Nucleon Electromagnetic Form Factors at Large Momentum from Lattice QCD

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Outline

- Nucleon vector form factors at large momentum
- Challenges for large-$Q^2$ hadron structure on lattice
- Preliminary results and comparison to experiment & phenomenology
- Examining systematic effects
- Summary and Outlook
Nucleon Elastic E&M Form Factors

Elastic e−p amplitude

$$\langle P + q | \bar{q} \gamma^\mu q | P \rangle = \bar{U}_{P+q} \left[ F_1(Q^2) \gamma^\mu + \frac{F_2(Q^2) i \sigma^{\mu\nu} q_\nu}{2M_N} \right] U_P$$

Sachs Electric

$$G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{4M^2} F_2(Q^2)$$

Magnetic

$$G_M(Q^2) = F_1(Q^2) + F_2(Q^2)$$

Elastic e−p cross-section

- $G_{E,M}$ from $\epsilon$-dep. at fixed $\tau(Q^2)$
  - "Rosenbluth separation"
- dominated by $G_M$ at large $Q^2$
- $2\gamma$ corrections at $Q^2 \approx 1$ GeV²

$$\frac{d\sigma}{d\Omega} = \frac{\sigma_{Mott}}{1 + \tau} \left[ G_E^2 + \frac{\tau}{\epsilon} G_M^2 \right]$$

$$\tau = \frac{Q^2}{4M_N^2}$$

$$\epsilon = \left[ 1 + 2(1 + \tau) \tan^2 \frac{\theta}{2} \right]^{-1}$$

Polarization transfer: polarized e−beam

+ detect polarization of recoil nucleon
  (alt.: transverse asymmetry on pol. target)

- $G_E/G_M$ ratio (only small radiative corrections)

$$P_t/P_l \propto G_E/G_M$$
Recent/Ongoing Experiments

Projected new precision on proton & neutron form factors

Experiments at JLab@12GeV
- Hall A (HRS, SBS):
  $G_{Mp}$ @ $Q^2 \leq 17.5$ GeV$^2$
  $G_{Ep}/G_{Mp}$ @ $Q^2 \leq 15$ GeV$^2$
  $G_{Mn}$ @ $Q^2 \leq 18$ GeV$^2$
  $G_{En}/G_{Mn}$ @ $Q^2 \leq 10.2$ GeV$^2$
- Hall B (CLAS12):
  $G_{Mn}$ @ $Q^2 \leq 14$ GeV$^2$
- Hall C:
  $G_{En}/G_{Mn}$ @ $Q^2 \leq 6.9$ GeV$^2$
Recent/Ongoing Experiments

Projected new precision on proton & neutron form factors

New $G_{Mp}$ data from Hall A
[Christy et al, PRL'22]
Nucleon Form Factors: Open Questions

- **Are model descriptions of the nucleon viable?**
  Nucleon models disagree beyond explored range.

- **Role of diquark correlations in elastic scattering?**
  Neutron & proton $G_E/G_M$ at/above $Q^2 = 8 \text{ GeV}^2$

- **Scale of transition to perturbative QCD?**
  $(F_2/F_1)$ scaling at large $Q^2$: $Q^2 F_{2p}/F_{1p} \propto \log^2(Q^2/\Lambda^2)$

- **What are contributions from $u$ and $d$ flavors?**
  Proton and neutron data needed in wide $Q^2$ range

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[Research Mgmt. Plan for SBS(JLab Hall A)]

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Dyson-Schwinger Eqns: quarks & $0^+, 1^+$ diquarks
($\alpha \approx$ rate of transition const.quarks $\rightarrow$ pQCD with $Q^2$)

[Cloet, Roberts, Prog.Part.Nucl.Phys 77:1 (2014)]
Challenges at Large $Q^2$

- Discretization effects:
  \[ (V_\mu)_I = [\bar{q} \gamma_\mu q] + c_V a \partial_\nu [\bar{q} i \sigma_{\mu\nu} q] \propto Q \]

- Stochastic noise grows faster with $T$ [Lepage’89]:
  - Signal: $\langle N(T)\bar{N}(0) \rangle$
  - Noise: $\langle |N(T)\bar{N}(0)|^2 \rangle - |\langle N(T)\bar{N}(0) \rangle|^2$
  - Signal/Noise

  \[ e^{-E_NT} \quad e^{-3m_\pi T} \quad e^{-(E_N - \frac{3}{2}m_\pi)T} \]

