}}^{\text{LO-HVP}}$ and $\delta(\Delta_{\text{had}}^{(5)}\alpha)$:

  On lattice side, result, from:

  - a $C(t)$ that is enhanced in $t \sim [0.4, 1.5]\,\text{fm}$
  - also probably for $t \gtrsim 1.5\,\text{fm}$
  - w/ possible suppression for $t \lesssim 0.4\,\text{fm}$ (mainly based on preliminary $\delta(\Delta_{\text{had}}^{(5)}\alpha)$)

  On data-driven side, could be explained by:

  - enhancing measured R-ratio around ρ-peak
  - or in any larger interval including ρ-peak
- Lattice and measured R-ratio correlations critical for drawing such conclusions

Conclusions

- Important to check that uncertainties on uncertainties and correlations do not spoil picture, especially for inverse problem

  $\rightarrow$ checked here for lattice stat and syst uncertainties

  $\rightarrow$ must do so for measured R-ratio uncertainties

- Also important not to share results between 2 approaches before they are final (mutual blinding)

- W/ more HVP observables, many generalizations possible, also including Φ constraints

- However, limit on independent HVP observables in data-driven and lattice approaches (not shown)

- Same methods can be used to combine determinations of lattice and data-driven results for HVP observables, once differences are understood

- No problems w/ EWP fits in case of 3-observable comparisons (not shown)

Some references to related work on HVP

- Windows proprosed in RBC/UKQCD arXiv:1801.07224
- Discussed in context of detailed comparison in Colangelo et al arXiv:2205.12963
- Consequences of rescaling of measured R-ratio studied in Crivellin et al arXiv:2003.04886, Keshavarzi et al arXiv:2006.12666, de Rafael arXiv:2006.13880, Malaescu et al arXiv:2008.08107
- Consequences of lattice $a_\mu^{\text{LO-HVP}}$ on $\pi^+\pi^-$ contributions to R-ratio w/ Φ constraints in Colangelo et al arXiv:2010.07943
- Use of Backus-Gilbert method for reconstruction of smeared R-ratio from lattice $C(t)$ in Hansen et al arXiv:1903.06476, Alexandrou et al arXiv:2212.08467
- Proposal for comparing measured R-ratio and lattice $C(t)$ via spectral-width sumrules in Boito et al arXiv:2210.13677
- . . . (many other references for reconstructing spectral functions from lattice correlators)