- Excited states: boosting "shrinks" the energy gap
  \[ E_1 - E_0 = \sqrt{M_1^2 + \vec{p}^2} - \sqrt{M_2^2 + \vec{p}^2} < M_1 - M_0 \]
  - N(\~1500): $pN \rightarrow 1.5$ GeV $\Rightarrow \Delta E = 500 \rightarrow 300$ MeV

- Quark-disconnected contributions:
  - negligible ($\lesssim 1\%$) at $Q^2 \leq 1$ GeV$^2$, unknown at large $Q^2$

- Large $p_N$: no reliable EFT/ChPT for $m_\pi$, lattice size-extrapolation

\textit{Large statistics required to suppress MC noise in lattice correlators}
Minimize $E_{in, out}$ for target $Q^2$:

$Q^2 = (p_{in} - p_{out})^2 - (E_{in} - E_{out})^2$

Back-to-back $Q^2 = 4p^2$

For $(Q^2)_{max} = 10 \text{ GeV}^2 (E_N \approx 1.9 \text{ GeV})$

$|p| = \frac{1}{2} \sqrt{Q^2_{max}} \approx 1.6 \text{ GeV}$

Nucleon momentum $\sim$ Brillouin zone

$$\langle N\bar{N} \rangle^{-1}(p) \equiv -i\gamma^\mu \bar{\psi}^{\text{lat}} + m_N$$

$$p^{\text{lat}}_\mu = k_\mu + O(k^3)$$

$\Rightarrow$ expect $O(a^2)$ corrections

from lattice nucleon spinor

"Brick-Wall" frame

lattice kinematics for $Q^2 \approx 10 \text{ GeV}^2$
Present QCD Calculation Parameters

- $N_F = 2+1$ clover-improved Wilson fermion ensembles (JLab / W&M / LANL / MIT)
- Lattice spacing $a \approx 0.073 \div 0.091$ fm
- Light quark masses approaching physical: $m_\pi = 170 \div 280$ MeV
- Large physical volume $L \gtrsim 3.7 (m_\pi)^{-1}$
- Source-sink separation $t_{\text{sep}} = 0.51 \div 1.09$ fm
- Momentum smearing, AMA sampling
- Estimate disconnected contributions

2022/23:
- MC Statistics ~250k on D6 ($48^3 \times 96$), E5 ($48^3 \times 128$)
- Disconnected contractions on D6 (1000+ configs)

Made possible by new nVidia A100 clusters
- Perlmutter [NERSC]
- Juwels-Booster [Fz. Juelich]

Many thanks to the QUDA team!
[ K. Clark, R.Babich, R.Brower, M.Wagner, E.Weinberg, and many others ]
E5: $m_\pi = 272$ MeV, spacing $a = 0.073$ fm, 266k MC samples

Effective energy and 2-state fits

$$E_{eff} = \frac{1}{a} \log \frac{C_{N\bar{N}}(t)}{C_{N\bar{N}}(t + a)}$$

Dispersion relation

Dashed lines: cont. $E^2(p) = E^2(0) + p^2$
Lattice Nucleon Energy & Dispersion Relation (D6)

D6: $m\pi = 166$ MeV, spacing $a = 0.091$ fm, 261k MC samples

Effective energy and 2-state fits

$E_{eff} = \frac{1}{a} \log \frac{C_{NN}(t)}{C_{NN}(t + a)}$

Dispersion relation
Dashed lines: cont. $E^2(p) = E^2(0) + p^2$
Nucleon Matrix Element & Form Factor Fits (D5)

2-state fit \( t_{\text{sep}} = 0.73 \pm 1.09 \text{ fm} \) \((8a \div 12a)\); energies fixed from 2-state fits to \( \langle N\bar{N} \rangle \)

PRELIMINARY

Sergey Syritsyn

Nucleon Form Factors at High \( Q^2 \) from LQCD

LATTICE 2023, Aug 1, Fermilab
Nucleon Form Factors

- 2-state fits to extract the ground state
- discrepancy $x(2..2.5)$ for $Q^2 > 2 GeV^2$: exc.states? discretization? quark mass

PRELIMINARY

- No disconnected diagrams
Proton $F_2/F_1$ Ratio

- Lattice data: 2-state fits
- Phenomenology curves: [Alberico et al, PRC79:065204 (2008)]
- Comparison to experimental data (black points)

![Graph showing $Q^2 F_2/F_1$ vs. $Q^2 [\text{GeV}^2]$ with various curves and data points.](image)

-algorithms

[Alberico:2009]

- D5 ($m_\pi = 278$ MeV, $a=0.094$ fm)
- E5 ($m_\pi = 272$ MeV, $a=0.073$ fm)
- D6 ($m_\pi = 166$ MeV, $a=0.091$ fm)

No disconnected diagrams

PRELIMINARY


- BJY - pQCD (2003)
Proton $G_E/G_M$ Ratio

- Lattice data: 2-state fits
- Phenomenology curves: [Alberico et al, PRC79:065204 (2008)]
- Comparison to experimental data (black points)

Earlier calculation: (a=0.074 fm, $m_\pi=470$ MeV)
Feynman-Hellman method
[Chambers et al (CSSM), PRD96: 114509]
Neutron $G_{En}/G_{Mn}$ Ratio

- Lattice data: 2-state fits
- Phenomenology curves: [Alberico et al, PRC79:065204 (2008)]
- Comparison to experimental data (black points)

$Q^2$ [GeV$^2$]

(\mu_{E}/G_{M})_n$

[Alberico]
D5 ($m_\pi = 278$ MeV, $a=0.094$ fm)
E5 ($m_\pi = 272$ MeV, $a=0.073$ fm)
D6 ($m_\pi = 166$ MeV, $a=0.091$ fm)

[No disconnected diagrams] PRELIMINARY
Light-Flavor Decomposition (Proton)

\[ Q^4 F_{1u} \]

\[ Q^4 F_{2u} \]

\[ (2.5) \cdot Q^4 F_{1d} \]

\[ (-0.75) \cdot Q^4 F_{2d} \]

\[ d \text{ quark} \times 2.5 \]

\[ d \text{ quark} \times 0.75 \]

Robust estimator from nucleon-current correlators:

avoid lattice correlators fits to $\sim \Sigma \exp(-Et)$

$$\begin{align*}
\text{Re} \langle p' \hat{x} | J_t | p \hat{x} \rangle &\propto \cosh \frac{\lambda' + \lambda}{2} G_E \\
\text{Re} \langle p' \hat{x} | J_y | p \hat{x} \rangle &\propto \sinh \frac{\lambda' - \lambda}{2} G_M
\end{align*}$$

where $\left( \begin{array}{c} p^{(t)} \\ E^{(t)} \end{array} \right) = m_N \sinh \lambda^{(t)}$

$$\left( \begin{array}{c} p^{(t)} \\ E^{(t)} \end{array} \right) = m_N \cosh \lambda^{(t)}$$

D5 ($m_{\pi} = 278$ MeV, $a = 0.094$ fm)

E5 ($m_{\pi} = 272$ MeV, $a = 0.073$ fm)

D6 ($m_{\pi} = 166$ MeV, $a = 0.094$ fm)

$$\left( \begin{array}{c} \sinh \frac{\lambda' - \lambda}{2} \\ \cosh \frac{\lambda' + \lambda}{2} \end{array} \right) \begin{array}{c}
\text{Re} \langle N_T^{(p_x', T)} | J_t (T/2) | \tilde{N}_T(p_x, 0) \rangle \\
\text{Re} \langle N_T^{(p_x', T)} | J_y (T/2) | \tilde{N}_T(p_x, 0) \rangle
\end{array} \xrightarrow{T \to \infty} G_E/G_M$$
Disconnected Quark Loops

- **Stochastic evaluation:**
  \[ \xi(x) = \text{random } Z_2\text{-vector} \]
  \[ E[\xi^\dagger(x)\xi(y)] = \delta_{x,y} \]

  \[ \sum_x e^{iqx} \mathcal{D}^{-1}(x,x) \approx \frac{1}{N_{MC}} \sum_i^{N_{MC}} \xi^\dagger_i(e^{iqx} \mathcal{D}^{-1}\xi_i) \]

  \[ \text{Var}(\sum_x \mathcal{D}^{-1}(x,x)) \sim \frac{1}{N_{MC}} \]

  \[ (\text{contributions from } \mathcal{D}^{-1}(x \neq y)) \]

- **Exploit** \( \mathcal{D}^{-1}(x, y) \) **falloff** to reduce \( \sum_{x \neq y} |\mathcal{D}^{-1}(x, y)|^2 \):

Hierarchical probing method [K.Orginos, A.Stathopoulos, ’13]:

In sum over \( N=2^{n+d+1} \) 3D(4D) Hadamard vectors, near-(x,y) terms cancel:

\[ \frac{1}{N} \sum_i z_i(x)z_i(y)^\dagger = \begin{cases} 
0, & 1 \leq |x - y| \leq 2^k, \\
1, & x = y \text{ or } 2^k < |x - y| 
\end{cases} \]

- Further decrease variance by deflating low-lying, long-range modes [A.Gambhir's PhD thesis]
Prior work: Disc. Light & Strange Quark F.F's

N_{f}=2+1 dynamical fermions, m_{\pi} \approx 320 \ MeV
(C13 ensemble)

\begin{align*}
| (G_{E}^{u/d})_{\text{disc}} | & \lesssim 0.010 \ \text{of} \ | (G_{E}^{u/d})_{\text{conn}} | \\
| (G_{E}^{s})_{\text{disc}} | & \lesssim 0.005 \ \text{of} \ | (G_{E}^{u/d})_{\text{conn}} | \\
| (G_{M}^{u/d})_{\text{disc}} | & \lesssim 0.015 \ \text{of} \ | (G_{M}^{u/d})_{\text{conn}} | \\
| (G_{M}^{s})_{\text{disc}} | & \lesssim 0.005 \ \text{of} \ | (G_{M}^{u/d})_{\text{conn}} | \\
\end{align*}

[J. Green, S. Meinel, S.S. et al; PRD92:031501 (2015)]
Disconnected Light, Strange vs. Connected

- D5 ensemble (mπ=280 MeV, a=0.094 fm), 1346 configs,
- 512 HP vectors; UD: also deflation with 500 DdagD evecs [Stathopoulos et al (2013); Gambhir et al 2017]
- s-, disconnected u,d- contributions are small also at high Q^2 up to \( \approx 10 \text{ GeV}^2 \)

<table>
<thead>
<tr>
<th>( F_{1}^{s} )</th>
<th>( F_{1}^{u/d} )_{disc}</th>
<th>( F_{2}^{u/d} )_{disc}</th>
</tr>
</thead>
<tbody>
<tr>
<td>10% of ( F_{2}^{u,d} )</td>
<td>20% of ( F_{2}^{u,d} )</td>
<td></td>
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</table>
Disconnected Light, Strange vs. Connected

- D5 ensemble (mπ=280 MeV, a=0.094 fm), 1346 configs,
- 512 HP vectors; UD: also deflation with 500 DdagD evecs
  [Stathopoulos et al (2013); Gambhir et al 2017]
- s-, disconnected u, d- contributions are small
  also at high Q² up to ≲ 10 GeV²

\[ |F_{1s}^s| \lesssim |(F_{1d}^{u/d})_{\text{disc}}| \lesssim 10\% \text{ of } |F_{2u,d}^u| \]
\[ |F_{2s}^s| \lesssim |(F_{2d}^{u/d})_{\text{disc}}| \lesssim 20\% \text{ of } |F_{2u,d}^u| \]
Improved vector current \( (V_\mu)_I = \bar{q}\gamma_\mu q + c_V a_\mu \bar{q}i\sigma_{\mu\nu}q \)

\( O(a^1) \) correction: form factors of \( a_\mu \langle N | \partial_\nu (\bar{q}i\sigma^{\mu\nu}q) | N \rangle \)

O(a\(^1\)) correction: form factors of \( a_\mu \)

Relative magnitude of \( O(a^1) \) effects: \( \{ O(a^1) \} / \{ O(a^0) \} \) form factors (assuming \( c_V = 0.05 \))

- improvement coefficient \( c_V \): must be computed on lattice from WI
- perturbation theory: \( c_V \approx -0.01C_F(g_0)^2 \)
Summary

- Preliminary results for high MC-statistics high-momentum form factors up to $Q^2 \lesssim 12 \text{ GeV}^2$, two lattice spacings $a \gtrsim 0.07 \text{ fm}$, two pion masses $m_\pi \gtrsim 170 \text{ MeV}$
  
  (*No quark-disconnected contributions yet*)

- Form factor results overshoot experimental data $x(2 \ldots 2.5)$;
  $G_E/G_M$ ratios in qualitative agreement

  *Discretization?*
  *Excited states?*
  *Non-physical quark masses?*
  *Quark-Disconnected contributions?*

- Comparison to experiment important to validate lattice methods for computing relativistic nucleon matrix elements

  *Impact on lattice methodology for TMDs, PDFs, DAs calculation*